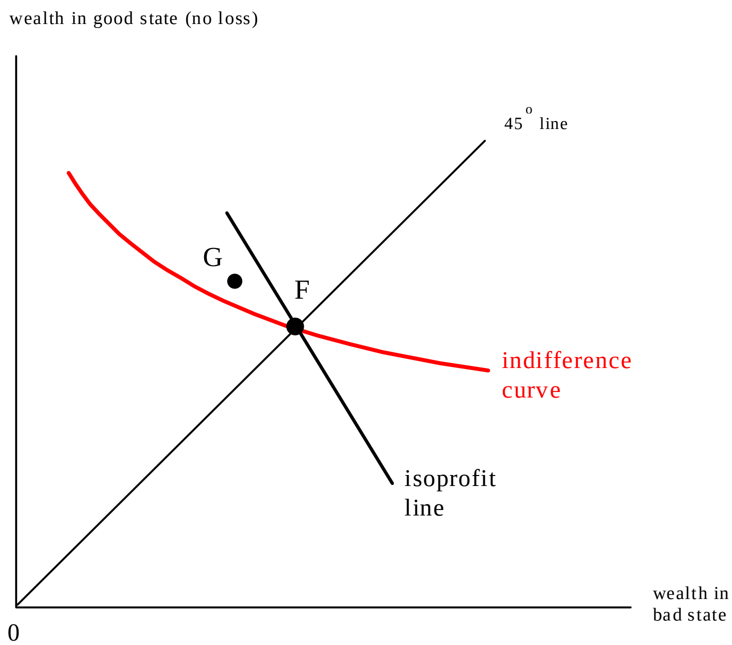
Micro Prelim June 20, 2023
QUESTION 1 ANSWER KEYS

(a) The absolute value of the slope of the indifference curve at point (W_1, W_2) is $\frac{p_L}{1-p_L}\frac{U'(W_1)}{U'(W_2)}$ and thus at a point on the 45^0 line it is $\frac{p_L}{1-p_L}$. On the other hand, the absolute value of the slope of any isoprofit line is $\frac{p}{1-p}$, which is greater than $\frac{p_L}{1-p_L}$ since $p > p_L$. Thus a full-insurance contract (such as point *F* in the figure below) cannot be profit-maximizing, because there are points (such as point *G*) that are below the isoprofit line (thus corresponding to higher profits) and above the indifference curve (thus preferred to *F* by the customers). The indifference curve shown in the diagram below could be the reservation indifference curve or a higher one. The argument is the same.



The above picture refers to the case where the isoprofit line that goes through contract F corresponds to non-negative profits. There is also the possibility that the zero isoprofit line (the line with slope $-\frac{p}{1-p}$ that goes through the NI point) is entirely below the "delusional" reservation indifference curve, in which case the monopolist would not be able to make positive profits with any contract (and thus would not offer any contract, in particular the full-insurance contract F).

(b.1) The subjective expected utility of no insurance is $\frac{1}{7}\ln(2,200) + \frac{6}{7}\ln(4,000) = 8.21$. The objective expected utility of no insurance is $\frac{1}{5}\ln(2,200) + \frac{4}{4}\ln(4,000) = 8.17$.

(b.2) The profit-maximizing contract is that point on the reservation indifference curve at which the absolute value of the slope of the indifference curve is equal to $\frac{p}{1-p}$, assuming that positive profits are possible: see the last paragraph of part (a). Let *h* be the premium and *d* the deductible. Then we need to solve

$$\frac{p_L}{1-p_L}\frac{U'(W-h-d)}{U'(W-h)} = \frac{p}{1-p}, \quad \text{and}$$

$$p_L U(W-h-d) + (1-p_L)U(W-h) = p_L U(W-L) + (1-p_L)U(W)$$

Given the parameter values, the first equation is $\boxed{\frac{1}{6}\left(\frac{4,000-h}{4,000-h-d}\right)=\frac{1}{4}}$ and the second equation is $\boxed{\tfrac{1}{7}\ln(4,000-h-d)+\tfrac{6}{7}\ln(4,000-h)=8.21}$. [The solution is h = 108.43, d = 1,297.19.]

(c) There must be a point on the reservation indifference curve at which the absolute value of the slope of the indifference curve is equal to $\frac{p}{1-p}$. This is the case if and only if the slope of the indifference curve at the no insurance point is greater than $\frac{p}{1-p}$. Thus the necessary and sufficient condition is:

$$\boxed{\frac{p_L}{1-p_L}\frac{U'(W-L)}{U'(W)}>\frac{p}{1-p}}$$

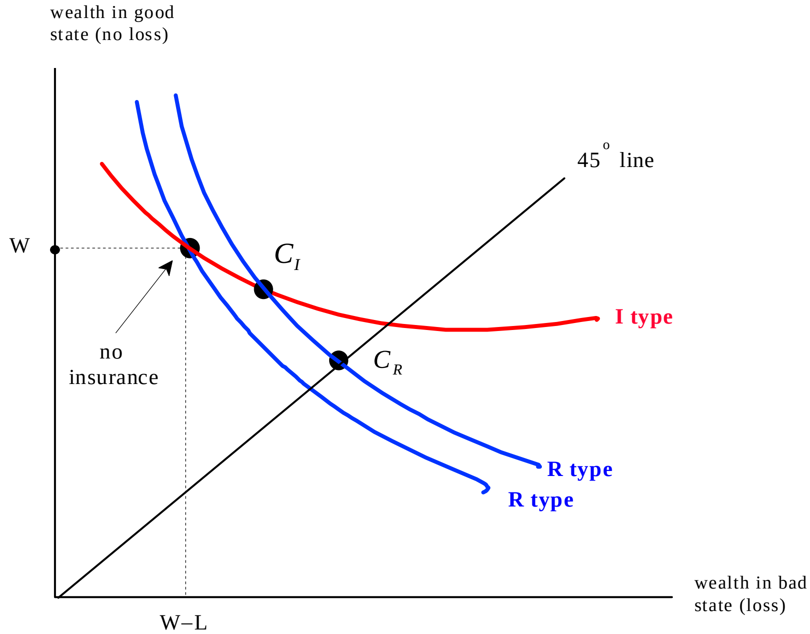
(d) To the *I* group the monopolist will offer the contract of part (b.2) (assuming that positive profits are possible: see the last paragraph of part (a)), while to the *R* group it will offer the full-insurance contract with premium *h* given by the solution to

$$U(W-h)=pU(W-L)+(1-p)U(W).$$

(e.1) $\underset{q_H,d_H,q_L,d_L}{\text{Max}}\ \Pi=\tfrac{n_R}{n}\left[h_H-p\left(L-d_H\right)\right]+\tfrac{n_L}{n}\left[h_L-p\left(L-d_L\right)\right]$ (note: p in both cases) subject to

$$\begin{array}{ll}
(IR_L) & p_LU(W-h_L-d_L)+\left(1-p_L\right)U(W-h_L)\geq p_LU(W-L)+\left(1-p_L\right)U(W) \\
(IC_L) & p_LU(W-h_L-d_L)+\left(1-p_L\right)U(W-h_L)\geq p_LU(W-h_H-d_H)+\left(1-p_L\right)U(W-h_H) \\
(IR_H) & pU(W-h_H-d_H)+\left(1-p\right)U(W-h_H)\geq pU(W-L)+\left(1-p\right)U(W) \\
(IC_H) & pU(W-h_H-d_H)+\left(1-p\right)U(W-h_H)\geq pU(W-h_L-d_L)+\left(1-p\right)U(W-h_L)
\end{array}$$

(e.2) If there are mostly type *R* people in the population then the monopolist will offer only the full-insurance contract with premium *h* given by the solution to $U(W-h)=pU(W-L)+(1-p)U(W)$ thus serving only type *R* individuals. If there is a "sufficient number" of *I* types, then the monopolist will offer a full-insurance contract targeted to the *R* types that gives them a positive surplus and as contract targeted to the *I* type it will offer a partial insurance contract at the intersection of two indifference curves (the reservation indifference curve of the *I* types and the indifference curve of the *R* types that goes through the contract targeted to them), as shown in the following figure (C_I is the contract targeted to the *I* types and C_R is the contract targeted to the *R* types).



(f) In case (b.2) the *I* types are the opposite of exploited: in terms of the objective probability of loss, they get higher expected utility than if they were not insured (using the objective probability to evaluate the latter).
In case (d) the *R* types are just as well off as they would be without insurance, while the I types are better off.
In case (e), the *R* types are at least as well off as they would be without insurance, so are not exploited (they are as well of as without insurance if the monopolist serves only the *R* types and better off if the monopolist serves both types). The *I* types get higher expected utility than if they were not insured (using the objective probability to evaluate the latter).
So, in all three cases no customer is exploited.

Question 2 Answer keys

Consider an exchange economy with two commodities. Suppose that for each individual, the utility function $u^i : \mathbb{R}^2_+ \to \mathbb{R}$ is continuous, strictly quasi-concave and strictly monotone, and her endowment is strictly positive: $w^i \in \mathbb{R}^2_{++}$.

If we fix the price of good 1 at 1 and denote the price of commodity 2 as $p \in \mathbb{R}_{++}$, the individual demands are

$$x^i(p) = \arg\max_{x \in \mathbb{R}^2_+} \left\{ u^i(x) : x_1 + px_2 \leq w^i_1 + pw^i_2 \right\}.$$

These functions are continuous and you can take for granted that each of them satisfies the following properties:

(i) for all $\Delta > 0$, there exists $\underline{\pi}^i > 0$ such that $x^i_2(p) > \Delta$ if $p \leq \underline{\pi}^i$; and

(ii) for all $\Delta > 0$, there exists $\bar{\pi}^i > 0$ such that $x^i_1(p) > \Delta$ if $p \geq \bar{\pi}^i$.

(These properties simply say that if one commodity becomes arbitrarily cheap while the other remains expensive, the individuals' demand for the cheap commodity becomes unboundedly large.)

Define the function $Z(p) = \sum_i [x^i_2(p) - w^i_2]$.

(a) How do you interpret the function Z?

(b) Argue that if $Z(p^*) = 0$, then $((1, p^*), (x^i(p^*))_{i \in \mathcal{I}})$ is a competitive equilibrium for the economy.

(c) Argue that there exist $\underline{\pi} > 0$ and $\bar{\pi} > 0$ such that $Z(p) > 0$ if $p \leq \underline{\pi}$ and $Z(p) < 0$ if $p \geq \bar{\pi}$.

(d) The intermediate value theorem says the following:

> *Suppose that $X \subseteq \mathbb{R}$ is an interval, function $f : X \to \mathbb{R}$ is continuous, and $\underline{x}, \bar{x} \in X$ are such that $\underline{x} < \bar{x}$, $f(\underline{x}) > 0$, and $f(\bar{x}) < 0$. Then, there exists $x^* \in (\underline{x}, \bar{x})$ for which $f(x^*) = 0$.*

Use this theorem to prove that there exists $p^* \in \mathbb{R}_{++}$ for which $Z(p^*) = 0$.

(e) Argue that this economy has at least one competitive equilibrium.

(f) Consider now the demand of agent $i = 1$ as a function also of her endowments:

$$x^1(p, w^1) = \arg \max_{x \in \mathbb{R}^2_+} \left\{ u^1(x) : x_1 + px_2 \leq w^1_1 + pw^1_2 \right\}.$$

Argue that

$$x^1(p, w^1 + \delta(p, -1)) = x^1(p, w^1).$$

(g) Redefining

$$Z(p, w^1) = [x^1_2(p, w^1) - w^1_2] + \sum_{i \neq 1} [x^i_2(p) - w^i_2],$$

argue that for all δ

$$Z(p, w^1 + \delta(p, -1)) = Z(p, w^1) + \delta.$$

(h) Formalize the following claim, which the previous point proves:

> Suppose that $(1, p)$ is a competitive equilibrium price vector of an exchange economy with continuous and strictly quasi-concave utility functions, and with strictly positive endowments for individual $i = 1$. There exists another economy where:
>
> (i) the only difference is the endowments of agent $i = 1$,
>
> (ii) the magnitude of this difference is arbitrarily small, and
>
> (iii) $(1, p)$ is not a competitive equilibrium price vector of this other economy.

Sketched Answers:

(a) This is the economy's aggregate excess demand function for good 2.

(b) Since the definition of each function x^i implies individual rationality, we only need to check market clearing.

Suppose that $Z(p^*) = 0$. By definition, the market for good 2 is clearing: $\sum_i x^i_2(p^*) = \sum_i w^i_2$. Since preferences are strictly monotone, we can invoke Walras's law to argue that the market for good 1 clears too. Alternatively, monotonicity implies that $x_1^i(p^*) + p^* x_2^i(p^*) = w_1^i + p^* w_2^i$ for all i. This implies that

$$\sum_i [x_1^i(p^*) - w_1^i] = \sum_i p^* [w_2^i - x_2^i(p^*)] = p^* Z(p^*) = 0.$$

Incidentally, observe that $Z(p^*) = 0$ is also a necessary condition for $(1, p^*)$ to be a competitive equilibrium price vector.

(c) For each i, the property (i) above implies that there is $\underline{\pi}^i > 0$ such that $x_2^i(p) > w_2^i$ when $p \leq \underline{\pi}^i$. Let $\underline{\pi} = \min_i \{\underline{\pi}^i\}$.

Similarly, property (ii) implies that there is $\bar{\pi}^i > 0$ such that $x_1^i(p) > w_1^i$ when $p \geq \bar{\pi}^i$. By local non-satiation of preferences, $x_1^i(p) - w_1^i = p[w_1^i - x_1^i(p)]$, so $x_2^i(p) < w_2^i$ when $p \geq \bar{\pi}^i$. Let $\bar{\pi} = \min_i \{\bar{\pi}^i\}$.

(d) Since all utility functions are continuous, so is function Z. By construction, it must be true that $\underline{\pi} < \bar{\pi}$: otherwise, for any $\bar{\pi} \leq p \leq \underline{\pi}$, $Z(p) < 0 < Z(p)$. Since, moreover, $Z(\underline{\pi}) > 0 > Z(\bar{\pi})$, the result follows immediately from the intermediate value theorem.

(e) This follows from parts (d) and (b).

(f) Just note that

$$(1, p) \cdot [w^1 + \delta(p, -1)] = (1, p) \cdot w^1 + \delta(1, p) \cdot (p, -1) = (1, p) \cdot w^1.$$

(The first individual's nominal income at prices p is the same under the two endowment vectors.)

(g) By direct computation, for all δ

$$\begin{aligned} Z(p, w + \delta(p, -1)) &= [x_2^1(p, w + \delta(p, -1)) - (w_2 - \delta)] + \sum_{i \neq 1} [x_2^i(p) - w_2^i] \\ &= [x_2^1(p, w) - w_2] + \sum_{i \neq 1} [x_2^i(p) - w_2^i] + \delta \\ &= Z(p, w) + \delta, \end{aligned}$$

where the second equality comes from the previous point.

(h) The previous point proves the following:

Let $(p, \mathbf{x})$ be a competitive equilibrium of economy $\{\mathcal{I}, \mathbf{u}, \mathbf{w}\}$, and let $\varepsilon > 0$. There exists a profile of endowments $\tilde{\mathbf{w}}$ such that:

(i) *for all $i \neq 1$, $\tilde{w}^i = w^i$;*

(ii) $\|\tilde{w}^1 - w^1\| < \varepsilon$*; and*

(iii) $x^1(p, \tilde{w}^1) + \sum_{i \neq 1} x^i(p) \neq \sum_i \tilde{w}^i$.

To see why this is the case, note that one can always find $\delta \neq 0$, with $|\delta|$ small enough that parts (ii) and (iii) are guaranteed.

Comment: Parts (a)–(e) constitute a simple existence argument for economies with two commodities. Unfortunately, this argument doesn't extend to an arbitrary number of commodities, and it becomes necessary to use a more sophisticated proof using fixed point theory. Parts (f)–(h) can be extended to any number of commodities, on the other hand. They constitute the first steps on the argument for local uniqueness of competitive equilibrium, which uses transversality theory. Existence and local uniqueness are some of the positive results that we didn't have time to cover in class.

Question 3 Answer Keys

The economics faculty is at lunch in the department. Professor Clark mentions that he taught Giffen goods in his principles class today. He laments that it is hard to find examples of Giffen goods. He suggests that wine could be a Giffen good as people sometimes buy it when the price is higher rather than lower. Professor Rapson interjects that when people buy somewhat pricier wine rather than cheap wine, it is most likely due to the fact that they cannot distinguish bad from good wine and take the price as a quality signal. Professor Schipper quips that if Professor Clark had paid attention to consumer theory when studying at Harvard, then he would know that Giffen good implies inferior good. And clearly wine is not an inferior good. There is a moment of silence. It is not clear whether economic logic stifled the conversation or Professor Schipper's arrogant undertone. Realizing latter, Professor Schipper asks (more rhetorically than seriously) how to overcome the argument that Giffen good implies inferior good.

a. Does the proposition that Giffen good implies inferior good depend on the existence of a utility function? Explain.

No. WARP implies that the Slutsky substitution matrix is negative semi-definite. This implies that the own-price substitution effect is non-positive (i.e., entries at the diagonal of the Slutsky substitution matrix). This in turn implies the proposition. (Look at my slides on WARP for details.)

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b. It dawns on Professor Schipper that we have to go beyond standard consumer theory in order to overturn the proposition that Giffen good implies inferior good. Wine is consumed in social settings. Could consumption externalities allow for a Giffen good that is not inferior? Being a slow thinker, Professor Schipper poses this as a prelim problem. Here is the problem description:

   Professor C spends his wealth w on wine and other goods. We denote by $x_1 \geq 0$ and $y_1 \geq 0$ Professor C's spending on other goods and wine, respectively. Professor C also cares about the wine that others drink. Since Professor Schipper only theorizes about alcohol, we enlist the help of Professor T who has non-trivial practical experience with wine. Denote by y_2 the amount of wine consumed by Professor T.[1] The price of wine is $p > 0$. The price of spending on other goods is normalized to 1. Professor C's problem is now

$$\max_{x_1, y_1} u(x_1, y_1, y_2)$$

   subject to the budget constraint

$$x_1 + py_1 \leq w.$$

[1]As always, names, characters, and incidents are either the products of the author's imagination or used in a fictitious manner. Any resemblance to actual persons, living or dead, or actual events is purely coincidental.

As usual, utility functions of economists are well-behaved, that is, the utility function of Professor C is concave and continuously differentiable with strict positive gradient in the interior of its domain for every y_2.

Ignoring the non-negativity constraints, write down the Lagrangian and state the first-order conditions. Assuming an interior and unique solution, simplify as much as possible and arrive at a system of equations that does not involve multipliers. At the moment, you do *not* need to solve for solutions $x_1(p, w, y_2)$ and $y_1(p, w, y_2)$. Are the first-order conditions also sufficient?

The Lagrangian is given by

$$\mathcal{L}(x_1, y_1, \lambda) = u(x_1, y_1, y_2) - \lambda_1(x_1 + py_1 - w).$$

The first-order conditions are

$$\begin{aligned} \frac{\partial u(x_1, y_1, y_2)}{\partial x_1} - \lambda &= 0 \\ \frac{\partial u(x_1, y_1, y_2)}{\partial y_1} - \lambda p &= 0 \\ \lambda(x_1 + py_1 - w) &= 0 \end{aligned}$$

Assuming that the budget constraint is binding, the first-order conditions can be reduced to

$$\begin{aligned} \frac{\partial u(x_1, y_1, y_2)}{\partial x_1} p - \frac{\partial u(x_1, y_1, y_2)}{\partial y_1} &= 0 \\ x_1 + py_1 &= w \end{aligned}$$

This system of equations does not involve any multipliers.

Since the utility function is assumed to be concave and the constraints are linear, the first-order conditions are also sufficient.

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c. From now on, assume that

$$u(x_1, y_1, y_2) := x_1 + \boldsymbol{a} \cdot \boldsymbol{y} - \frac{1}{2} \boldsymbol{y} \cdot B \boldsymbol{y} \tag{1}$$

with

$$\boldsymbol{a} := \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} >> \boldsymbol{0} \tag{2}$$

$$\boldsymbol{y} := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{3}$$

$$B := \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{4}$$

We assume that B is a positive definite symmetric matrix.

Specialize the system of equations from part a. to the utility function given by equations (1) to (4). We restrict prices, wealth, and parameters such that a satiation point of the utility function is outside any budget sets we consider.

Observe that

$$\frac{\partial u(x_1, y_1, y_2)}{\partial x_1} = 1$$
$$\begin{pmatrix} \frac{\partial u(x_1,y_1,y_2)}{\partial y_1} \\ \frac{\partial u(x_1,y_1,y_2)}{\partial y_2} \end{pmatrix} = \boldsymbol{a} - B\boldsymbol{y}$$

Thus, the system of equations from a. above specializes to

$$\begin{aligned} p - (a_1 - b_{11}y_1 - b_{12}y_2) &= 0 \\ x_1 + py_1 &= w \end{aligned}$$

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d. Use the system of equations from part c. to solve for Professor C's Walrasian demand functions.

From the first equation, we get

$$y_1(p, w, y_2) = \frac{a_1 - b_{12}y_2 - p}{b_{11}}$$

Substituting this equation into the second equation we obtain

$$\begin{aligned} x_1(p, w, y_2) &= w - py_1(p, w, y_2) \\ &= w - p\left(\frac{a_1 - b_{12}y_2 - p}{b_{11}}\right) \end{aligned}$$

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e. Check whether wine is a Giffen good for Professor C. How does his demand for wine change with the price?

We are interested in the $s_{y_1y_1}$-entry of the Slutsky substitution matrix. Since he has quasi-linear preferences that are linear in spending on other goods, there is no wealth effect for wine. Thus,

$$s_{y_1y_1} = \frac{\partial y_1(p, w, y_2)}{\partial p} + \frac{\partial y_1(p, w, y_2)}{\partial w} y_1(p, w, y_2) = \frac{\partial y_1(p, w, y_2)}{\partial p} = -\frac{1}{b_{11}}.$$

Since B is positive definite, we must have $b_{11} > 0$. Thus, Professor C' demand for wine goes up when the price goes down. It appears that wine is *not* a Giffen good for Professor C.

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f. How is Professor C's consumption of wine affected by Professor T's consumption of wine?

Again, since B is positive definite, we must have $b_{11} > 0$. Now

$$\frac{\partial y_1(p, w, y_2)}{\partial y_2} = -\frac{b_{12}}{b_{11}}$$

Thus,

$$\frac{\partial y_1(p, w, y_2)}{\partial y_2} > 0$$

if and only if $b_{12} < 0$.

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g. Professor C thinks that our earlier answer about how his consumption of wine responds to changes of the price of wine is wrong. He argues that if he cares about Professor T's consumption of wine, then there should also be an effect of the price change via Professor T's change of consumption of wine. Let's analyze this argument. Obviously, we need a model for Professor T's consumption. Assume that his utility function is given by

$$v(y_1, x_2, y_2) = y_1 x_2 y_2$$

That is, Professor T also cares about Professor C's consumption. Here x_2 denotes Professor T's spending on other goods. His budget set is given by

$$x_2 + p y_2 \leq m$$

for $m > 0$. Derive a condition on the parameters under which Professor C's consumption of wine (not a typo; we care about Professor C's consumption) is decreasing in the price of wine and explain. For this, you will have to derive explicitly Professor T's demand for wine.

Note that Professor T has Cobb-Douglas preferences. Using as optimality condition that the marginal rate of substitution equal to the price ratio,

$$\begin{aligned}
\frac{\frac{\partial v(y_1, x_2, y_2)}{\partial x_2}}{\frac{\partial v(y_1, x_2, y_2)}{\partial y_2}} &= \frac{1}{p} \\
\frac{y_2}{x_2} &= \frac{1}{p} \\
\frac{p y_2}{x_2} &= 1 \\
1 + \frac{p y_2}{x_2} &= 2 \\
\frac{x_2}{x_2} + \frac{p y_2}{x_2} &= 2 \\
\frac{m}{x_2} &= 2 \\
\frac{m}{2} &= x_2
\end{aligned}$$

which implies $py_2 = \frac{m}{2}$. Thus, $y_2(p, m, y_1) = \frac{m}{2p}$. The function is constant in y_1. Thus, we omit y_1 as argument. Now using the chain rule

$$D_p y_1(p, w, y_2(p, m)) = \frac{\partial y_1(p, w, y_2(p, m))}{\partial p} + \frac{\partial y_1(p, w, y_2(p, m))}{\partial y_2}\frac{\partial y_2(p, m)}{\partial p} = \frac{b_{12}\frac{m}{2p^2} - 1}{b_{11}} < 0$$

if $b_{12} \leq 0$. This is kind of intuitive because we know that when $b_{12} < 0$, then he would decrease his consumption of wine when Professor T decreases his. Since a price increase lets Professor T decrease his consumption of wine, it also decreases Professor C's consumption. So clearly, under this condition wine is not a Giffen good.

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h. Find a condition on the parameters such that Professor C's consumption of wine increases in the price of wine. Argue that wine is not an inferior good to him. Why do we have a counterexample to the proposition? Explain what's going on.

If $b_{12} > 0$, then it possible that he increases his consumption of wine when the price goes up. For instance, if Professor T's wealth is sufficiently large, i.e.,

$$m > \frac{2p^2}{b_{12}},$$

then Professor Clark's consumption of increases in p. This is also intuitive. If $b_{12} > 0$, then Professor Clark wants to increase his consumption of wine when Professor T decreases his, i.e., $\frac{\partial y_1(p,w,y_2(p,m))}{\partial y_2} < 0$. If Professor T's wealth is sufficiently large, Professor T also reacts more to price changes in absolute terms. This may overcompensate his "intrinsic" desire to reduce his consumption of other goods when the price increases. In such a case, wine behaves like a Giffen good for Professor C even though it is not inferior because there is no wealth effect! That is, with consumption externalities, it is not true anymore that Giffen good implies inferior good. So Professor Clark is right after all although perhaps in a way he did not anticipate.

Notice though that the Giffen good effect is not due to both professors merely dine and wine together. Rather, it is due to Professor C getting so upset about Professor T's large negative price response that he needs extra wine to cope with it.

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i. Did we miss something in our analysis? We know that Professor T also cares about Professor C's consumption of wine. Shouldn't Professor T's consumption of wine also depend on Professor C's consumption of wine? In other words, aren't both professors' consumption decisions interdependent so that we would need game theory to solve the problem? Explain why we do not need a fixed-point argument like Nash equilibrium (or an iterated best "wine consumption" response argument) to solve the problem.

Although Professor T cares about Professor C's consumption of wine because y_1 enters Professor T's utility function, it just scales his utility up or down but it does not affect Professor T's wine consumption behavior. The fact that Professor T's consumption of wine does not depend on Professor C's consumption is an artifact of his Cobb-Douglas utility function. We can see this clearly when we go from the first to the second equation of the answer to f. where y_1 cancels out in the numerator and denominator.
