MACROECONOMICS PRELIM, JULY 2023

ANSWER KEY FOR QUESTIONS 1 AND 2 (ECN 200D)

Question 1

a) A RCE for this economy is a list of:

$$\text{pricing functions: } r(\bar{k}), w(\bar{k})$$
$$\text{policy function: } k' = g(k, \bar{k})$$
$$\text{value function: } V(k, \bar{k})$$
$$\text{transition function: } \bar{k}' = G(\bar{k})$$

such that:

1) $g(k, \bar{k})$ and $V(k, \bar{k})$ solve the agent's recursive problem

$$\begin{aligned} V(k, \bar{k}) &= \max_{c,k'}\{u(c) + \beta V(k', \bar{k}')\}, \\ s.t.\ \ c + k' &= w(\bar{k}) + \left[r(\bar{k}) + 1 - \delta\right] k, \\ \bar{k}' &= G(\bar{k}). \end{aligned}$$

2) Prices are competitively determined:

$$\begin{aligned} r(\bar{k}) &= F_1(\bar{k}, \bar{k}) = aA\bar{k}^{a+\gamma-1}, \\ w(\bar{k}) &= F_2(\bar{k}, \bar{k}) = (1-a)A\bar{k}^{a+\gamma}. \end{aligned}$$

3) Aggregate consistency:

$$G(\bar{k}) = g(\bar{k}, \bar{k}).$$

Solving the agent's recursive problem (in any way you find convenient) will lead to the following Euler equation:

$$u'(c) = \beta \left[r(\bar{k}') + 1 - \delta\right] u'(c').$$

Typically, at this point we can replace the consumption expressions above from the budget constraint and describe the RCE more sharply (as we did in class). However, since the question explicitly asks you to describe only the steady state capital, we can focus on steady states right away, and re-write the last Euler equation as

$$1 = \beta \left[r(\bar{k}) + 1 - \delta \right].$$

We can now replace the rental rate of capital from the definition of the RCE above, and after a little algebra we will arrive at the desired result. More specifically, we will find that the steady state level of capital in this economy with production externalities will be given by

$$\bar{k} = \left[\frac{1 - \beta(1-\delta)}{a\beta A} \right]^{\frac{1}{a+\gamma-1}}.$$

b) We now move on to the problem of the social planner who, as we described in the question, internalizes the production externality. The recursive problem for the social planner is as follows:

$$\begin{aligned} V(\bar{k}) &= \max_{c,\bar{k}'} \{u(c) + \beta V(\bar{k}')\}, \\ s.t. \;\; c + k' &= A\bar{k}^{a+\gamma} + (1-\delta)k. \end{aligned}$$

Solving the social planner's recursive problem (in any way you find convenient) will lead to the following Euler equation:

$$u'(c) = \beta \left[(a+\gamma) A \bar{k}^{a+\gamma-1} + 1 - \delta \right] u'(c').$$

Once again we will focus right away on steady states, and this will allow us (after some algebra) to immediately calculate the steady state level of capital that the hypothetical social planner would have chosen:

$$k^S = \left[\frac{1 - \beta(1-\delta)}{(a+\gamma)\beta A} \right]^{\frac{1}{a+\gamma-1}}.$$

Our last task in this part is to compare this level of capital with the one attained in the decentralized economy (of part (a)), where agents do not internalize the production externality. Of course, our intuition strongly suggests that the social planner, who internalizes the externality, would choose a higher level of capital. Our intuition will be confirmed if and only if

$$\bar{k} < k^S \Leftrightarrow \left[\frac{1 - \beta(1-\delta)}{a\beta A} \right]^{\frac{1}{a+\gamma-1}} < \left[\frac{1 - \beta(1-\delta)}{(a+\gamma)\beta A} \right]^{\frac{1}{a+\gamma-1}}.$$

Notice that one of the maintained assumptions of the model is that $\gamma < 1 - a$, implying that $a + \gamma - 1 < 0$, which, in turn, means that the powers to which these expressions are risen are negative, and the result will be confirmed if and only if

$$\frac{1-\beta(1-\delta)}{a\beta A} > \frac{1-\beta(1-\delta)}{(a+\gamma)\beta A} \Leftrightarrow a+\gamma > a,$$

which, of course, is true. Thus, the math confirms our strong conjecture that the social planner's steady state capital should be higher. In fact, it also tells us that it will be higher as long as $\gamma > 0$, since $\gamma = 0$ would simply capture the case where the externality completely disappears.

c) We now move to the (decentralized) economy where the government imposes a lump-sum tax, and uses the tax revenue to subsidize investment (hoping that in this way they will be able to incentivize agents to make the "correct" investment decisions). In this new environment, a RCE is a list of:

$$\begin{gathered}
\text{pricing functions: } r(\bar{k}), w(\bar{k}) \\
\text{policy function: } k' = g(k, \bar{k}) \\
\text{value function: } V(k, \bar{k}) \\
\text{transition function: } \bar{k}' = G(\bar{k}) \\
\text{taxation function: } T(\bar{k})
\end{gathered}$$

such that:

1) $g(k, \bar{k})$ and $V(k, \bar{k})$ solve the agent's recursive problem:

$$\begin{aligned}
V(k, \bar{k}) &= \max_{c,k'} \{u(c) + \beta V(k', \bar{k}')\}, \\
s.t.\ c + (1-\tau)\left[k' - (1-\delta)k\right] &= w(\bar{k}) + r(\bar{k})k - T(\bar{k}), \\
\bar{k}' &= G(\bar{k}).
\end{aligned}$$

2) Prices are competitively determined:

$$\begin{aligned}
r(\bar{k}) &= F_1(\bar{k}, \bar{k}) = aA\bar{k}^{a+\gamma-1}, \\
w(\bar{k}) &= F_2(\bar{k}, \bar{k}) = (1-a)A\bar{k}^{a+\gamma}.
\end{aligned}$$

3) Aggregate consistency:

$$G(\bar{k}) = g(\bar{k}, \bar{k}).$$

4) The government's budget constraint is satisfied in every period:

$$T(\bar{k}) = \tau\left[\bar{k}' - (1-\delta)\bar{k}\right].$$

d) In the last part, we need to calculate the subsidy rate that would incentivize agents to fully internalize the production externality, thus, equating the steady state level of capital in the decentralized economy with that of the social planner (described in part (b)). First, we need to solve the new recursive problem of the typical agent, described earlier in part (1) of the definition of the RCE. Solving that problem (in any way you find convenient) will lead to the following Euler equation:

$$(1-\tau)u'(c) = \beta\left[r(\bar{k}') + (1-\tau)(1-\delta)\right]u'(c').$$

Once again, here there is no reason to develop this Euler equation any further in order to characterize the RCE more sharply. We will simply focus on steady states right away, and re-write the last Euler equation as

$$1-\tau = \beta\left[aA\bar{k}^{a+\gamma-1} + (1-\tau)(1-\delta)\right].$$

We can now solve for the steady state level of capital in this economy with production externalities *and* government intervention, and, eventually, find that it is given by:

$$\bar{k} = \left[\frac{1-\tau-\beta(1-\tau)(1-\delta)}{a\beta A}\right]^{\frac{1}{a+\gamma-1}}.$$

The level of government subsidies that would equalize this level of capital with the one chosen by the social planner, is the τ that equalizes the above expression with the expression for k^S provided in part (b). That is, we are looking for the τ that solves:

$$\left[\frac{1-\tau-\beta(1-\tau)(1-\delta)}{a\beta A}\right]^{\frac{1}{a+\gamma-1}} = \left[\frac{1-\beta(1-\delta)}{(a+\gamma)\beta A}\right]^{\frac{1}{a+\gamma-1}}.$$

After some easy algebra, we finally obtain the unique value of τ that achieves the desired result, and it is given by:[1]

$$\tau = \frac{\gamma}{a+\gamma}.$$

[1] In questions like this, it is always good to perform a logical consistency test. Logic and intuition suggest that if γ was equal to 0, then the government should have no reason to intervene. Indeed, we can see in the formula above that if γ is equal to zero, the same should be true for τ. This type of test should make you very confident that the result you found is correct. (Or, if the result makes no sense, it could help you identify that you have made a mistake and try again!)

Question 2

Notice that this question is similar to one of the questions in your midterm. The difference is that in the midterm type-0 workers did not earn unemployment benefits, while here they are less productive.

a) This part is identical to the midterm. (Type-0 workers are different in a different dimension compared to the midterm, but that does not change their equilibrium measure.) As we saw in the midterm, the Beveridge curves will be given by:

$$u_0 = \frac{\delta}{\theta q(\theta) + \delta}$$

$$u_1 = \frac{\lambda(1-u_0)}{\theta q(\theta) + \lambda + \delta} = \frac{\lambda}{\theta q(\theta) + \lambda + \delta} \cdot \frac{\theta q(\theta)}{\theta q(\theta) + \delta}$$

For future reference, we will define $\gamma(\theta)$ to be the probability of matching with a type-0 worker, conditional on matching with some worker. Then,

$$\gamma(\theta) = \frac{u_0}{u_0 + u_1} = \frac{\delta(\theta q(\theta) + \lambda + \delta)}{\lambda \theta q(\theta) + \delta(\theta q(\theta) + \lambda + \delta)}$$

b) We have the following value functions for workers:

$$(r+\delta)U_0 = z + \theta q(\theta)(W_0 - U_0)$$

$$(r+\delta)U_1 = z + \theta q(\theta)(W_1 - U_1)$$

$$(r+\delta)W_0 = w_0 + \lambda(U_1 - W_0)$$

$$(r+\delta)W_1 = w_1 + \lambda(U_1 - W_1)$$

c) For firms, the value functions in the various states are as follows:

$$rV = -c + q(\theta)\left[\gamma(\theta)J_0 + (1-\gamma(\theta))J_1\right]$$

$$rJ_0 = p - \kappa - w_0 - (\lambda+\delta)J_0$$

$$rJ_1 = p - w_1 - (\lambda+\delta)J_1$$

The free entry condition still requires $V = 0$. Then, we can derive the free entry condition:

$$c = \frac{q(\theta)}{r+\lambda+\delta}\left[\gamma(\theta)\left(p-\kappa-w_0\right) + \left(1-\gamma(\theta)\right)\left(p-w_1\right)\right] \tag{1}$$

d) Using standard methods, we find that the Nash bargaining solution will require

$$(1-\beta)(W_1 - U_1) = \beta J_1$$

Replace the value functions from previous parts in the expression above, and after some algebra we find the Wage Curve for workers of type 1:

$$w_1 \quad = \quad \frac{r+\lambda+\delta}{r+\lambda+\delta+\beta\theta q(\theta)}\left[\beta p + (1-\beta)z + \frac{p\beta\theta q(\theta)}{r+\lambda+\delta}\right] \tag{2}$$

e) Using standard methods, we find that the Nash bargaining solution will require

$$(1-\beta)(W_0 - U_0) = \beta J_0$$

Again, our first task is to replace these value functions with the expressions provided earlier. After some algebra we find that

$$w_0 = \beta(p-\kappa) + (1-\beta)(r+\delta)U_0 - (1-\beta)\lambda(U_1 - U_0). \tag{3}$$

In this expression, we will replace U_0 directly from part (b). For the term $U_1 - U_0$, notice, again from part (b), that we can write it as

$$U_1 - U_0 = \frac{\theta q(\theta)}{r+\delta+\theta q(\theta)}(W_1 - W_0)$$

With a little more work, we can get rid of the term $W_1 - W_0$ (replace it from the value functions), and, eventually, realize that

$$U_1 - U_0 = \frac{\theta q(\theta)}{(r+\delta+\theta q(\theta))(r+\delta+\lambda)}(w_1 - w_0)$$

This is precisely what I was suggesting you do in the hint. We can now replace U_0 from part (b) and $U_1 - U_0$ from the last expression directly into equation (3). This will give us the desired wage curve, which is a relationship between w_0 and θ, but also w_1. More precisely the wage curve for type-0 workers is as follows:

$$\begin{aligned} w_0\left[1 + \frac{\beta\theta q(\theta)}{r+\delta+\lambda} - \frac{(1-\beta)\lambda\theta q(\theta)}{(r+\delta+\lambda)(r+\delta+\theta q(\theta))}\right] &= \\ \beta(p-\kappa) + (1-\beta)z + \frac{(p-\kappa)\beta\theta q(\theta)}{r+\delta+\lambda} - (1-\beta)\lambda\frac{\theta q(\theta)}{(r+\delta+\lambda)(r+\delta+\theta q(\theta))}w_1 & \end{aligned} \tag{4}$$

While unusual, the fact that here w_0 is also a function of w_1 is not strange. Type-0 workers know that their only chance to leave this bad state is if they get their first job.

After that, they will automatically be type-1 workers. Thus, a higher w_1, the wage they will be able to enjoy *after* they get (and lose) their first job provides extra incentives for workers to be willing to accept a lower w_0. (Hence, the minus in the last expression.)

f) We have five equilibrium (endogenous) variables: $\{\theta, w_0, w_1, u_0, u_1\}$.

The free entry condition together with the two wage curves is a self-containing block that allows us to solve for θ, w_0, w_1. Once θ has been obtained, the two Beveridge curves can be used to find the unemployment rates for the two types of workers.

g) It is possible. Here type-0 workers can make at least z when they are unemployed. This increases their outside option and makes them pickier. On the other hand, getting that first job allows the workers to "transform" to type-1 and enjoy the much higher w_1 thereafter. Thus, one can come up with parameter values so extreme, that a type-0 worker would be willing to pay just to get his/her first job! What parameters could make that happen? A very low z and a very high κ. (Just take the limit as $\kappa \to p$.)

Regardless of whether this is realistic or not, this is exactly the channel I expected you to discuss: the effect of getting that first job and moving into the better state "1".