University of California, Davis
Department of Economics
**Microeconomic Theory**

Date: June 22, 2015
Time: 5 hours
Reading Time: 20 minutes

**PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE**
**Please answer four questions (out of five)**

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**Question 1.**

We postulate the utility function

$$\tilde{u} : \Re^L \to \Re: \ \tilde{u}(x) = a \bullet x \ - [1/2]\, x \bullet B\, x \,, \qquad (1.1)$$

where $a = (a_1, ..., a_L) \in \Re^L_{++}$, and $B$ is an $L \times L$ symmetric, positive definite matrix.

**1.1.** Compute the gradient vector $\nabla \tilde{u}(x)$.

**1.2.** Compute the Hessian matrix $D^2 \tilde{u}(x)$.

**1.3**. Is the preference relation represented by $\tilde{u}$ continuous? Homothetic? Quasilinear? No need to justify your answers.

**1.4.** Is the preference relation represented by $\tilde{u}$ convex? Strictly convex? Locally nonsatiated? Explain your answers.

**1.5.** In order to develop your intuition, graphically represent the indifference map for $L = 2$, for the special case where $a_1 = a_2 \equiv a$ and $B = \begin{bmatrix} b & c \\ c & b \end{bmatrix}$. Separately consider the cases $c = 0$ and $c > 0$. Be as precise as possible.

**1.6.** You can assume that, for $(p, w) >> 0$, a solution to the UMAX[$p$, $w$] problem exists and is unique. Without attempting at this point to explicitly solve the UMAX[$p$, $w$], show that for $w$ above a certain level, that you should made explicit, demand is insensitive to increases in wealth, whereas for values of $w$ below this level Walrasian demand is affine in wealth. (Hint: For the second part, use the Implicit Function Theorem.)

**1.7.** We now consider a society with $I$ consumers, each endowed with the preference relation represented by $\tilde{u}$ in (1.1). We denote by $P := \Re_{++}^{L+I}$ the domain of prices and individual wealth vectors $(p; w^1, ..., w^I)$. Is there a subset of $P$ for which a positive representative consumer exists? Explain.

**1.8.** We return to the one-consumer case. Solve the UMAX[$p$, $w$] problem for good 1 in the case where $L = 2$, $a_1 = a_2 = a$ and $B = \begin{bmatrix} b & c \\ c & b \end{bmatrix}$, for $c > 0$. Can one of the goods be inferior at a point of the domain of the Walrasian demand function? (Hint. Consider the wealth expansion paths in ($x_1$, $x_2$) space.)

**1.9.** We now consider a consumer who faces uncertainty. Her *ex ante* preference relation satisfies the expected utility hypothesis with von Neumann-Morgenstern-Bernoulli utility function $u : [0, \bar{a} / \bar{b}] \to \Re$, where $\bar{a}$ and $\bar{b}$ are positive parameters, and with coefficient of absolute risk aversion equal to $\dfrac{\bar{b}}{\bar{a} - \bar{b}x}$. She is facing the contingent-consumption optimization problem of maximizing her expected utility subject to a budget constraint. Let there be two states of the world, $s_1$ and $s_2$, with probabilities $\pi$ and $1 - \pi$, respectively

**1.9.1.** Comment on her *ex ante* preferences.

**1.9.2.** To what extent is her problem formally a special case of the UMAX problem of 1.6 above? Explain.

**1.9.3.** Draw a plausible wealth expansion path in contingent commodity space. As her wealth increases, does the consumer bear absolutely more or less risk? Relatively more or less risk?

**Question 2.**

Let there be two goods: good 1 is an input, and good 2 is an output. We set at $p_1 = 1$ the price of the input and denote by $p \equiv p_2 / p_1$ the relative price of the output. Throughout this question we postulate an industry comprised of two firms, named Firm 1 and Firm 2. We denote by $z_1 \geq 0$ (resp. $q_1 \geq 0$) the amount of input used by (resp. output produced by) Firm 1, and similarly we denote by $z_2 \geq 0$ (resp. $q_2 \geq 0$) the amount of input used by (resp. output produced by) Firm 2. We assume that both firms are price takers in the input market.

Buyers of output are assumed to be price-taking consumers. The aggregate demand-for-output function in this market is given by $y = 2 - p$, where $y$ is the aggregate quantity of output demanded by consumers, assumed to be independent from their wealth levels.

We will be concerned about productive efficiency. For the purposes of this question, we say that the vector $(z_1, q_1; z_2, q_2) \in \Re^4_+$ is *productively efficient* if with the total amount of input $z_1 + z_2$ it is not possible to produce an aggregate amount of output higher than $q_1 + q_2$.

We consider three alternative models, named Market *A*, Market *B* and Market *C*. The information provided above applies to all three markets.

**2A. MARKET *A*: PRICE TAKING, NO EXTERNALITIES**

Both firms are price takers in the output market. Firm 1's direct production function is

$$f_1 : \Re_+ \to \Re_+ : f_1(z_1) = z_1 \quad . \qquad (2.1)$$

**2A.1.** Obtain Firm 1's cost function, and graph its average cost and its marginal cost. Obtain Firm 1's supply-of-output correspondence (or function).

**2A.2.** Firm 2's direct production function is

$$f_2 : \Re_+ \to \Re_+ : f_2(z_2) = \sqrt{z_2} \quad . \qquad (2.2)$$

Obtain Firm 2's cost function, and graph its average cost and its marginal cost. Obtain Firm 2's supply-of-output correspondence (or function).

**2A.3.** Find the vector $(\tilde{z}_1, \tilde{q}_1; \tilde{z}_2, \tilde{q}_2)$ of input and output quantities for the two firms at the equilibrium of Market *A*.

**2A.4.** Is $(\tilde{z}_1, \tilde{q}_1; \tilde{z}_2, \tilde{q}_2)$ productively efficient? Argue your answer.

**2B. MARKET *B*: DUOPOLY, NO EXTERNALITIES**

We now consider Market *B*, with two firms, again named Firm 1 and Firm 2 with the same technologies as above, i. e., with the technology given by (2.1) for Firm 1, and that of (2.2) for Firm 2. The difference is that now the two firms behave as a Cournot duopoly.

**2B.1.** Find the vector $(\overline{z}_1, \overline{q}_1; \overline{z}_2, \overline{q}_2)$ of input and output quantities for the two firms at the Cournot Equilibrium of Market *B*.

**2B.2.** Is $(\overline{z}_1, \overline{q}_1; \overline{z}_2, \overline{q}_2)$ productively efficient? Argue your answer.

**2C. MARKET *C*: PRICE TAKING, EXTERNALITIES**

We now consider Market *C*, with two firms, again named Firm 1 and Firm 2. The technology of Firm 1 is as given by (2.1) above, but that of Firm 2 involves a positive production externality generated by Firm 1. More precisely, the output of Firm 2 depends on the level of output of Firm 1, in addition to depending on Firm 2's use of the input, $z_2$, according to the expression

$$q_2 = \sqrt{q_1}\sqrt{z_2} \quad . \tag{2.3}$$

(For instance, increasing the output of Firm 1 generates technical knowledge that spills over to Firm 2.)

As in Market *A*, both firms operate independently as price takers.

**2C.1.** Find the vector $(\hat{z}_1, \hat{q}_1; \hat{z}_2, \hat{q}_2)$ of input and output quantities for the two firms at the equilibrium of Market *C*.

**2C.2.** Is $(\hat{z}_1, \hat{q}_1; \hat{z}_2, \hat{q}_2)$ productively efficient? Argue your answer.

**2D.** Briefly comment on the productive efficiency in the three markets studied.

## Question 3

Consider an economy with 2 factors of production, labor and capital, and 2 produced consumption goods, food (good1) and electricity (good 2). The production functions for goods 1 and 2 are

$$y_1 = \sqrt{L_1 K_1}, \qquad y_2 = (\min\{L_2, K_2\})^2$$

that is, the production of good 2 exhibits increasing returns to scale. There is a representative agent for the economy who consumes the produced goods, has no disutility for labor, has preferences represented by the utility function $u(x_1, x_2) = \sqrt{x_1 x_2}$, and is endowed with one unit of labor and one unit of capital.

(a) Find the Pareto optimal allocation of this economy.

(b) The second Theorem of Welfare Economics does not apply to this economy since the production set of firm 2 exhibits increasing returns to scale. An alternative to a Competitive Equilibrium for an economy in which some firms have increasing returns to scale is a Marginal Cost Pricing equilibrium. All goods have market prices that consumers and firms take as given. The behavior of consumers and firms with convex production sets are the same as in a competitive equilibrium. The only difference is that the firms with increasing returns to scale are regulated: they are required to minimize their production costs given the prices of the factor of production, and to produce a quantity of output such that the marginal cost of the last unit produced is equal to the price of this output on the market. The regulator uses taxes to finance the deficit of the regulated firms. In equilibrium the prices of the goods must be such that all markets clear. Show that in a Robinson Crusoe economy (one agent, two goods, one technology) in which the production set exhibits increasing returns to scale but the preferences are convex, every Pareto optimal allocation can be obtained as a Marginal Cost Pricing equilibrium (draw an example with a unique Pareto optimal allocation).

(c) Coming back to the economy studied in (a), the planner who regulates firm 2 taxes capital in order to finance the subsidy to firm 2. Write the definition of a Marginal Cost Pricing equilibrium for this economy.

(d) Show that the Pareto optimal allocation found in question (a) can be obtained as a Marginal Cost Pricing equilibrium. Indicate the prices of the good and the tax which is needed to subsidize firm 2.

## Question 4

Let $u(x, y)$ be a utility function, where $x$ is the consumption of a private good and $y$ the consumption of a public good. $u$ is continuously differentiable, increasing, strictly quasi concave and satisfies the Inada conditions ($\lim \frac{\partial u}{\partial x}(x, y) \to \infty$ if $x \to 0, \forall y > 0$, $\lim \frac{\partial u}{\partial y}(x, y) \to \infty$ if $y \to 0, \forall x > 0$).

Consider the maximum problem

$$\max\{u(x, y) \mid x + y \leq W\} \tag{1}$$

where the non negativity constraints, being unnecessary, are omitted. Let $(\widetilde{x}(W), \widetilde{y}(W))$ be the solution to (1) as a function of the income $W$. We assume that both the private and the public goods are normal goods for the preferences represented by $u$: $\widetilde{x}(W)$ and $\widetilde{y}(W)$ are increasing functions of $W$. Since $\widetilde{y}(W)$ is increasing, it is invertible and we denote by $\phi$ the inverse function (for your own sake interpret the relation $W = \phi(y)$). Note that $\phi(y) - y > 0$.

Consider now an economy with a private good and a public good, and $I$ agents with the same preferences represented by the utility function $u$ satisfying the assumptions introduced above. The production of public good exhibits constant returns, one unit of private good producing one unit of public good. Agents differ by their incomes. Let $w_i$ denote the income of agent $i$. We want to show that in a voluntary contribution equilibrium of such an economy there is a level of income $\bar{w}$ such that all agents with income lower than $\bar{w}$ do not contribute, and agents with income above $\bar{w}$ contribute $w_i - \bar{w}$.

(a) Let $\left((\bar{x}_i)_{i=1}^I, (\bar{z}_i)_{i=1}^I, \bar{y}\right)$ be a voluntary contribution equilibrium of the economy described above, where $\bar{z}_i$ denote the contribution of agent $i$ ($\bar{z}_i = \omega_i - \bar{x}_i$), and $\bar{y} = \sum_i \bar{z}_i$. Let $\bar{Z}^{-i} = \sum_{j \neq i} \bar{z}_j$ be the contribution of the agents other than agent $i$. Show that

- if $\bar{z}_i > 0$, then $\bar{y} = \widetilde{y}\left(w_i + \bar{Z}^{-i}\right)$;
- if $\bar{z}_i = 0$, then $\bar{y} \geq \widetilde{y}\left(w_i + \bar{Z}^{-i}\right)$;

[Hint: show that you can express the maximum problem of agent $i$ choosing his optimal contribution to the public good under the form of problem (1) with one additional inequality constraint on $y$.]

(b) Let $\bar{w} = \phi(\bar{y}) - \bar{y}$. Deduce from (a) that

(i) If $w_i \leq \bar{w}$, then it must be that $\bar{Z}^{-i} = \bar{y}$ and $\bar{z}_i = 0$.

(ii) If $w_i > \bar{w}$, then it must be that $\bar{Z}^{-i} < \bar{y}$ and $\bar{z}_i > 0$.

(iii) Deduce from (ii) that if $w_i > \bar{w}$, $\bar{z}_i = w_i - \bar{w}$.

(c) Let us apply the previous result to an economy with two types of agents, the rich and the poor. All agents have the same utility function

$$u(x_i, y) = \ln(x_i) + \gamma \ln(y), \qquad 0 < \gamma < 1.$$

There are $n_1$ "poor" agents with income $w_1$, and $n_2$ "rich" agents with income $w_2$, where $w_1 < \frac{n_2}{1+n_2} w_2$. Show that the voluntary contribution equilibrium cannot be such that all agents contribute (use (b)(iii) and proceed by contradiction). Calculate the equilibrium with voluntary contributions.

## QUESTION 5

Chef Yepbu is opening a new restaurant in Davis next month. He is **risk neutral**. He is planning to run an all-you-can-eat buffet. He knows that each consumer in Davis can be characterized in terms of a pair of constants $(r,d)$, where *r* is the consumer's reservation price for a buffet meal (he/she is willing to go to the buffet if and only if the price is less than or equal to *r*) and *d* is the number of dishes that he/she consumes at an all-you-can-eat buffet (note that *d* is a constant, thus independent of the price). Let $(r_i,d_i)$ be the characteristic of consumer *i* and let *W* be the set of possible characteristics (that is, $W=\{(r,d):(r,d)=(r_i,d_i) \text{ for some } i\in N\}$, where *N* is the set of potential consumers). Chef Yepbu cannot tell, when somebody shows up, how many dishes that person will consume (of course, by showing up, the consumer reveals that, for her, $r \geq p$, where *p* is the posted price of a buffet meal). The cost of producing each dish is constant and equal to $c > 0$. The restaurant has a capacity of *K*, that is, only *K* customers can be accommodated, with *K* smaller than the total number of potential consumers, that is, $K<|N|$ (where $|N|$ denotes the cardinality of the set *N*).

**(a)** Suppose first that $W=\underbrace{\{10,11\}}_{\text{reservation prices}}\times\underbrace{\{1,2\}}_{\text{\#of dishes consumed}}=\{(10,1),(10,2),(11,1),(11,2)\}$, $K=60$ and $|N|=120$.

Construct an example, i.e. find a value of *c* and a probability distribution *P* over *W*, where $P(r,d)$ is the proportion of the population of consumers that is characterized by the pair $(r,d)$, such that Chef Yepbu prefers to charge each customer \$10 despite the fact that at least 60 consumers would show up at the restaurant even if the price were \$11. Prove your claim with appropriate calculations.

**(b)** For the general case (that is, for a general *W*) what properties would the probability distribution *P* have to satisfy for the phenomenon of part (a) to arise? What is that phenomenon called?

For the remaining questions assume that $W=\underbrace{\{1,2,...,n\}}_{\text{reservation prices}}\times\underbrace{\{1,2,...,m\}}_{\text{\# of dishes consumed}}$ (with $2\leq m<n$) and $c\leq 1$. Let $F(i,j)\geq 0$ be the *number* of people characterized by $(i,j)$ (i.e., the number of people who have a reservation price equal to *i* and a consumption level equal to *j* dishes) and let *a* be a positive integer.

**(c)** For this question assume that $1\leq K\leq a\cdot m$ and $F(i,j)=a$, for every $(i,j)\in W$. Find the profit-maximizing price for the restaurant, for every *K* in the given range.

**(d)** For this question assume that $K=a\cdot m$ and $F(i,j)=\begin{cases} a\cdot m & \text{if either } (i<n \text{ and } j=1) \text{ or } (i=n \text{ and } j=m) \\ 0 & \text{otherwise} \end{cases}$.

**(d1)** Write an expression that gives the restaurant's expected total profit if it charges $p=n-3$ for a buffet meal.

**(d.2)** For the case where $m\leq 6$, prove that the restaurant would never choose to charge a price $p\leq n-2$.

**(d.3)** For the case $m\leq 6$, find necessary and sufficient conditions on the values of *m* and *c* for the restaurant to want to charge a price at which demand exceeds capacity, instead of a higher price at which demand equals capacity.