MACROECONOMICS PRELIM, JULY 2019

ANSWER KEY FOR QUESTIONS 1 AND 2

Question 1

a) Since getting the budget constraint right here is very important let's write that separately first. The household has an income which it can allocate to either c or i. However, as I clearly implied in the hint, for any extra unit that goes into i the household's income (or resources) will go down by τ. Hence, we have

$$c + i = w + rk - \tau i + T,$$

which can be re-written as

$$c + i(1+\tau) = w + rk + T,$$

and since we know that $k' = (1-\delta)k + i$ (or, if you prefer $i = k' - (1-\delta)k$), the budget constraint becomes

$$c = w + [r + (1-\delta)(1+\tau)]k - (1+\tau)k' + T.$$

Hence, the typical household's problem can be written recursively as

$$V(k, K) = \max_{c,k'} \left\{ u(c) + \beta V(k', K') \right\}$$

$$s.t. \;\; c = w + [r + (1-\delta)(1+\tau)]k - (1+\tau)k' + T, \quad (1)$$

$$K' = H(K), \quad (2)$$

$$w = w(K) = F_2(K, 1), \quad (3)$$

$$r = r(K) = F_1(K, 1), \quad (4)$$

$$T = T(K) = \tau[H(K) - (1-\delta)K], \quad (5)$$

where k is the individual capital, and K is the aggregate capital. Moreover, (1) is the household's budget constraint, (2) is the aggregate law of motion of capital, (3) and (4) follow directly from market clearing, and (5) is the government revenue as a function of the aggregate state.

The definition of a RCE is standard, and can be found in the lecture notes (or in problem 4 of PS 6). In the definition of RCE, the most important part is to clarify that consistency requires $g(K, K) = H(K)$, where H was defined above, and g is the typical household's policy function.

b) The Euler equation for the typical household is given by

$$u'(c)(1+\tau) = \beta[F_1(K', 1) + (1-\delta)(1+\tau)]\, u'(c').$$

But the objective here was to express everything as a function of the aggregate capital stock only. To that end, notice that we can use the budget constraint to write consumption in a more useful form. First impose consistency (i.e., $k = K$) to obtain

$$c = w + [r + (1-\delta)(1+\tau)]K - (1+\tau)K' + T.$$

Next, replace the prices and the total revenue T with the terms that include the aggregate capital stock (i.e., $r = F_1(K,1)$, $w = F_2(K,1)$, and $T = \tau[K' - (1-\delta)K]$). We obtain:

$$\begin{aligned} c &= F_2(K,1) + [F_1(K,1) + (1-\delta)(1+\tau)]K - (1+\tau)K' + \tau[K' - (1-\delta)K] = \\ &= F_2(K,1) + F_1(K,1)K + (1-\delta)K - K'. \end{aligned}$$

Finally, exploiting Euler's Theorem and the definition of $f(K)$, we find that

$$c = f(K) - K'.$$

Using this expression back into the Euler equation, and noticing that $f'(K) = F_1(K,1) + 1 - \delta$, we obtain the second-order difference equation, which describes the law of motion of aggregate capital:

$$u'(f(K) - K')(1+\tau) = \beta[f'(K) + \tau(1-\delta)]\, u'(f(K') - K'').$$

c) In the steady-state equilibrium $c = c'$ and $K = K' = K''$. Imposing this condition on the Euler equation, it is easy to show that the steady state capital stock is given by:

$$K^*(\tau) \equiv \left\{ K : f'(K) = \frac{1 + \tau[1 - \beta(1-\delta)]}{\beta} \right\}. \tag{6}$$

d) When $\tau = 0$, it is easy to check that $K^*(0)$ solves $f'(K^*(0)) = 1/\beta$, which, of course, is exactly the same as the steady-state level of capital in the baseline model with no government and taxes (or, if you prefer, the same as in the social planner's problem). If $\tau = 1$, we have $f'(K^*) = [2 - \beta(1-\delta)]/\beta$.

e) No. As we just saw, for $\tau = 1$ we have $f'(K^*) = [2 - \beta(1-\delta)]/\beta$, which means that $K^*(1) > 0$. The reason why a proportional tax (equal to 100%) on capital income and on investment do not lead to the same result is simple. The tax on capital income is a tax on ALL the household's capital. The tax on investment is just a tax on new capital that comes to replace the depreciated one, so clearly it is not as severe.

f) With $F(K,N) = K^a N^{1-a}$, we have $f(K) = K^a + (1-\delta)K$ and $f'(K) = aK^{a-1} + 1 - \delta$. Using these functional forms in (6), we get

$$K^*(\tau) = \left[\frac{a\beta}{(1+\tau)[1 - \beta(1-\delta)]} \right]^{\frac{1}{1-a}}. \tag{7}$$

g) In the steady state, $T = T(K^*) = \tau\delta K^*$, i.e., investment is just enough to replace the depreciated capital. Replacing for K^* from (7), yields

$$T = \tau\delta K^*(\tau) = B\tau(1+\tau)^{\frac{1}{a-1}}, \tag{8}$$

where we have defined

$$B \equiv \delta\left[\frac{a\beta}{1-\beta(1-\delta)}\right]^{\frac{1}{1-a}},$$

a positive constant. It is easy to show that

$$\frac{\partial T}{\partial \tau} = B\frac{1}{(1+\tau)^{\frac{1}{1-a}+1}}\frac{1-a(1+\tau)}{1-a}.$$

.Both B and the first fraction in this expression are positive. Thus, the sign of $\partial T/\partial\tau$ will coincide with the sign of the numerator of the second fraction. We have two cases:

- Case 1: If $a \leq 1/2$, then $\partial T/\partial\tau > 0$ for all $\tau \in [0,1]$, and the maximization of T has a "corner solution", i.e., $\tau = 1$.

- Case 2: If $a > 1/2$, then T obtains an interior maximum at

$$\tau^* = \frac{1-a}{a}.$$

Question 2

a) The typical buyer's enters the CM with some money, m, and some debt, say b. Her value function is given by

$$W(m,b) = \max_{X,H,\hat{m}} \left\{ U(X) - H + \beta V(\hat{m}) \right\},$$
$$s.t.\ \ X + \phi\hat{m} = \phi m + H - b + T,$$

where T is the lump-sum monetary transfer to the buyer by the monetary authority. Using the standard procedure described in class, we will find that

$$W(m,b) = \Lambda + \phi m - b,$$

where Λ just summarizes a number of terms that are not related to the state variables.

b) As we know, in these types of models the seller never wants to carry money out of the CM. The reason is very simple: money is costly to carry due to its liquidity, but the sellers, by default, can never take advantage of that liquidity. (So why pay the unnecessary cost?) Thus, as in the case of the buyer, the seller's VF will be linear in its argument(s). What are these arguments? If a seller is of type 1, she only accepts money, and the argument of that seller's value function is m. More precisely,

$$W_S^1(m) = \Lambda_S^1 + \phi m,$$

where Λ_S^1 is a constant.

If the seller si of type 2, she accepts credit and money and, typically, that seller's VF will have both m and b as arguments (where b just represents the amount of CM good that the buyer has promised to repay in the CM). Thus,

$$W_S^2(m,b) = \Lambda_S^0 + \phi m + b,$$

where Λ_S^0 is a constant of no direct interest to us.

This is all we need in order to solve the bargaining problem that follows.

c) Bargaining in a type-1 meeting is identical to the one we saw in class. The buyer wishes to purchase the first-best amount q^*, but she may be constrained by her cash holdings. Thus, the bargaining solution is as follows: $q_1 = \min\{\phi m, q^*\}$ and $d_1 = \min\{m, q^*/\phi\}$. In words, given the price of money, ϕ, q^*/ϕ is the amount of money that allows the buyer to buy the first best q^*. Then, either $m \geq q^*/\phi$ and the buyer gets $q_1 = q^*$, or $m < q^*/\phi$, and the buyer gives up all her money just to get $q_1 = \phi m < q^*$.

In a type-2 meeting things are very similar, except that the buyer can also purchase the day good using some credit, up to the amount $C < q*$. Thus, it turns out that there exist a critical level of money, call it m_2^*, such that buyers who carry that amount of money or more, can get q^* in the type-2 meeting. What is that amount? Clearly, it is[1]

$$m_2^* \equiv \frac{q^* - C}{\phi}.$$

Then, $q_2 = \min\{C + \phi m, q^*\}$ and $d_2 = \min\{m, m_2^*\}$.

d) As I explained, there is no need to formally derive the objective because it is quite intuitive (assuming one has understood the environment well). The one thing to notice (highlighted in the footnote), is that the objective function will behave differently around the point $m_2^* = \frac{q^* - C}{\phi'}$.[2] Intuitively, if the buyer chooses an m' higher than m_2^*, she can afford q^* in the event of meeting a type-2 seller (but not in the event of meeting a type-1 seller). If the buyer chooses an m' lower than m_2^*, then she cannot afford q^* even if she met a type-2 seller.[3] Therefore, the objective function can be summarized as follows.

- If $m' \geq m_2^*$, then

$$J(m') = (-\phi + \beta\phi')m' + \beta(1-\sigma)[u(\phi'm') - \phi'm'] + \beta\sigma[u(q^*) - (q^* - C)],$$

- If $m' < m_2^*$, then

$$J(m') = (-\phi + \beta\phi')m' + \beta(1-\sigma)[u(\phi'm') - \phi'm'] + \beta\sigma[u(\phi'm' + C) - \phi'm'].$$

e) The next step is to obtain the FOC and evaluate it in equilibrium.

For $m' \geq m_2^*$, the FOC yields:

$$\phi = \beta\phi'\Big\{1 + (1-\sigma)[u'(\phi'm') - 1]\Big\}.$$

For $m' < m_2^*$, the FOC yields:

$$\phi = \beta\phi'\Big\{1 + (1-\sigma)[u'(\phi'm') - 1] + \sigma[u'(\phi'm' + C) - 1]\Big\}.$$

In the steady state equilibrium, we have $z = \phi M = \phi' M'$. Moreover, using the Fisher equation, we can replace the term $(1 + \mu - \beta)/\beta$ with the nominal interest rate on an illiquid bond, i. Hence, the two earlier FOCs will become:

$$i = (1-\sigma)[u'(z) - 1], \tag{9}$$

$$i = (1-\sigma)[u'(z) - 1] + \sigma[u'(C + z) - 1], \tag{10}$$

[1] Notice that here, to make things interesting, we assume $C < q^*$; thus, even type 2 buyers cannot rely completely on credit. If we had $C \geq q^*$, then the credit limit would not be an issue and the question would be identical to the one in the August 2018 Prelim.

[2] Clearly, m_2^* is defined in an identical fashion as above, except it contains ϕ' in the denominator, because the objective function summarizes the optimal money holdings *for the next period.*

[3] As we discussed in class, with $i > 0$, the buyer will never choose $m' > q^*/\phi'$. Another way of saying this is that in the type-1 meeting the buyer will always spend all her money, or that the bargaining solution will be in the "binding branch", i.e., $d_1 = m$ and $q_1 = \phi m$.

which give us the equilibrium real balances, z, as a function of i. Only one question remains: for which i's is (9) the relevant equilibrium condition, and for which ones should we refer to (10)? (This is precisely the question I directed you to ask in the Hint.)

Condition (10) contains an extra term, which is the marginal utility of an additional dollar to a buyer who meets with a type-2 seller. That term is there (and is positive) precisely because that buyer did not carry too many dollars, so any additional dollar has a high valuation because it is helpful in BOTH types of meetings. (For the sake of discussion, let's call this the "scarce money" case). Of course, as the buyer carries more money, that term eventually disappears, because beyond a certain level of z, the buyer who meets a type-2 seller will be able to afford q^*. For that buyer the marginal benefit of an extra dollar, in the event of meeting a type-2 seller, is zero. (For the sake of discussion, let's call this the "plentiful money" case). What is the level of i that separates the space between the scarce and plentiful cases? Well, of course, the i for which $z + C = q^*$, or, equivalently,

$$\bar{\imath} \equiv (1-\sigma)[u'(q^* - C) - 1].$$

To summarize,

- For any $i \leq \bar{\imath}$, equilibrium z is given by (9);

- For any $i > \bar{\imath}$, equilibrium z is given by (10).

f) Given (e) this is very easy. Of course, $q_1 = z$: with TIOLI offers by the buyer, the amount she will purchase in a type-1 meeting is exactly equal to her real balances.

How about q_2? If the interest rate is relatively low, the buyer will carry enough balances so that these balances together with the credit will get her q^*. If the interest rate is high, the buyer will give up all her z and still not afford q^*. Formally,

- If $i \leq \bar{\imath}$, $q_2 = q^*$;

- If $i > \bar{\imath}$, $q_2 = z + C < q^*$.

g) As I explained in the question, the welfare function here is

$$\mathcal{W} = \sigma[u(q_2) - q_2] + (1-\sigma)[u(q_1) - q_1],$$

where q_1, q_2 were determined in the previous parts. Also, we assume a quadratic utility. It is easy to verify that with these specific preferences, we have $q^* = \gamma$ and $\bar{\imath} = (1-\sigma)C$. Hence, equilibrium is as follows:

- If $i \leq \bar{\imath}$, $q_1 = z = \gamma - \frac{i}{1-\sigma}$ and $q_2 = q^* = \gamma$;

- If $i > \bar{\imath}$, $q_1 = z = \gamma - i - \sigma C$ and $q_2 = \gamma - i + (1-\sigma)C$.

Now, recall that the question concerns the derivative of $\mathcal{W}$ with respect to C. If $i \leq \bar{\imath}$, then the equilibrium is irrelevant from C, so this is not an interesting case. So let's focus on the "scarce" case where interest rates are relatively high. Here,

$$\mathcal{W} = \sigma[u(\gamma - i + (1-\sigma)C) - (\gamma - i + (1-\sigma)C)] + (1-\sigma)[u(\gamma - i - \sigma C) - (\gamma - i - \sigma C)].$$

Exploiting the quadratic utility form one more time, and differentiating with respect to C yields (after some algebra):

$$\frac{\partial \mathcal{W}}{\partial C} = -\sigma(1-\sigma)C,$$

which is negative for any parameter values. The intuition is similar to the question in July 2017 Prelim (although, there, we were interested in $\partial \mathcal{W}/\partial \sigma$). A higher credit limit, C, sounds like a good idea because it implies that *ex post*, buyers who meet type-2 sellers will be able to get a very high q_2. But *ex ante* it discourages agents from carrying money. It turns out that with this functional form, the second force is so powerful that welfare ends up being decreasing in C for any parameter values.

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PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE,

Answer Key: 200E Questions

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Question 3 (Sketch Answer) *(20 points)*

Consider the following decentralized business cycle model with no trend growth. The household makes consumption (C) decisions to maximize lifetime utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t (\ln C_t) \tag{1}$$

subject to their budget constraint:

$$C_t + I_t = w_t N_t + r_t^k K_t + \Pi_t \tag{2}$$

where w is the real wage, N is hours worked, K is capital, I is investment, r^k is the rental price of capital and Π are profits from firms. Assume that labor is supplied inelastically so $N_t = 1$. There are adjustment costs to capital. As a result, capital evolves according to the following capital production function:

$$K_{t+1} = K_t^{\delta} I_t^{1-\delta} \tag{3}$$

When $\delta = 0$, this becomes the simple model we saw in class with full depreciation (i.e. where $K_{t+1} = I_t$).

Competitive firms produce output, Y_t, using capital and labor. The production function is given by:

$$Y_t = Z_t K_t^{\alpha} (N_t)^{1-\alpha} \tag{4}$$

where TFP, Z_t, is stochastic and follows a Markov process. There is no trend growth.

a) Write down the household's problem in recursive form and write down the firm's maximization problem. Derive the household's first order conditions and the firm's optimal hiring rules. For households, use two constraints: the budget constraint and the capital production function. Denote the Lagrange multiplier on the budget constraint as λ_t and the one on the capital production constraint as $\lambda_t q_t$ (where q_t is the relative price of capital goods).

Answer:

Households:

Denoting k_t^S as capital chosen by the representative household and K_t as the aggregate capital stock: The recursive formulation is:

$$V(k_t^S, K_t, Z_t) = \max_{C_t, I_t, K_{t+1}^S} \left(\ln C_t + \beta E_t V(k_{t+1}^S, K_{t+1}, Z_{t+1}) \right)$$

subject to

$$C_t + I_t = w_t N_t + r_t^k K_t + \Pi_t$$
$$K_t^{\delta} I_t^{1-\delta} = K_{t+1}$$

so the recursive formulation of the household's problem is:

$$V(k_t^S, K_t, Z_t) = \max_{C_t, I_t, K_{t+1}} \ln C_t + \beta E_t V(k_{t+1}^S, K_{t+1}, Z_{t+1}) \tag{5}$$

$$+ \lambda_t(w_t N_t^S + r_t^k K_t + \Pi_t - C_t - I_t) + \lambda_t q_t (K_t^\delta I_t^{1-\delta} - K_{t+1}) \tag{6}$$

FOCs:

$$\frac{1}{C_t} = \lambda_t$$

$$\lambda_t q_t = \beta E \partial V(k_{t+1}^S, K_{t+1}, Z_{t+1}) / \partial K_{t+1}$$

$$\frac{1}{q_t} = (1-\delta) K_t^\delta I_t^{-\delta}$$

q_t is the price of capital in terms of consumption goods. This is Tobin's q that we discussed in the lectures.

After using the Envelope theorem and tidying up we get:

$$q_t = \frac{1}{(1-\delta)} \frac{I_t}{K_{t+1}}$$

$$q_t = \beta E_t \left[\frac{C_t}{C_{t+1}} \left(r_{t+1}^k + \frac{\delta}{1-\delta} \frac{I_{t+1}}{K_{t+1}} \right) \right]$$

where the first expression has made use of the capital production function again.

Firms:

Denote k_t^d and N_t^d the capital stock and labor input demanded by the representative firm:

$$\max_{N_{d,t}, k_{d,t}} Z_t k_{D,t}^\alpha N_{D,t}^{1-\alpha} - w_t N_{D,t} - r_t^k k_{D,t} \tag{7}$$

$$r_t^k = \alpha Z_t k_{d,t}^{\alpha-1} N_{d,t}^{1-\alpha}$$

$$w_t = (1-\alpha) Z_t k_{d,t}^\alpha N_{d,t}^{-\alpha}$$

b) Carefully define a recursive competitive equilibrium.

Answer:

A recursive competitive equilibrium is a value function $V(Z_t, k_t^s, K_t)$ decision rules $k_{t+1} = k(Z_t, k_t^s, K_t)$, $c_t = c(Z_t, k_t^s, K_t)$, a law of motion for the aggregate capital stock $K_{t+1} = g(Z_t, K_t)$ and prices $\{w(Z_t, K_t), r^k(Z_t, K_t)\}$

Such that

- Given the pricing functions and the law of motion for capital, the value function and decision rules solve the households problem (the allocation satisfies all the first order conditions)
- The firms optimality conditions are satisfied
- All markets clear:

$$k_t^d = k_t^s = K_t$$
$$N_t^d = 1 = N_t$$
$$Y_t = C_t + I_t$$

and the law of motion for the individual state is consistent with the law of motion for the aggregate state (rational expectations):

$$g\left(Z_t, K_t\right) = k^s\left(Z_t, k_t^s, K_t\right)$$

c) Show that the equilibrium conditions in parts (a) and (b) are equivalent to the following equilibrium conditions from the social planner's problem:

$$q_t = \beta E_t\left[\frac{C_t}{C_{t+1}}\left(\alpha\frac{Y_{t+1}}{K_{t+1}} + \frac{\delta}{1-\delta}\frac{I_{t+1}}{K_{t+1}}\right)\right]$$
$$q_t = \frac{1}{(1-\delta)}\frac{I_t}{K_{t+1}}$$
$$Y_t = C_t + I_t$$
$$Y_t = Z_t K_t^\alpha$$

Answer:

Impose market clearing conditions and combine the Euler Equations with the firm's FOC. This yields

$$q_t = \beta E_t\left[\frac{C_t}{C_{t+1}}\left(\alpha\frac{Y_{t+1}}{K_{t+1}} + \frac{\delta}{1-\delta}\frac{I_{t+1}}{K_{t+1}}\right)\right]$$

and (as above)

$$q_t = \frac{1}{(1-\delta)}\frac{I_t}{K_{t+1}}$$

Substituting the optimal hiring rules for capital and labor into the profit function shows that the firm makes zero profit.

Substituting these optimal hiring rules into the household budget constraint, together with the zero profit result implies:

$$Y_t = C_t + I_t$$

These equilibrium conditions are what we would obtain from the solving the social planner's problem.

d) Solve this model using guess and verify. In particular, find the policy functions for K_{t+1}, I_t and C_t (**Hint:** start by guessing that investment and consumption are a constant share of output). Briefly discuss how, and why, the responses of consumption and investment to TFP shocks vary with δ.

Answer:

Let's guess that the solution for investment is as follows:

$$I_t = BY_t$$

where B is a constant to be found. From the resource constraint, this implies

$$C_t = (1 - B)Y_t$$

Combine the FOCs to eliminate q_t and substitute in the guesses:

$$\frac{I_t}{K_{t+1}}\frac{1}{1-\delta} = \beta E_t \left[\frac{(1-B)Y_t}{(1-B)Y_{t+1}} \left(\frac{\alpha Y_{t+1}}{K_{t+1}} + \frac{\delta}{1-\delta}\frac{BY_{t+1}}{K_{t+1}} \right) \right]$$

K_{t+1} is predetermined and cancels on the LHS and the RHS. Y_{t+1} also cancels out on the RHS. Everything that's left only depends on period t, so the E term doesn't matter. Using the guess again for I_t on the LHS implies:

$$BY_t = \beta(1-\delta)\left(\alpha + \frac{B\delta}{1-\delta}\right) Y_t$$

We can therefore solve for B

$$B = \frac{\beta\alpha(1-\delta)}{1-\beta\delta}$$

This makes sense: if $\delta = 0$, the capital production function therefore looks like the standard model we've seen before with full depreciation and the policy function for K_{t+1} would have the familiar form $K_{t+1} = I_t = \beta\alpha Y_t$. The full solution is:

$$I_t = \frac{\beta\alpha(1-\delta)}{1-\beta\delta} Z_t K_t^{\alpha}$$

and

$$K_{t+1} = \left(\frac{\beta\alpha(1-\delta)}{1-\beta\delta}\right)^{1-\delta} Z_t^{1-\delta} K_t^{(\delta+\alpha(1-\delta))}$$

$$C_t = \left(1 - \frac{\beta\alpha(1-\delta)}{1-\beta\delta}\right) Z_t K_t^{\alpha}$$

Examining the solution above we can see that when $\delta > 0$, I_t and K_{t+1} are less responsive to TFP shocks. Consumption also responds more. As δ falls towards 0 we can see that the response of investment gets larger. This means that δ controls how strongly investment and new capital respond to TFP shocks. When δ is larger, each unit of investment produces fewer units of new capital. When $\delta = 0$ investment will jump a lot of impact.

e) Briefly explain how you would solve this model computationally using a linearization-based technique. Give one advantage of this method over value function iteration.

Answer:

The answer could outline the method of Blanchard and Kahn. The full details were covered in the lectures but to summarize: (1) linearize all equilibrium conditions (2) calibrate steady state and structural parameters (3) rewrite the linearized system in matrix format such that

$$\underbrace{\mathbf{B}^{-1}\mathbf{A}}_{C} E_t \begin{bmatrix} \hat{x}_{t+1} \\ \hat{w}_{t+1} \end{bmatrix} = \begin{bmatrix} \hat{x}_t \\ \hat{w}_t \end{bmatrix} \tag{8}$$

(4) Check the Blanchard-Kahn conditions are satisfied. The number of unstable eigenvalues needs to equal the number of controls/jumps for there to be a unique stable solution. (5) Applying the Blanchard Kahn algorithm yields matrices **F** and **P** where **P** represents the transition matrix for state variables and **F** contains the policy functions both as percentage deviations from steady state.

$$\hat{x}_{t+1} = P\hat{x}_t \tag{9}$$

$$\hat{w}_t = F\hat{x}_t \tag{10}$$

One advantage of this method of VFI is that it can easily handle larger state spaces and many variables.

Question 4 (Sketch Answer) *(20 points)*

This question considers a time-varying inflation target in the New Keynesian model. Consider the following set of linearized equilibrium conditions for the standard New Keynesian model, but where the policy rule includes a stochastic inflation target $\bar{\pi}_t$.

$$E_t\tilde{y}_{t+1} - \tilde{y}_t = (\hat{i}_t - E_t\hat{\pi}_{t+1}) \tag{11}$$

$$\hat{\pi}_t = \beta E_t(\hat{\pi}_{t+1}) + \kappa\tilde{y}_t \tag{12}$$

Monetary policy follows a rule:

$$\hat{i}_t = \bar{\pi}_t + \phi(\hat{\pi}_t - \bar{\pi}_t) \tag{13}$$

where $\bar{\pi}_t$ is the inflation target, which is stochastic and follows an AR(1) process

$$\bar{\pi}_t = \rho\bar{\pi}_{t-1} + e_t \tag{14}$$

e_t is i.i.d.

$\tilde{y}_t$ is the output gap. In deviations from steady state: $\hat{i}_t$ is the nominal interest rate and $\hat{\pi}_t$ is inflation. Although this model is written in terms of a time-varying inflation target, everything has still been linearized around the usual New Keynesian steady state with zero inflation. κ is a function of model parameters, including the degree of price stickiness.[1] Assume that $\phi > 0$ and $0 < \beta < 1$. The elasticity of intertemporal substitution is 1.

a) Using the method of undetermined coefficients, find the response of the output gap and inflation to an exogenous reduction in the inflation target $\bar{\pi}_t$ when prices are sticky and monetary policy follows the policy rule above. To do this, guess that the solution for each variable is a linear function of $\bar{\pi}_t$.

Answer:

Assume the solution is a linear function of the shock:

$$\tilde{y}_t = \Lambda_y\bar{\pi}_t$$

$$\hat{\pi}_t = \Lambda_\pi\bar{\pi}_t$$

First, substitute the policy rule and the guesses into the dynamic IS curve to remove the endogenous variables. Also make use of the following property of the shock process: $E_t\bar{\pi}_{t+1} = \rho\bar{\pi}_t$ to remove the expectations terms. This yields an expression relating Λ_y and Λ_π:

$$\Lambda_y(\rho - 1) = \Lambda_\pi(1 + \phi - \rho) - \phi$$

[1] $\kappa = (1+\psi)\frac{(1-\theta)(1-\beta\theta)}{\theta}$ where θ is the probability that a firm cannot adjust its price and ψ is the inverse Frisch elasticity of labor supply.

By substituting the guesses into the Phillips Curve and again making use of the following property of the shock process: $E_t\bar{\pi}_{t+1} = \rho\bar{\pi}_t$, we obtain another express:

$$\kappa\Lambda_y = \Lambda_\pi(1 - \beta\rho)$$

Combing these allows us to solve for Λ_y and Λ_π:

$$\Lambda_\pi = \frac{\phi}{(1-\rho)\left(1 + \frac{1-\beta\rho}{\kappa}\right) + \phi}$$

$$\Lambda_y = \frac{\phi(1-\beta\rho)}{\kappa\left((1-\rho)\left(1 + \frac{1-\beta\rho}{\kappa}\right) + \phi\right)}$$

b) What is the solution for the path of the nominal interest rate following a shock to the inflation target? (**Hint**: one way to find this is by using the policy rule together with the solution you found in part (a)).

Answer:

Substituting the guesses into the policy rule:

$$\hat{i}_t = (\Lambda_\pi(1+\phi) - \phi)\bar{\pi}_t$$

$$\hat{i}_t = \left(\frac{\phi(1+\phi)}{(1-\rho)\left(1 + \frac{1-\beta\rho}{\kappa}\right) + \phi} - \phi\right)\bar{\pi}_t$$

c) First assume $\rho = 0$. Discuss how, and why, a reduction in the inflation target affects nominal interest rates, the output gap and inflation. How would your results change if $\rho = 1$?

Answer: First consider $\rho = 0$. This is a purely transitory shock to the inflation target and works a bit like a typical monetary policy shock.

$$\Lambda_\pi = \frac{\phi}{\left(1 + \frac{1}{\kappa}\right) + \phi} > 0$$

$$\Lambda_y = \frac{\phi}{\kappa\left(\left(1 + \frac{1}{\kappa}\right) + \phi\right)} > 0$$

$$\hat{i}_t = -\frac{\phi/\kappa}{\left(1 + \frac{1}{\kappa}\right) + \phi} < 0$$

A temporarily higher inflation target looks like a positive monetary policy shock. Interest rates fall to stimulate the economy. Because some firms adjust prices and others adjust quantities, GDP rises above potential, the output gap is positive and inflation increases temporarily.

Now consider $\rho = 1$. In this case:

$$\Lambda_\pi = 1$$

$$\Lambda_y = \frac{(1-\beta\rho)}{\kappa}$$

$$\hat{i}_t = \bar{\pi}_t$$

Shocks to the inflation target are permanent and inflation responds one-for-one. Because this is a permanent increase in the inflation target, for a given real interest rate, the nominal interest rate is also *higher.* Nominal interest rates also adjust one-for-one with the inflation target. A higher target requires higher nominal interest rates. This is sometimes called a Fisherian effect: interest rates and inflation move in the same direction. It also underpins the logic of some commentators who would like to raise the target inflation rate to reduce the chance of nominal rates hitting the Zero Lower Bound.

d) Now suppose that prices are flexible. Under flexible prices, the real interest rate is exogenous and constant. This model can be written as:

$$\hat{i}_t = E_t\hat{\pi}_{t+1} \tag{15}$$

$$\hat{i}_t = \bar{\pi}_t + \phi(\hat{\pi}_t - \bar{\pi}_t) \tag{16}$$

Show that $\phi > 0$ is required to ensure a unique stable solution for inflation in this flexible price model and find the stable solution for inflation.

Answer:

First combine the two equations in part (d) to produce one equation for the dynamics of inflation.

$$\hat{\pi}_t = \frac{1}{1+\phi}E_t\hat{\pi}_{t+1} + \frac{\phi}{1+\phi}\bar{\pi}_t$$

Solving this forward yields:

$$\hat{\pi}_t = \frac{\phi}{1+\phi}\sum_{k=0}^{\infty}\left(\frac{1}{1+\phi}\right)^k E_t\bar{\pi}_{t+k} + \lim_{T\to\infty}\left(\frac{1}{1+\phi}\right)^T E_t\hat{\pi}_{t+T} \tag{17}$$

To ensure a stable solution, we need the second term:

$$\lim_{T\to\infty}\left(\frac{1}{1+\phi_\pi}\right)^T E_t\hat{\pi}_{t+T}=0 \tag{18}$$

which shows that we need $\phi>0$.

Imposing this and using the fact that $E_t\bar{\pi}_{t+k}=\rho^k\bar{\pi}_t$ yields:

$$\hat{\pi}_t=\frac{\phi}{1+\phi-\rho}\bar{\pi}_t$$

e) When $\rho=1$, how does the result for inflation in part (d) compare to part (c) under sticky prices. Briefly discuss.

Answer:

When $\rho=1$, movements in the inflation target go one-for-one into inflation. This is the same result as in part (c). Also note, that flexible prices implies $\kappa\to\infty$, so the general condition found in this part of the equation is then equivalent to the condition found in part (c) as $\kappa\to\infty$.

Prelim 2019: Answer Key

Nicolas Caramp

Question 5 (*20 points*)

a) *State the entrepreneurs' problem. Take the first order condition with respect to l_1. Show that the entrepreneurs will always choose l_1 to maximize profits in period 1, that is, they choose l_1 to solve*

$$w_1 = (1-\alpha)k_1^\alpha l_1^{-\alpha}$$

Show that this implies that profits are a linear function of the capital stock k_1, that is,

$$y_1 - w_1 l_1 = R(w_1)k_1$$

for some function $R(\cdot)$ that depends only on w_1 and parameters.

The entrepreneurs' problem is

$$\max_{c_0, c_1, k_1 \geq 0, b_1} c_0 + c_1$$

subject to

$$c_0 + k_1 \leq n_0 + b_1$$
$$c_1 \leq k_1^\alpha l_1^{1-\alpha} - w_1 l_1 - b_1$$
$$b_1 \leq \lambda(k_1^\alpha l_1^{1-\alpha} - w_1 l_1)$$

Let μ_t be the Lagrange multiplier associated to the budget constraint in period t and η the Lagrange multiplier with respect to the borrowing constraint. The first order condition with respect to l_1 is

$$((1-\alpha)k_1^\alpha l_1^{-\alpha} - w_1)(\mu_1 + \eta) = 0$$

Since $\mu_1 > 0$ (the budget constraint *always* binds) and $\eta \geq 0$,

$$w_1 = (1-\alpha)k_1^\alpha l_1^{-\alpha}$$

This implies

$$w_1 l_1 = (1-\alpha)y_1$$

and

$$l_1 = \left(\frac{1-\alpha}{w_1}\right)^{\frac{1}{\alpha}} k_1$$

Therefore,

$$y_1 - w_1 l_1 = R(w_1) k_1$$

where $R(w_1) = \alpha \left(\frac{1-\alpha}{w_1}\right)^{\frac{1-\alpha}{\alpha}}$.

b) *Using the result from a), restate the entrepreneurs' problem as a simpler linear problem, replacing $y_1 - w_1 l_1$ by $R(w_1)k_1$. Characterize the solution to the entrepreneurs' problem with first order conditions.*

The simplified problem for the entrepreneurs is

$$\max_{c_0, c_1, k_1 \geq 0, b_1} \quad c_0 + c_1$$

subject to

$$c_0 + k_1 \leq n_0 + b_1$$
$$c_1 \leq R(w_1)k_1 - b_1$$
$$b_1 \leq \lambda R(w_1)k_1$$

The FOCs are

$$(c_0): \quad 1 - \mu_0 \leq 0 \tag{1}$$
$$(c_1): \quad 1 - \mu_1 \leq 0 \tag{2}$$
$$(k_1): \quad -\mu_0 + \mu_1 R(w_1) + \eta \lambda R(w_1) \leq 0 \tag{3}$$
$$(b_1): \quad \mu_0 - \mu_1 - \eta = 0 \tag{4}$$

c) *Show that if $\lambda R(w_1) < 1 \leq R(w_1)$ the entrepreneurs' investment in capital is strictly positive and finite. What happens if $\lambda R(w_1) \geq 1$? What if $R(w_1) < 1$?*

From (4) we have

$$\mu_0 = \mu_1 + \eta$$

Plugging in (3)

$$\mu_1(R(w_1) - 1) + \eta(\lambda R(w_1) - 1) \leq 0$$

If $R(w_1) < 1$, the FOC with respect to capital is strictly negative, so $k_1 = 0$. If $\lambda R(w_1) \geq 1$, the FOC with respect to capital is strictly positive, so investment is infinite. Now suppose $\lambda R(w_1) < 1 \leq R(w_1)$. Using that $\mu_0 = \mu_1 + \eta$, we have that FOC (3) is satisfied with equality if $\eta = \mu_1 \frac{1-\lambda R(w_1)}{R(w_1)-1}$, which is strictly positive since $\mu_1 \geq 1$. Finally, noting that $\mu_0 = \mu_1 + \eta > 1$, we have that $c_0 = 0$, hence $k_1 = n_0 + b_1 = n_0 + \lambda R(w_1)k_1 > 0$.

d) *Show that if the entrepreneurs are constrained, their choice of capital is*

$$k_1 = \frac{n_0}{1 - \lambda R(w_1)}$$

When the entrepreneurs are constrained, $\eta > 0$, the borrowing constraint binds and therefore

$$b_1 = \lambda R(w_1)k_1$$

From (4) we have

$$\mu_0 = \mu_1 + \eta > 1$$

hence $c_0 = 0$. From the budget constraint

$$k_1 = n_0 + \lambda R(w_1)k_1$$

or

$$k_1 = \frac{n_0}{1 - \lambda R(w_1)}$$

e) *Show that the equilibrium level of capital solves*

$$k_1 = \frac{n_0}{1 - \lambda \frac{\alpha}{k_1^{1-\alpha}}}$$

Show that k_1 and the entrepreneurs' profits are increasing in n_0. ***Hint:*** *Remember that we limit attention to the case in which $\lambda R(w_1) < 1 \leq R(w_1)$.*

From a) we know that

$$R(w_1) = \alpha \left(\frac{1-\alpha}{w_1}\right)^{\frac{1-\alpha}{\alpha}}$$

$$w_1 = (1-\alpha)k_1^{\alpha} l_1^{-\alpha} = (1-\alpha)k_1^{\alpha}$$

where the second equality uses the market clearing condition for labor. Hence, $R(w_1)$ as function of k_1 is given by

$$\alpha \frac{1}{k_1^{1-\alpha}}$$

Therefore, k_1 solves

$$k_1 = \frac{n_0}{1 - \lambda \alpha \frac{1}{k_1^{1-\alpha}}}$$

The LHS is increasing in k_1, continuous, and goes from zero to infinity. For the RHS we need to use the condition that $\lambda R(w_1) < 1$, which implies that the RHS is well defined only for $k_1 > (\lambda \alpha)^{\frac{1}{1-\alpha}}$. Under this condition, the RHS is decreasing in k_1, continuous, and goes from infinity to n_0. Hence, the LHS and RHS intersect and that intersection is unique. Since the RHS is increasing in n_0, it is immediate to see that k_1 is increasing in n_0.

f) *Assume that the entrepreneurs are constrained. Suppose the government taxes a lump-sum τ to consumers in period 0, and transfers the receipts from the tax* *to the entrepreneurs. Derive an expression for the utility of consumers and entrepreneurs as a function of τ.*

The utility of consumers is

$$U_c = e_0 - b_1 - \tau + e_1 + w_1 + b_1 = e_0 + e_1 + w_1 - \tau$$

where e_0 and e_1 are the consumers' endowments.

The utility of the entrepreneurs is

$$U_e = (1-\lambda)R(w_1)k_1 = (1-\lambda)\alpha k_1^{\alpha}$$

where

$$k_1 = \frac{n_0 + \tau}{1 - \lambda\alpha\frac{1}{k_1^{1-\alpha}}} \tag{5}$$

and k_1 is increasing in τ.

g) *Show that if n_0 is sufficiently small, a small positive tax increases the utility of both consumers and entrepreneurs.*

Since k_1 is increasing in τ, and U_e is increasing in τ, that the utility of entrepreneurs is increasing in τ is immediate. On the other hand, for consumers an increase in τ has a negative direct wealth effect and a positive indirect effect through wages. If $\frac{\partial w_1}{\partial \tau} - 1 > 0$, the tax increases the utility of the consumers. But τ affects w_1 only through k_1. Thus

$$\frac{\partial w_1}{\partial \tau} = \frac{\partial w_1}{\partial k_1}\frac{\partial k_1}{\partial \tau}$$

We have

$$\frac{\partial w_1}{\partial k_1} = \alpha(1-\alpha)k_1^{\alpha-1}$$

Note that if n_0 is very small, $k_1 \simeq (\lambda\alpha)^{\frac{1}{1-\alpha}}$. Hence, for small k_1

$$\frac{\partial w_1}{\partial k_1} \simeq \frac{1-\alpha}{\lambda}$$

Moreover. we can use the implicit function theorem to obtain $\frac{\partial k_1}{\partial \tau}$. Rewrite (5) as

$$k_1 - \lambda\alpha k_1^{\alpha} = n_0 + \tau$$

Differentiating with respect to τ

$$\frac{\partial k_1}{\partial \tau} - \lambda\alpha^2 k_1^{\alpha-1}\frac{\partial k_1}{\partial \tau} = 1$$

Evaluating at $k_1 = (\lambda\alpha)^{\frac{1}{1-\alpha}}$

$$\frac{\partial k_1}{\partial \tau} = \frac{1}{1-\alpha}$$

Putting everything together

$$\frac{\partial w_1}{\partial \tau} = \frac{1-\alpha}{\lambda}\frac{1}{1-\alpha} = \frac{1}{\lambda} > 1$$

So a small tax when n_0 is small increases the utility of both consumers and entrepreneurs.

Question 6 (*20 points*)

a) *Suppose agents have access to a market for an asset, denoted by a_{t+1}, which pays a rate of return $1+r_t$. Assume that agents face no borrowing constraint except for the natural borrowing limit. Define a competitive equilibrium for this economy.*

A competitive equilibrium is an allocation $\{c_t^o, c_t^e, a_{t+1}^o, a_{t+1}^e\}_{t=0}^{\infty}$ and prices $\{r_t\}_{t=0}^{\infty}$ such that

1. Given prices $\{r_t\}_{t=0}^{\infty}$ and initial asset holdings a_0^i, $\{c_t^i, a_{t+1}^i\}_{t=0}^{\infty}$ solves agents type $i \in \{odd, even\}$ problem

$$\max_{\{c_t \geq 0, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

   subject to

$$c_t + a_{t+1} \leq y_t^i + (1+r_{t-1})a_t$$
$$a_0^i \text{ given}$$

2. Markets clear

$$a_t^o + a_t^e = 0, \quad c_t^o + c_t^e = 1 \quad \forall t \geq 0$$

b) *Assume that initial asset holdings are zero for all agents. Compute all the equilibria of the economy. That is, find all the prices and quantities (including asset holdings) consistent with your definition of equilibrium. Are the equilibria Pareto optimal? Conclude that Pareto optimality requires that each agent's consumption is constant over time.* ***Hint:*** *It might be useful to compute the agents' intertemporal budget constraint.*

The FOCs of an agent's problem are

$$(c_t): \quad \beta^t \frac{1}{c_t} = \lambda_t$$
$$(a_{t+1}): \quad \lambda_t = (1+r_t)\lambda_{t+1}$$

Combining the two equations, we get

$$c_{t+1} = \beta(1+r_t)c_t$$

Since this equation holds for both types of agents at all times, we can combine them and impose market clearing

$$\underbrace{c^o_{t+1} + c^e_{t+1}}_{=1} = \beta(1 + r_t)\underbrace{(c^o_t + c^e_t)}_{=1}$$

which implies that in any equilibrium,

$$1 + r_t = \frac{1}{\beta} \quad \forall t$$

Therefore, from the Euler equation we get

$$c^i_{t+1} = c^i_t = c^i \quad \forall t$$

Plugging this in the intertemporal budget constraint we get

$$\sum_{t=0}^{\infty} \beta^t c^i = \sum_{t=0}^{\infty} \beta^t y^i_t \equiv Y^i$$

hence

$$c^o = (1-\beta)\frac{\beta}{1-\beta^2} = \frac{\beta}{1+\beta}$$
$$c^e = (1-\beta)\frac{1}{1-\beta^2} = \frac{1}{1+\beta}$$

and

$$a^o_t = \begin{cases} -\frac{\beta}{1+\beta} & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases}$$
$$a^e_t = \begin{cases} \frac{\beta}{1+\beta} & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases}$$

By now it should be clear that Pareto optimality requires equalization of the marginal rate of substitution across agents, that is

$$\beta\frac{c^o_{t+1}}{c^o_t} = \beta\frac{c^e_{t+1}}{c^e_t}$$

which is satisfied in equilibrium. Moreover, Pareto optimality would require both agents to have identical growth paths of consumption (either increasing, decreasing or constant). But only constant paths satisfy the resource constraint. Hence, Pareto optimal allocations have constant consumption for each type of agent.

c) *Define and compute a steady state with a positive and constant price for the useless asset. Show that this equilibrium is not Pareto optimal.* ***Hint:*** *Guess and verify that agents' consumption takes only two values,* $\overline{c}$ *and* $\underline{c}$, *with* $\overline{c} \geq \underline{c}$.

An equilibrium with a positive and constant price for the useless asset is a constant allocation $\{c^o_t, c^e_t, m^o_t, m^e_t\}_{t=0}^{\infty}$ and a price p such that

1. Given the price p and initial asset holdings m^i_{-1}, $\{c^i_t, m^i_t\}_{t=0}^{\infty}$ solves agents type $i \in \{odd, even\}$ problem

$$\max_{\{c_t, m_t \geq 0\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to

$$c_t + pm_t \leq y^i_t + pm_{t-1}$$
$$m^i_{-1} \text{ given}$$

2. Markets clear

$$m^o_t + m^e_t = M, \quad c^o_t + c^e_t = 1 \quad \forall t \geq 0$$

The FOCs of the agents' problem are

$$(c_t): \quad \beta^t \frac{1}{c_t} = \lambda_t$$
$$(m_t): \quad -p\lambda_t + p\lambda_{t+1} \leq 0$$

Combining the two conditions, we get

$$c_{t+1} \geq \beta c_t$$

with equality if $m_t > 0$. Note that this condition can be satisfied with inequality only when the agent decides to hold zero units of the useless asset. Hence, we guess and verify an equilibrium of the form

$$\underline{c} = pM \text{ when } y^i_t = 0$$
$$\overline{c} = 1 - pM \text{ when } y^i_t = 1$$

with $\overline{c} \geq \underline{c}$. Plugging in the Euler equation, we get

$$\underline{c} = \beta \overline{c} \tag{6}$$
$$\overline{c} \geq \beta \underline{c} \tag{7}$$

Using (6)

$$pM = \beta(1 - pM)$$

or

$$p = \frac{\beta}{1+\beta} \frac{1}{M}$$

Plugging in (7)

$$1 - pM = \frac{1}{1+\beta} > \frac{\beta}{1+\beta} = pM$$

which is satisfied. Note that in this equilibrium $\underline{c} < \overline{c}$ so, by the argument in b), the equilibrium is not Pareto optimal.

d) *Assume that the government hands out the new currency (or withdraws old currency if $\mu < 1$) proportionally to initial holdings of money. That is, if an agent holds m_{t-1} units of currency from $t-1$ to t, the government augments her holdings by $(\mu - 1)m_{t-1}$ for a total of μm_{t-1} units at the beginning of period t. Compute a stationary equilibrium with a positive price for money that grows (or decreases) at the constant rate γ. Is there any μ and γ such that the equilibrium is Pareto optimal? Explain.*

The problem of an agent of type i is

$$\max_{\{c_t, m_t \geq 0\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to

$$c_t + p_t m_t \leq y_t^i + p_t \mu m_{t-1}$$

The FOCs are

$$(c_t): \quad \beta^t \frac{1}{c_t} = \lambda_t$$

$$(m_t): \quad -p_t\lambda_t + p_{t+1}\mu\lambda_{t+1} \leq 0$$

Let's guess and verify an equilibrium of the form

$$\underline{c}_t = p_t M_t \text{ when } y_t^i = 0$$

$$\overline{c}_t = 1 - p_t \mu M_{t-1} \text{ when } y_t^i = 1$$

Plugging in and combing the FOCs

$$p_{t+1}\mu\beta \frac{1}{p_{t+1}\mu M_t} = p_t \frac{1}{1 - p_t M_t}$$

or

$$\beta(1 - p_t M_t) = p_t M_t$$

Hence

$$p_t = \frac{\beta}{1+\beta} \frac{1}{M_t}$$

This implies that

$$\underline{c}_t = \underline{c} = \frac{\beta}{1+\beta}$$

$$\overline{c}_t = \overline{c} = \frac{1}{1+\beta}$$

so the allocation cannot be Pareto optimal.

e) *Assume that the transfers are lump sum. That is, each agent, regardless of type, receives $(\mu - 1)M_{t-1}/2$ units of money from the government at the beginning of t. Compute a stationary equilibrium with a positive price for money that grows at the constant rate γ. Is there any μ and γ such that the equilibrium is Pareto optimal? Explain.*

The problem of an agent of type i is

$$\max_{\{c_t, m_t \geq 0\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to

$$c_t + p_t m_t \leq y_t^i + p_t m_{t-1} + p_t(\mu - 1)\frac{M_{t-1}}{2}$$

The FOCs are

$$(c_t): \quad \beta^t \frac{1}{c_t} = \lambda_t$$
$$(m_t): \quad -p_t\lambda_t + p_{t+1}\lambda_{t+1} \leq 0$$

Let's guess and verify an equilibrium of the form

$$\underline{c}_t = p_t M_{t-1} + p_t(\mu - 1)\frac{M_{t-1}}{2} \text{ when } y_t^i = 0$$
$$\bar{c}_t = 1 - p_t M_t + p_t(\mu - 1)\frac{M_{t-1}}{2} \text{ when } y_t^i = 1$$

Plugging in and combing the FOCs

$$p_{t+1}\beta \frac{1}{p_{t+1}M_t + p_{t+1}(\mu - 1)\frac{M_t}{2}} = p_t \frac{1}{1 - p_t M_t + p_t \frac{\mu-1}{2\mu} M_t}$$

After some algebra, we get

$$p_t M_t = pM = \frac{\beta}{1 + \frac{\mu-1}{2} + \beta\left(1 - \frac{\mu-1}{2\mu}\right)}$$

This implies that

$$\underline{c}_t = \underline{c} = \frac{1}{\mu}pM + \frac{\mu - 1}{2\mu}pM$$
$$\bar{c}_t = \bar{c} = 1 - pM + \frac{\mu - 1}{2\mu}pM$$

Pareto optimality requires $\bar{c} = \underline{c}$, which is achieved if

$$\frac{1}{\mu}pM + \frac{\mu - 1}{2\mu}pM = 1 - 1 - pM + \frac{\mu - 1}{2\mu}pM$$

or

$$\frac{\mu}{1 + \mu} = pM = \frac{\beta}{1 + \frac{\mu-1}{2} + \beta\left(1 - \frac{\mu-1}{2\mu}\right)}$$

which is satisfied for $\mu = \beta < 1$. In that case, $1 + \gamma = \frac{1}{\beta} > 1$.