

**PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE**

**Please answer four parts (out of five)**

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**QUESTION 1. AMBIGUITY AVERSION AND INSURANCE**

**ANSWER KEY**

Assume that nature randomly chooses one of two “states of the world,” numbered 1 and 2, where state 1 is interpreted as the bad state, involving a loss. Two experts, named  $G$  and  $R$ , have estimated the probabilities of the two states. Expert  $G$  (for “Gloomy”) estimates the probability of state 1 as  $\pi^G \in (0, 1)$  (and that of state 2 as  $1 - \pi^G$ ), whereas Expert  $R$  (for “Rosy”) estimates the probability of state 1 as  $\pi^R \in (0, 1)$  (and that of state 2 as  $1 - \pi^R$ ). We postulate that

$$\pi^G > \pi^R .$$

We also postulate that Expert  $G$  is correct with probability  $\mu \in (0, 1)$ , and Expert  $R$  with probability  $1 - \mu$ .

The outcome set is  $\mathfrak{R}_+$ , i. e., there is a single good, interpreted as “consumption.” We will alternatively consider decision maker (DM) Leonard or Frank. Leonard and Frank will be endowed with the same utility function over outcomes  $u: \mathfrak{R}_+ \rightarrow \mathfrak{R}$ , concave, strictly increasing and twice differentiable. Denote by  $x_i$  the DM’s consumption contingent to state  $i$  ( $i = 1, 2$ ).

In the absence of insurance, the DM ends up with the level of consumption  $\omega_1$  in state 1 and  $\omega_2$  in state 2, where  $\omega_2 - \omega_1 > 0$  is the amount of the loss. The insurance market offers contracts with variable *coverage*  $\alpha \in [0, \omega_2 - \omega_1]$ , and linear *premium*, i. e., the premium per unit of coverage is  $q$ , which the DM takes as given while choosing  $\alpha$  (hence the total premium is  $q\alpha$ ). Leonard and Frank will face the same  $q$  and will have the same endowments.

Given  $q$ , an amount of coverage  $\alpha$  yields the contingent consumption vector  $(x_1, x_2) = (\omega_1 + \alpha(1 - q), \omega_2 - q\alpha)$ . The premium rate  $q$  and the endowments  $(\omega_1, \omega_2)$  induce a budget equality of the form  $P_1x_1 + P_2x_2 = W$ , where  $W = P_1\omega_1 + P_2\omega_2$ .

Leonard and Frank differ in the following respect. Leonard combines the probability estimates of the two experts, and his objective function is

$$U^L(x_1, x_2) \equiv \beta u(x_1) + (1 - \beta)u(x_2), \quad (1)$$

where  $\beta \equiv \mu\pi^G + (1 - \mu)\pi^R$ , (2)

and, accordingly,  $1 - \beta = \mu(1 - \pi^G) + (1 - \mu)(1 - \pi^R)$ .

Frank's objective function is, instead

$$U^F(x_1, x_2) \equiv \mu\varphi(\pi^G u(x_1) + (1 - \pi^G)u(x_2)) + (1 - \mu)\varphi(\pi^R u(x_1) + (1 - \pi^R)u(x_2)), \quad (3)$$

where  $\varphi: \mathfrak{R} \rightarrow \mathfrak{R}$  is a strictly increasing, twice differentiable, concave function. Accordingly,  $U^F$  is a concave function.

**1(a).** Consider a given vector  $(x_1, x_2) \in \mathfrak{R}_{++}^2$ . Find, by implicit differentiation, expressions for the marginal rates of substitution of Leonard and of Frank at  $(x_1, x_2)$ . Show that Leonard and Frank have the same marginal rates of substitution at any point  $(x_1, x_2)$  satisfying  $x_1 = x_2$  (i. e., any point on the *certainty line*.)

ANSWER.

Leonard. From (1),  $\frac{dx_2}{dx_1} \Big|_{U^L \text{ cons}} = -\frac{\frac{\partial U^L}{\partial x_1}}{\frac{\partial U^L}{\partial x_2}} = -\frac{\beta u'(x_1)}{(1 - \beta)u'(x_2)}$  (A.1)

Frank. From (3),

$$\begin{aligned} \frac{dx_2}{dx_1} \Big|_{U^F \text{ cons}} &= -\frac{\frac{\partial U^F}{\partial x_1}}{\frac{\partial U^F}{\partial x_2}} \\ &= -\frac{\mu\varphi'(\pi^G u(x_1) + (1 - \pi^G)u(x_2))\pi^G u'(x_1) + (1 - \mu)\varphi'(\pi^R u(x_1) + (1 - \pi^R)u(x_2))\pi^R u'(x_1)}{\mu\varphi'(\pi^G u(x_1) + (1 - \pi^G)u(x_2))(1 - \pi^G)u'(x_2) + (1 - \mu)\varphi'(\pi^R u(x_1) + (1 - \pi^R)u(x_2))(1 - \pi^R)u'(x_2)}. \end{aligned} \quad (A.2)$$

Now let  $x_1 = x_2$ . From (A.1),  $\frac{dx_2}{dx_1} \Big|_{U^L \text{ cons}} = -\frac{\beta}{(1 - \beta)}$

From (A.2), for  $x_1 = x_2 \equiv x$ ,

$$\begin{aligned}
\left. \frac{dx_2}{dx_1} \right|_{U^f \text{ cons}} &= - \frac{\mu \phi'(u(x)) \pi^G u'(x) + (1-\mu) \phi'(u(x)) \pi^R u'(x)}{\mu \phi'(u(x)) (1-\pi^G) u'(x) + (1-\mu) \phi'(u(x)) (1-\pi^R) u'(x)} \\
&= - \frac{\mu \pi^G + (1-\mu) \pi^R}{\mu(1-\pi^G) + (1-\mu)(1-\pi^R)} \\
&= - \frac{\beta}{1-\beta},
\end{aligned}$$

by the definition (2) of  $\beta$ .

**1(b).** Assume that  $\phi$  is affine, but that  $u$  is strictly concave. Do Leonard and Frank always make the same choice? If NO, who buys more insurance when they make a different choice? Prove your answer.

**ANSWER.** YES, Leonard and Frank always make the same choice when  $\phi$  is affine. If  $\phi(y) = a + by$ ,  $b > 0$ , then (3) becomes

$$\begin{aligned}
U^f(x_1, x_2) &= \mu[a + b[\pi^G u(x_1) + (1-\pi^G)u(x_2)]] + (1-\mu)[a + b[\pi^R u(x_1) + (1-\pi^R)u(x_2)]] \\
&= a + b[[\mu\pi^G + (1-\mu)\pi^R]u(x_1) + [\mu(1-\pi^G) + (1-\mu)(1-\pi^R)]u(x_2)] \\
&= a + b[\beta u(x_1) + (1-\beta)u(x_2)],
\end{aligned}$$

i. e.,  $U^f$  is an increasing transformation of  $U^L$ . Hence, the solutions to the maximization problems of Leonard and Frank coincide.

**1(c).** Assume now that  $\phi$  is strictly concave. Do Leonard and Frank always make the same choice? If NO, who buys more insurance when they make different choices? Does the answer depend on whether  $u$  is affine or strictly concave? Prove your answer.

**ANSWER.** Leonard and Frank do NOT always make the same choice when  $\phi$  is strictly concave, irrespective of whether  $u$  is affine or strictly concave. Consider the following “interior” cases, where Leonard satisfies his tangency condition

$$\frac{\beta u'(x_1)}{(1-\beta)u'(x_2)} = \frac{P_1}{P_2}. \quad (\text{A.3})$$

Case A. Suppose that Leonard chooses a point where  $x_1 = x_2$ , i. e., a point on the certainty line. Then Frank will indeed choose the same point, because, as seen in 1(b) above, Frank satisfies his tangency condition there, which condition is sufficient for Frank’s constrained maximization due to the concavity of  $U^f$  proved in 1(a) above.

**Case B.** Suppose now that Leonard chooses a point where  $x_1 < x_2$  (and satisfies (A.3)). Then Frank will not choose the same point. To prove this, we show that Frank's indifference curve at that point is steeper than Leonard's.

Because  $\pi^G > \pi^R$  and  $x_1 < x_2$ ,  $\pi^G u(x_1) + (1 - \pi^G)u(x_2) < \pi^R u(x_1) + (1 - \pi^R)u(x_2)$ . By the strict concavity of  $\varphi$ ,  $\varphi^*$  is decreasing and hence

$$\lambda^G \equiv \varphi'(\pi^G u(x_1) + (1 - \pi^G)u(x_2)) > \varphi'(\pi^R u(x_1) + (1 - \pi^R)u(x_2)) \equiv \lambda^R, \\ \text{i. e., } \lambda \equiv \frac{\lambda^R}{\lambda^G} < 1. \quad (\text{A.4})$$

The absolute value of Frank's marginal rate of substitution (MRS) can then be written, from (A.2),

$$\frac{\mu \lambda^G \pi^G u'(x_1) + (1 - \mu) \lambda^R \pi^R u'(x_1)}{\mu \lambda^G (1 - \pi^G) u'(x_2) + (1 - \mu) \lambda^R (1 - \pi^R) u'(x_2)} = \frac{\mu \pi^G u'(x_1) + (1 - \mu) \lambda \pi^R u'(x_1)}{\mu (1 - \pi^G) u'(x_2) + (1 - \mu) \lambda (1 - \pi^R) u'(x_2)}.$$

Subtracting from it the absolute value of Leonard's MRS leads to

$$\frac{\mu \pi^G u'(x_1) + (1 - \mu) \lambda \pi^R u'(x_1)}{\mu (1 - \pi^G) u'(x_2) + (1 - \mu) \lambda (1 - \pi^R) u'(x_2)} - \frac{[\mu \pi^G + (1 - \mu) \pi^R] u'(x_1)}{[\mu (1 - \pi^G) + (1 - \mu) (1 - \pi^R)] u'(x_2)} \\ = \frac{u'(x_1) [\mu \pi^G + (1 - \mu) \lambda \pi^R] [\mu (1 - \pi^G) + (1 - \mu) (1 - \pi^R)] - [\mu \pi^G + (1 - \mu) \pi^R] [\mu (1 - \pi^G) + (1 - \mu) \lambda (1 - \pi^R)]}{u'(x_2) [\mu (1 - \pi^G) + (1 - \mu) \lambda (1 - \pi^R)] [\mu (1 - \pi^G) + (1 - \mu) (1 - \pi^R)]}.$$

Dividing through by the first fraction and the denominator of the second one (both positive), the sign of this expression is that of the numerator of the second fraction, namely:

$$\begin{aligned} & [\mu \pi^G + (1 - \mu) \lambda \pi^R] [\mu (1 - \pi^G) + (1 - \mu) (1 - \pi^R)] - [\mu \pi^G + (1 - \mu) \pi^R] [\mu (1 - \pi^G) + (1 - \mu) \lambda (1 - \pi^R)] \\ & \equiv \mu \pi^G \mu (1 - \pi^G) + \mu \pi^G (1 - \mu) (1 - \pi^R) + (1 - \mu) \lambda \pi^R \mu (1 - \pi^G) + (1 - \mu) \lambda \pi^R (1 - \mu) (1 - \pi^R) \\ & \quad - \mu \pi^G \mu (1 - \pi^G) - \mu \pi^G (1 - \mu) \lambda (1 - \pi^R) - (1 - \mu) \pi^R \mu (1 - \pi^G) - (1 - \mu) \pi^R (1 - \mu) \lambda (1 - \pi^R) \\ & = \mu \pi^G (1 - \mu) (1 - \pi^R) [1 - \lambda] + (1 - \mu) \lambda \pi^R \mu (1 - \pi^G) [\lambda - 1] \\ & = [1 - \lambda] \mu (1 - \mu) [\pi^G (1 - \pi^R) - \pi^R (1 - \pi^G)] \\ & > 0, \end{aligned}$$

because  $\pi^G > \pi^R$  and, by (A.4),  $\lambda < 1$ .

Hence, as long as  $\varphi$  is strictly concave, at a point where  $x_1 < x_2$  and Leonard's indifference curve is tangent to the budget line, Frank's indifference curve is steeper than the budget line, and therefore Frank will choose a point closer to the certainty line, buying more insurance.

We assume, for the remainder of this question, that both  $\varphi$  and  $u$  are strictly concave. The strict concavity of  $\varphi$  has been interpreted in the literature as (strict) *ambiguity aversion*, in a

manner parallel to the interpretation of the strict concavity of  $u$  as (strict) *risk aversion*. Similarly, DM is *ambiguity neutral* when  $\varphi$  is affine.

**1(d).** Let the insurance premium rate be  $q = \beta$  ( $\beta$  as defined by (2)). Do Leonard and Frank always make the same choice? If NO, who buys more insurance when they make different choices? Prove and graphically illustrate your answer.

ANSWER. YES, they make the same choice. The prices that appear in the budget equality can be written as  $(P_1, P_2) = \left( \frac{1}{1-q}, \frac{1}{q} \right)$ . If  $q = \beta$ , then the premium is actuarially fair, and the slope

of the budget line is  $\frac{P_1}{P_2} = \frac{\frac{1}{1-q}}{\frac{1}{q}} = \frac{q}{1-q} = \frac{\beta}{1-\beta}$ , which is Leonard's MRS at any point of the

certainty line. Now Leonard's only solution is to fully insure, and hence, as in Case A of the answer to 1(c) above, this is also Frank's choice. See Figure 1.

**1(e).** Assume now that  $q > \beta$ . Do Leonard and Frank always make the same choice? If NO, who buys more insurance when they make different choices? Prove and graphically illustrate your answer.

ANSWER. NO. Now  $\frac{P_1}{P_2} > \frac{\beta}{1-\beta}$ . Because  $-\frac{\beta}{1-\beta}$  is Leonard's MRS at the certainty line,

Leonard's only solution is not to fully insure: he either partially insures or does not insure at all. If Leonard partially insures, then we are in Case B of the answer to 1(c) above, and hence Frank buys more insurance than Leonard, see Figure 2. But the premium could be actuarially so unfavorable that neither Leonard nor Frank insure, see Figure 3.

**1(f).** Comment on your results.

ANSWER. In our simple model, strict ambiguity aversion reinforces risk aversion, tending to increase the demand for insurance when the premium is actuarially unfavorable. When the premium is actuarially fair, then all strictly risk-averse or strictly ambiguity-averse DMs choose to fully insure. Of course, DMs who are both risk neutral and ambiguity neutral just maximize expected consumption, being indifferent among all points on the budget line when the premium is fair.

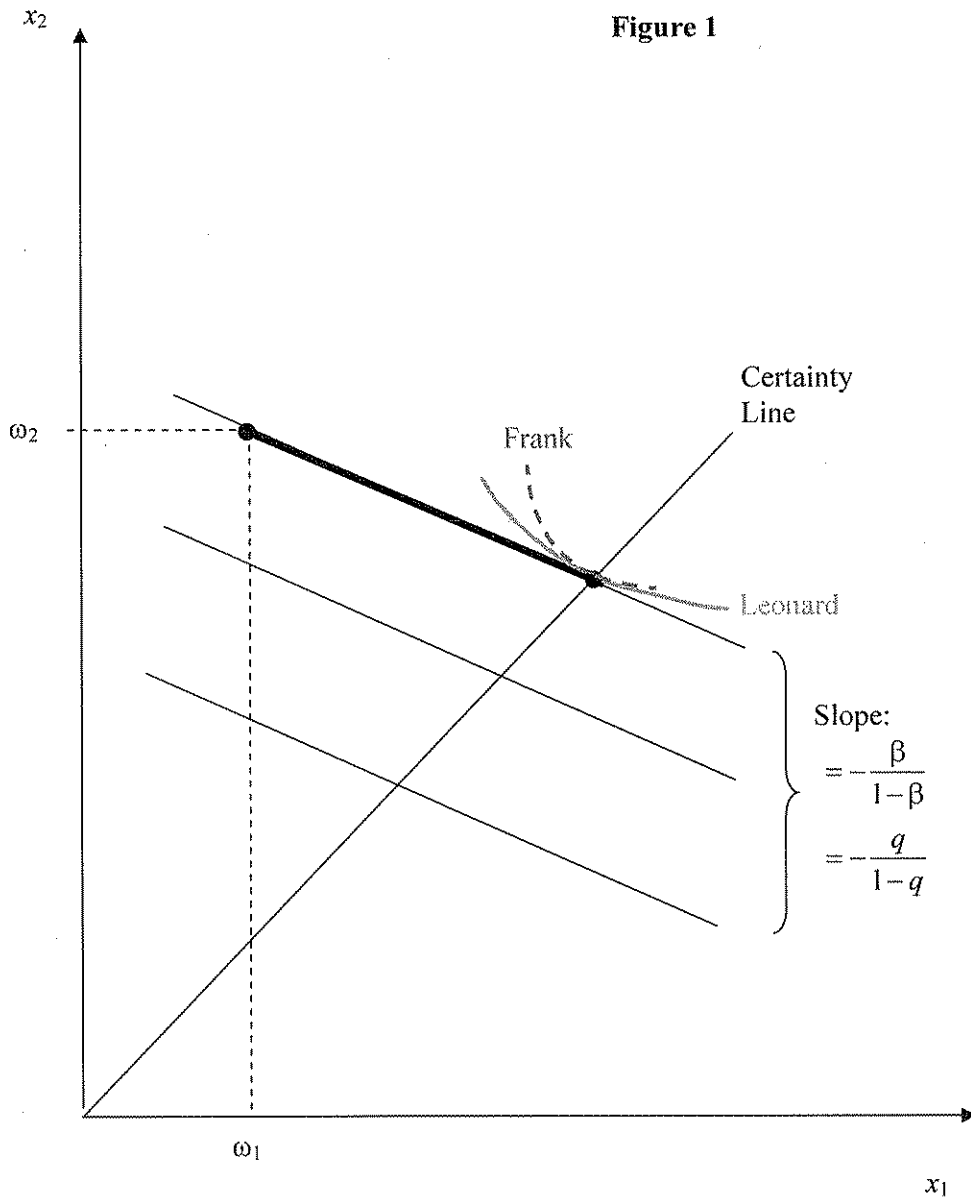
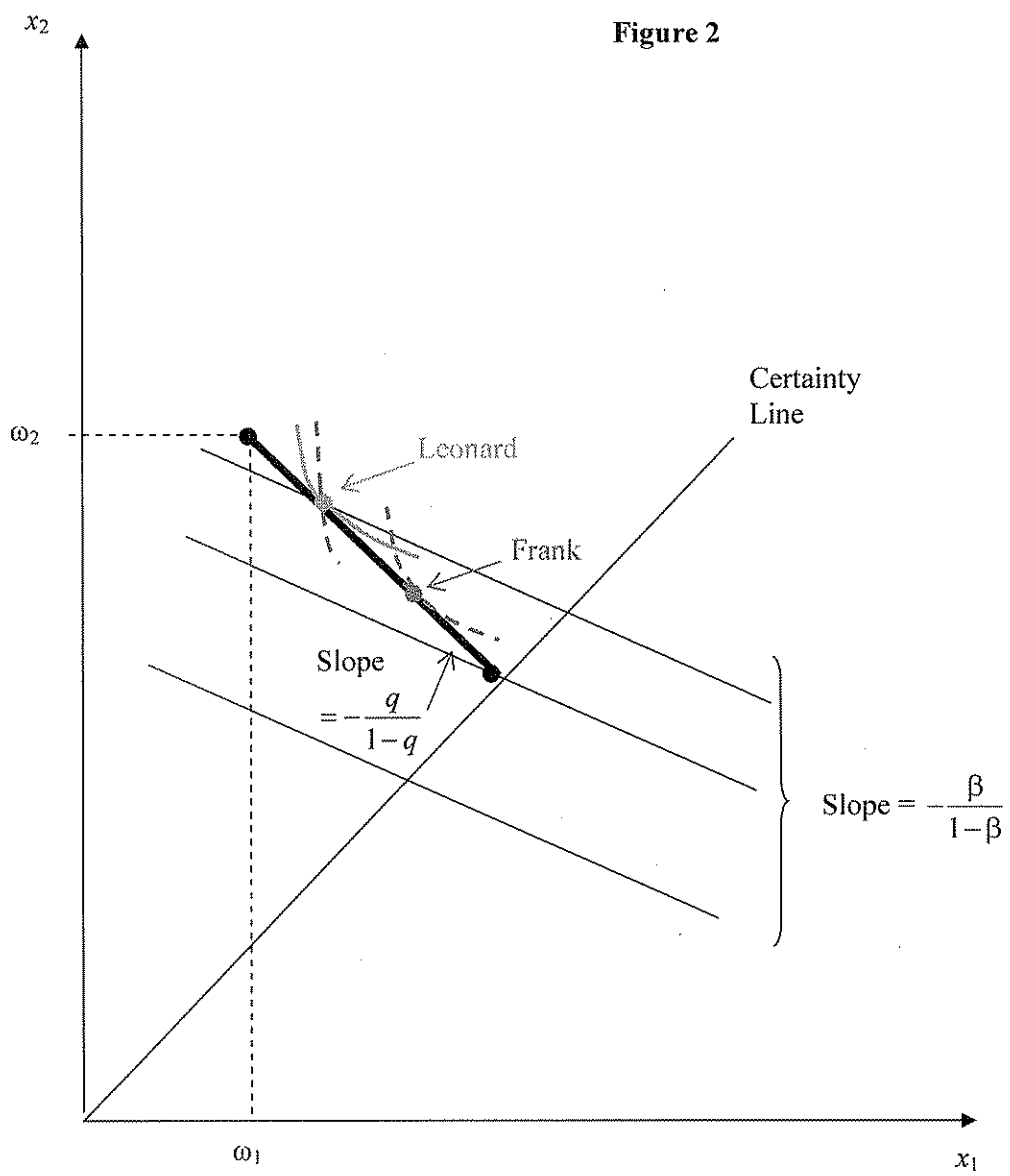
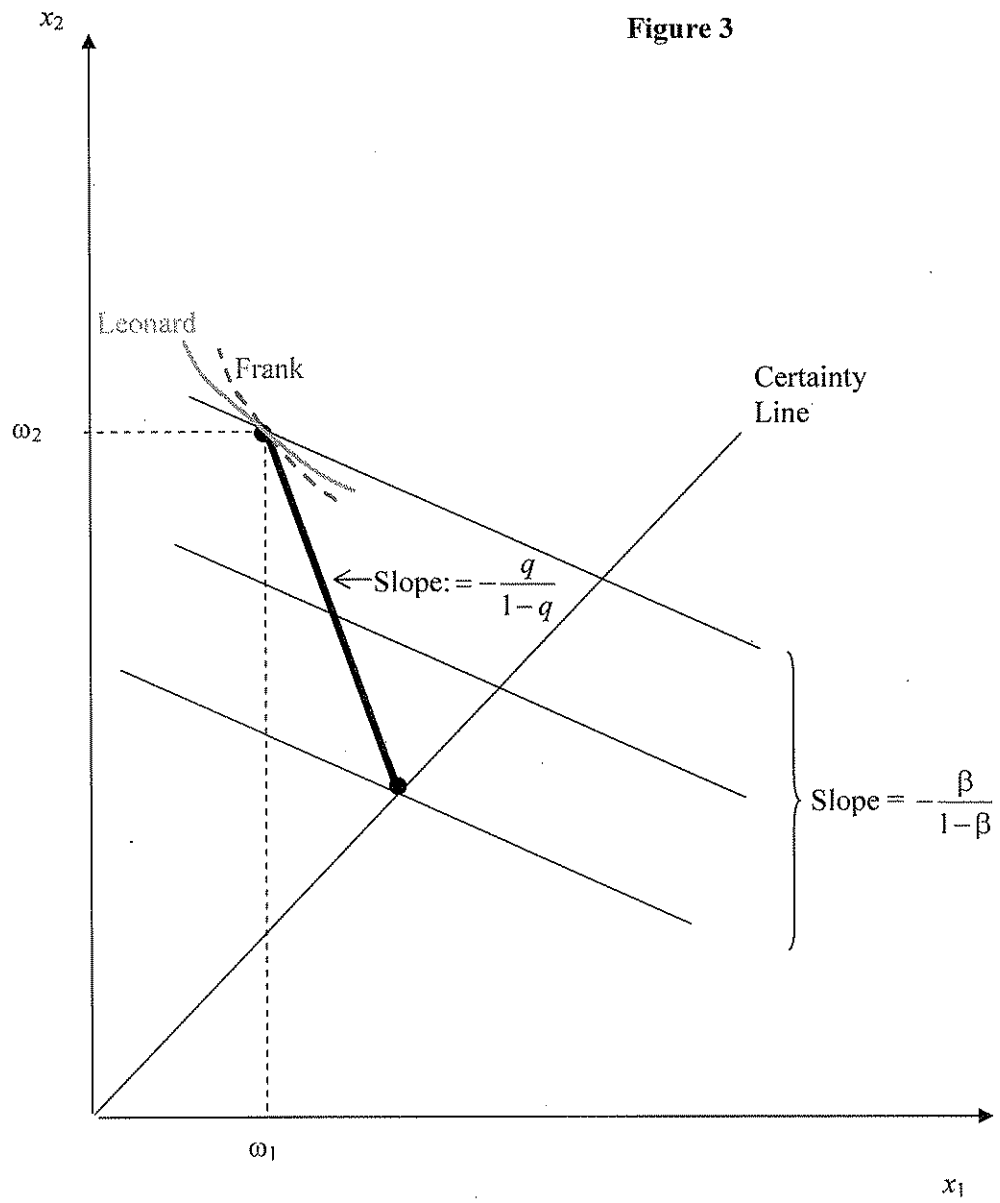


Figure 2





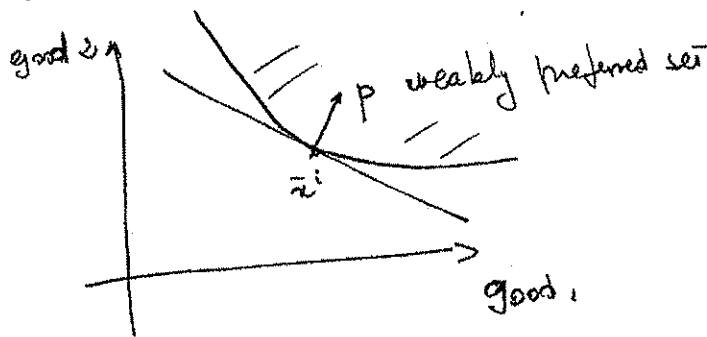


Answer Key for questions 2 and 3

①

**Question 2**

(a) In geometric terms, a price vector  $p$  "supports" a Pareto allocation  $\bar{x} = (\bar{x}^i)_{i=1}^I$  if for each  $i$ , the hyperplane through  $\bar{x}^i$  and orthogonal to  $p$  "supports" the preferred set of agent  $i$  at  $\bar{x}^i$ . This in turn means that all the weakly preferred consumption bundles  $x^i$  for agent  $i$  are all of the same side of the hyperplane  $p(x^i - \bar{x}^i) = 0$ .



Note: the a.k has been written without assuming the Inada conditions

In economic terms it means that all the consumption bundles weakly preferred by agent  $i$  to  $\bar{x}^i$  cost more than  $x^i$ : preferred cost more.

Since the utilities are strictly increasing, an allocation  $\bar{x}$  is Pareto optimal iff it solves the constrained maximum problem

$$\max u^i(x^i)$$

$$u^i(x^i) \geq v_i, \quad i=1, \dots, I \quad \alpha_i$$

$$\sum_L x_L^i \leq \sum_L w_L^i \quad l=1, \dots, L \quad p_l$$

(P)

$$x_L^i \geq 0 \quad \forall i \quad \forall l \quad v_L^i$$

for some guaranteed utility levels  $(v_2, \dots, v_I) \in \mathbb{R}^{I-1}$ .  $\textcircled{2}$   
 If  $v_i \leq u^i(0)$  for some agent(s), then any solution to (P) involves  $x^i = 0$  for these agents, who can then be omitted. Thus w.l.o.g. we assume  $v_i > u^i(0) \forall i=2, \dots, I$ . We also assume there exist  $\bar{x}$  satisfying the constraints with strict inequalities so that the qualification constraints are satisfied. Since the utility are quasi-concave, the solution of the problem (P) (unique by strict quasi-concavity) is characterized by the FOCs.

$$\alpha_i \frac{\partial u^i}{\partial x_l^i}(\bar{x}^i) = p_l - v_l^i \quad v_l^i \bar{x}_l^i = 0, \quad i=1, \dots, I, \quad l=1, \dots, L$$

$$\alpha_i (u^i(\bar{x}^i) - v_i) = 0 \quad i=2, \dots, I$$

$$p_l \left( \sum_{i=1}^I \bar{x}_l^i - \bar{w}_l \right) = 0 \quad l=1, \dots, L$$

where  $\alpha_1 = 1$ , and  $(\alpha_2, \dots, \alpha_I, p, v)$  are non-negative multipliers associated to the constraints of (P) as shown above. Given we have assumed  $v_i > u^i(0)$ ,  $\alpha^i > 0$  for all  $i$ , and since for each  $l$ , there exists an agent  $i$  with  $\alpha^i > 0$  and  $\bar{x}_l^i > 0$ ,  $p_l > 0$  since  $\frac{\partial u^i}{\partial x_l^i}(\bar{x}^i) > 0$ .

Note that the FOCs of the problem

$$\min \{ p \cdot x^i \mid u^i(x^i) \geq v_i, x^i \geq 0 \}$$

are satisfied at  $\bar{x}^i$ , so that consumption bundles preferred by agent  $i$  at  $\bar{x}^i$  cost (weakly) more, and  $p$  supports the allocation from  $\bar{x}$ .

(c) Let  $\bar{x}^i$  be the solution of  $\min \{ p \cdot x^i \mid u^i(x^i) \geq v_j^i, x_j^i \geq 0 \}$  (unique by strict quasi-concavity of  $u^i$ ). Let

$$\tilde{x}^i = v_j^i \frac{\bar{x}^i}{v_j^i}$$

Since  $u^i$  is homogeneous of degree 1  $u^i(\tilde{x}^i) = \frac{v_j^i}{v_j^i} u^i(\bar{x}^i) = v_j^i$

( $u^i(\bar{x}^i) = v_j^i$  since  $v_j^i \geq u^i(0)$  and giving more than the utility level  $v_j^i$  would cost more). Since  $\tilde{x}^i$  satisfies the constraint of the minimum expenditure problem of agent  $i$ ,  $p \cdot \tilde{x}^i \geq p \cdot \bar{x}^i$ . If  $\tilde{x}^i \neq \bar{x}^i$ , then  $p \cdot \tilde{x}^i > p \cdot \bar{x}^i$  (because the solution to the minimum expenditure problem is unique). But then

$$p \cdot \frac{\tilde{x}^i}{v_j^i} v_j^i > p \cdot \frac{\bar{x}^i}{v_j^i} v_j^i \Leftrightarrow p \cdot \tilde{x}^i > p \cdot \bar{x}^i$$

$u^i\left(\frac{\tilde{x}^i}{v_j^i} v_j^i\right) = v_j^i$ , so this inequality contradicts the assumption that  $\bar{x}^i$  minimizes the cost for agent  $i$ . Thus all the consumption bundles  $(\bar{x}^1, \dots, \bar{x}^I)$  are collinear

(d) Applying (b) and (c) for agents  $1, \dots, I$  imply that all their consumption bundles are collinear (agent 1 can be treated like the other agents by setting  $v_1 = u^1(\bar{x}^1)$ ). Thus all the consumption bundles of the agents must be collinear to the aggregate endowment:  $\bar{x}^i = \beta_i \bar{w}$  for some  $\beta_i > 0$ .

(which implies that the non-negativity constraints are not binding). ④

since the marginal utilities are homogeneous of  $d^0 0$ ,

$$p_l = \frac{\partial u}{\partial x_l}(p, \bar{w}) = \frac{\partial u}{\partial x_l}(\bar{w}), \quad l=1 \dots L$$

All the Pareto optimal allocations are supported by the price vector  $p = \left( \frac{\partial u}{\partial x_l}(\bar{w}) \right)_{l=1}^L$  in any collinear vector.

(e) The common utility function is a CES utility function homogeneous of  $d^0 1$ . So we can apply the previous analysis and all the equilibria have the same relative prices.

$$\bar{w} = (4, 2, 4) \quad \frac{\partial u}{\partial x_l} \left( \sum_{l=1}^4 (x_l)^{-2} \right)^{-3/2} x_l^{-3}$$

the  $p$  is collinear to  $(x_l^{-3})_{l=1}^L$ . If  $p_1 = 1$ , the price vector at any competitive equilibrium is

$$p = \left( 1, \frac{2^{-3}}{4^{-3}}, \frac{4^{-3}}{4^{-3}} \right) = (1, 8, 1)$$

The aggregate income is  $4 + 16 + 4 = 24$ , so each agent needs an income of 8 to buy a third of the endowment.

$p \cdot w^1 = 2$ , thus agent 1 must receive a transfer of 6  
 $p \cdot w^2 = 11$  thus agent 2 needs to give up 3  
 $p \cdot w^3 = 11$  thus agent 3 " " " " 3

QUESTION 3

①

(a) constrained demand function:

$$\text{agent 1: } \max \frac{1}{2} \log x_1 + \frac{1}{2} \log y_1 \quad \text{s.t.} \quad \begin{aligned} x_1 + p y_1 &\leq 4 & \lambda_1 \\ y_1 &\leq \bar{y} & \mu_1 \end{aligned}$$

$$\text{FOCs: } \frac{1}{2x_1} = \lambda_1, \quad \frac{1}{2y_1} = \lambda_1 p + \mu_1$$

$$\mu_1 = 0 \Leftrightarrow x_1 = \frac{1}{2\lambda_1}, \quad y_1 = \frac{1}{2\lambda_1 p}, \quad \frac{2}{2\lambda_1} = 4 \Rightarrow \lambda_1 = \frac{1}{4}, \quad x_1 = 2, \quad y_1 = \frac{2}{p}$$

$$\text{if } \frac{2}{p} \leq \bar{y}$$

$$\mu_1 \geq 0 \quad y_1 = \bar{y} \quad x_1 = 4 - p\bar{y} \quad \mu_1 = \frac{1}{2\bar{y}} - \frac{p}{2(4-p\bar{y})} = \frac{4-2p\bar{y}}{2\bar{y}(4-p\bar{y})} > 0$$

$$\text{if } \frac{2}{p} > \bar{y}$$

constrained demand of agent 1

$$\begin{cases} x_1 = 2, \quad y_1 = \frac{2}{p} & \text{if } \frac{2}{p} \leq \bar{y} \\ x_1 = 4 - p\bar{y}, \quad y_1 = \bar{y} & \text{if } \frac{2}{p} > \bar{y} \end{cases}$$

$$\text{agent 2: } \max \frac{1}{3} \log x_2 + \frac{2}{3} \log y_2 \quad \text{s.t.} \quad \begin{aligned} x_2 + p y_2 &\leq 2 \\ y_2 &\leq \bar{y} \end{aligned}$$

$$\text{same method as above leads to: } \begin{cases} x_2 = \frac{2}{3}, \quad y_2 = \frac{4}{3p} & \text{if } \frac{4}{3p} \leq \bar{y} \\ x_2 = 2 - p\bar{y}, \quad y_2 = \bar{y} & \text{if } \frac{4}{3p} > \bar{y} \end{cases}$$

(b) equilibrium  $((\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2); (\bar{x}, \bar{y}, \bar{p}))$  such that

$$\bullet (\bar{x}_i, \bar{y}_i) = (x_i(\bar{y}, \bar{p}), y_i(\bar{y}, \bar{p})), \quad i=1, 2 \quad (\text{agents optimize})$$

$$\bullet \bar{p}(\bar{y}_1 + \bar{y}_2) = \frac{2}{3} \quad (\text{break even price}) \quad \bar{y} = \bar{z} \quad (\text{feasibility})$$

$$\bullet \bar{x}_1 + \bar{x}_2 + \bar{z} = 6, \quad \bar{y}_1 \leq \bar{y}, \quad \bar{y}_2 \leq \bar{y} \quad (\text{feasibility})$$

Since  $\frac{2}{p} > \frac{4}{3p}$ , there are 3 possibilities for  $(\bar{y}, \bar{y})$  and  $\bar{x}$  for the equilibria with exclusion.

(i)  $\bar{y} \leq \frac{4}{3p}$ : then  $\bar{y}_1 = \bar{y}_2 = \bar{y}$ ,  $\bar{x}_2 = 4 - p\bar{y}$ ,  $\bar{x}_1 = 2 - p\bar{y}$

$$p(2\bar{y}) = \bar{z}, \quad \bar{y} = \bar{z}$$

$$4 - p\bar{y} + 2 - p\bar{y} + \bar{z} = 6$$

(ii)  $\frac{4}{3p} \leq \bar{y} \leq \frac{2}{p}$  then  $\bar{y}_1 = \bar{y}$ ,  $\bar{x}_1 = 4 - p\bar{y}$ ,  $\bar{x}_2 = \frac{2}{3}$ ,  $\bar{y}_2 = \frac{4}{3p}$

$$p\bar{y} + p\bar{y}_2 = \bar{z}, \quad \bar{y} = \bar{z}$$

$$4 - p\bar{y} + \frac{2}{3} + \bar{z} = 6.$$

(iii)  $\frac{2}{p} \leq \bar{y}$  then  $\bar{y}_1 = \frac{2}{p}$ ,  $\bar{x}_1 = 2$ ,  $\bar{y}_2 = \frac{4}{3p}$ ,  $\bar{x}_2 = \frac{2}{3}$

$$p(\bar{y}_1 + \bar{y}_2) = \bar{z}, \quad \bar{y} = \bar{z}$$

$$2 + \frac{2}{3} + \bar{z} = 6.$$

(c) equilibria of type (i): mean-even condition implies

$$2p\bar{y} = \bar{y} \Rightarrow \bar{p} = \frac{1}{2} \Rightarrow \bar{y} \leq \frac{8}{3}$$

feasibility condition  $4 - \frac{1}{2}\bar{y} + 2 - \frac{1}{2}\bar{y} + \bar{y} = 6$  is satisfied for all  $\bar{y}$

There is thus a continuum of equilibria with  $p=0.5$  and  $\bar{y} \leq \frac{8}{3}$ . Let us show that  $\bar{y} = \frac{8}{3}$  is Pareto dominated

(3)

the others. The utility of agent 1 is  $\frac{1}{2} \log(4 - \frac{1}{2} \bar{y}) + \frac{1}{2} \log \bar{y}$   
 and it is easy to see that this function is increasing in  $\bar{y}$ . In the same  
 way the utility of agent 2,  $\frac{1}{3} \log(4 - \frac{1}{2} \bar{y}) + \frac{2}{3} \log \bar{y}$  is increasing  
 in  $\bar{y}$ .

(d) equilibria of type (iii):  $\frac{2}{\bar{p}} \leq \bar{y}$

break even condition implies:  $\bar{p} \times \frac{2}{\bar{p}} + \bar{p} \times \frac{4}{3\bar{p}} = \bar{y} \Rightarrow \bar{y} = \frac{10}{3}$

which is compatible with the feasibility condition for the private  
 good. Any price  $\bar{p} \geq 0.6$  gives an equilibrium, but prices  
 $\bar{p} > 0.6$  are obviously inefficient, since a part of the public  
 good produced is not consumed by any of the agents and is  
 thus wasted.

(e) equilibria of type (ii)  $\frac{4}{3\bar{p}} \leq \bar{y} \leq \frac{2}{\bar{p}}$

break even condition:  $\bar{p} \bar{y} + \frac{4}{3} = \bar{y} \quad \frac{4}{3} = (1-\bar{p}) \bar{y}$

feasibility in private good:  $4 - \bar{p} \bar{y} + \frac{2}{3} \bar{y} = 6 \Leftrightarrow \frac{4}{3} = (1-\bar{p}) \bar{y}$

$\frac{4}{3\bar{p}} \leq \bar{y} \leq \frac{2}{\bar{p}}$  and  $(1-\bar{p}) \bar{y} = \frac{4}{3} \Rightarrow \frac{4}{3\bar{p}} \leq \frac{4}{3(1-\bar{p})} \leq \frac{2}{\bar{p}}$

The first inequality gives  $3\bar{p} \geq 3(1-\bar{p}) \Rightarrow \bar{p} \geq \frac{1}{2}$

The second inequality gives  $3(1-\bar{p}) \geq 2\bar{p} \Leftrightarrow \bar{p} \leq \frac{3}{5} = 0.6$

At  $\bar{p} = 0.5$   $\bar{y} = \frac{8}{3}$ ,  $\bar{x}_1 = 4 - \frac{4}{3} = \frac{8}{3}$ ,  $\bar{y}_1 = \bar{y}_2 = \frac{8}{3}$ ,  $\bar{x}_2 = \frac{2}{3}$  identical  
 to the "best" equilibrium of type (i)

(4)

At  $p=0.6$   $\bar{y} = \frac{10}{3}$   $\bar{y}_1 = \frac{10}{3}$   $\bar{z}_1 = \frac{8}{3}$ ,  $\bar{y}_2 = \frac{4}{3} \cdot \frac{5}{3} = \frac{20}{9} < \bar{y}$

this is the "best" equilibrium of type (ii)

$0.5 < \bar{p} < 0.6$   $\bar{y} = \frac{4}{3(1-\bar{p})}$ ,  $y_1 = \bar{y}$ ,  $z_1 = 4 - \frac{4\bar{p}}{3(1-\bar{p})}$ ,  $y_2 = \frac{4}{3\bar{p}}$ ,  $z_2 = \frac{2}{3}$

utility of the agents: at  $\bar{p}$ , for  $0.5 \leq \bar{p} \leq 0.6$

$$U_1 = \frac{1}{2} \log \left( 4 - \frac{4\bar{p}}{3(1-\bar{p})} \right) + \frac{1}{2} \log \frac{4}{3(1-\bar{p})}$$

$$= \frac{1}{2} \log \frac{4}{3} \frac{3-4\bar{p}}{1-\bar{p}} + \frac{1}{2} \log \frac{4}{3} (1-\bar{p})$$

$$= \frac{1}{2} \log (3-4\bar{p}) + \log \frac{4}{3}$$

which is obviously decreasing in  $\bar{p}$

$$U_2 = \frac{1}{3} \log \frac{2}{3} + \frac{2}{3} \log \frac{4}{3\bar{p}}$$

which is obviously decreasing in  $\bar{p}$

Thus the equilibrium with  $\bar{p} = 0.5$  Pareto dominates the others

(f) The best equilibrium, which Pareto dominates all the other is  $\bar{p} = 0.5$   $\bar{y} = \frac{8}{3}$ .

If the price is lower both agents are constrained: they would like to consume more of the public good at this price but they do not pay enough to finance more: increasing the price improves both agents. If the price is higher then



there is an inefficiency in the sense that agent 2 - who is <sup>(5)</sup> poorer than agent 1 but has higher relative preference for the public good - consumes less than what is produced, so that the public good is "underutilized". At  $p=0.5$  there is no "underutilization" and increasing the price would hurt both agents.

Additional comments:  
Let us check whether the Samuelson condition is satisfied

$$MRS_2 = \frac{\frac{2}{3} \cdot \frac{1}{8/3}}{\frac{1}{3} \cdot \frac{1}{2/3}} = \frac{1}{2} \quad MRS_1 = \frac{\frac{1}{2} \cdot \frac{1}{8/3}}{\frac{1}{2} \cdot \frac{1}{8/3}} = 1$$

$MRS_1 + MRS_2 = 1 + \frac{1}{2} > 1$ . The Samuelson condition is not satisfied and the equilibrium is not P.O. Thus the best "market solution" still does not reach Pareto

optimality: it equalizes the sum of the constrained marginal propensity to <sup>pay</sup> to the marginal cost, but if a constraint is binding the "constrained marginal propensity to pay, which includes the multiplier, is different from the true "marginal propensity to pay".

Answer keys for Questions 4 and 5 Micro Prelim June 2011

4. (a.1) The game is as follows:

		B	
		To me	To her
A	To me	0 , 0	a , 0
	To her	0 , b	$\frac{a+b}{2}$ , $\frac{a+b}{2}$

Since  $a < b$ ,  $a < \frac{a+b}{2} < b$  and thus “To her” is a weakly dominant strategy for player A and “To me” is a weakly dominant strategy for player B.

(a.2) The pure-strategy Nash equilibria are (To me, To me) and (To her, To me).

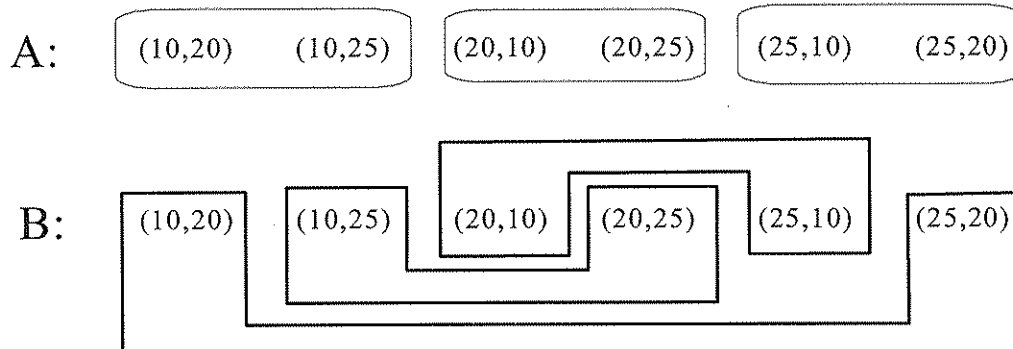
(b.1) The game is as follows:

		B	
		To me	To her
A	To me	-d , -d	a , 0
	To her	0 , b	$\frac{a+b}{2}$ , $\frac{a+b}{2}$

For A “To her” is a strictly dominant strategy. On the other hand, B does not have a dominant strategy.

(b.2) There is a unique Nash equilibrium given by (To her, To me).

(c.1) Let a state be denoted by a pair  $(x,y)$  where  $x$  is the valuation of A and  $y$  the valuation of B. Then the possible states are  $(10,20)$ ,  $(10,25)$ ,  $(20,10)$ ,  $(20,25)$ ,  $(25,10)$  and  $(25,20)$ . The information partitions are as follows (within each information set the probabilities are  $\frac{1}{2}$  on each state):



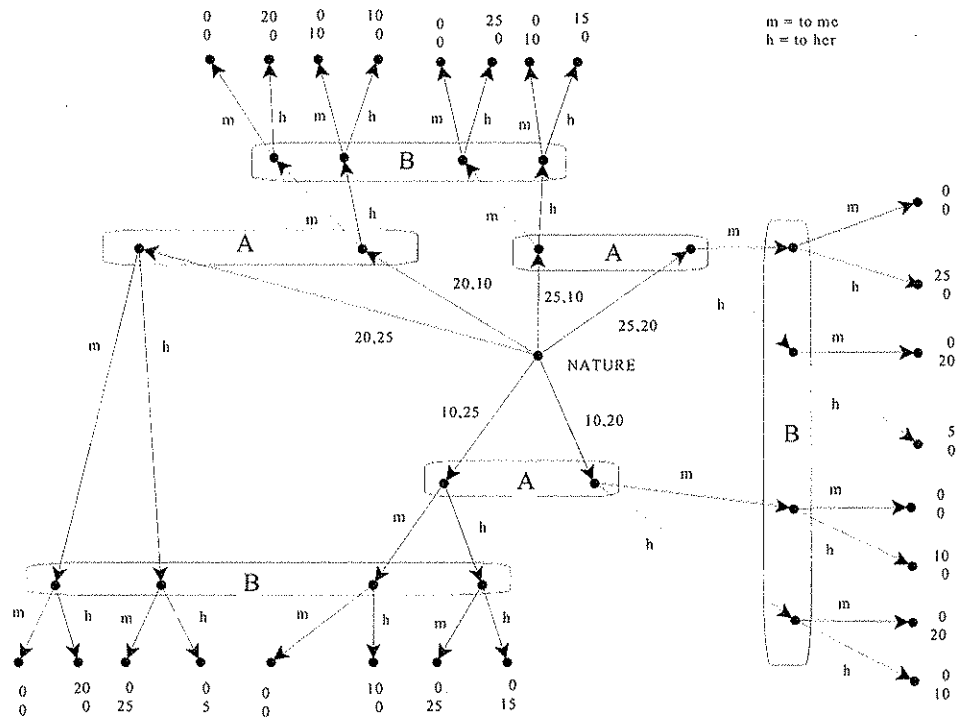
(c.2)  $E = \{(20,10), (25,10), (25,20)\}$ .  $K_A E = \{(25,10), (25,20)\}$ ,  $K_B E = \{(25,10), (20,10)\}$   
 $K_A K_B E = K_B K_A E = \emptyset$ .

(c.3) The game is as follows:

		B		
		(valuation 20)		
		10	20	25
A	10	15 , 0	0 , 10	0 , 10
	20	15 , 0	5 , 0	0 , 0
	25	15 , 0	5 , 0	0 , 0

For each player both 20 and 25 are weakly dominant strategies. Thus there are four dominant-strategy equilibria: (20,20), (20,25), (25,20) and (25,25). The “truthful” bids are 25 for player A and 20 for player B, thus the “truthful” dominant-strategy equilibrium is (25,20).

(c.4) The extensive-form is as follows (where all of Nature’s choices have the same probability, namely  $\frac{1}{6}$ ):



(c.5) (c.5.1) At every information set of player B “to me” is a weakly dominant choice (if also player A played “to me” then both choices of player B give player B a payoff of 0, while if player A played “to her” then with “to me” player B gets the object for free, while with “to her” either she does not get it or she gets it for a payment). Hence a weakly dominant strategy is for player B to play “to me” no matter what her valuation is (that is, at each of her three information sets).

(c.5.2) Given this strategy of player B, every strategy of player A gives the same payoff to player A (namely, zero). Hence every pair  $(s_1, s_2)$ , where  $s_1$  is any pure strategy of player A and  $s_2$  is the above strategy of player B, is a Nash equilibrium.

(c.5.3) Hence there are 8 such equilibria.

5. (a)  $u_1(10, 2) = 184$ . Thus the indifference curve of type 1 is the solution to  $u_1(m, t) = 184$ , which is  $\left\{ (m, t) : m = \frac{2}{5}t^2 + \frac{42}{5} \right\}$ .  $u_2(10, 2) = 192$ . Thus the indifference curve of type 2 is the solution to  $u_2(m, t) = 192$ , which is  $\left\{ (m, t) : m = \frac{t^2}{5} + \frac{46}{5} \right\}$ .

(b) Type 1 chooses  $t$  to maximize  $u_1(m, t)$  subject to  $m = 8 + 2t$ , that is, chooses  $t$  to maximize  $180 + 20t - 4t^2$ . The solution is  $t = 2.5$  with  $m = 13$  and  $u_1 = 205$ .

Type 2 chooses  $t$  to maximize  $u_2(m, t)$  subject to  $m = 8 + 2.5t$ , that is, chooses  $t$  to maximize  $180 + 25t - 2t^2$ . The solution is  $t = 6.25$  with  $m = 23.625$  and  $u_2 = 258.125$ .

(c) (c.1) In this case ( $t^* = 4$ ) there is no separating equilibrium. At such an equilibrium, in order for type 1 individuals to be paid according to their true productivity, they would have to choose  $t < t^*$ . From part (b) we know that the highest utility a type 1 person can get subject to  $t < t^*$  is obtained when  $t = 2.5$  and is given by  $u_1 = 205$ . On the other hand, if a type 1 chose  $t = 4$ , then he would be paid  $8 + 2.5(4) = 18$  and his utility would be  $u_1 = 216$ . Thus he would prefer “masquerading as a type 2 person” by choosing a  $t$  of at least 4 and would therefore not be paid according to his true productivity (e.g. if  $t = 4$  his true productivity is  $8 + 2(4) = 16$  rather than 18).

(c.2) In this case ( $t^* = 9$ ) we have a separating equilibrium where type 1 workers choose  $t = 2.5$ , are paid  $m = 13$  and their utility is  $u_1 = 205$  and type 2 workers choose  $t = 9$ , are paid  $m = 30.5$  and their utility is  $u_2 = 243$ . Proof that this is an equilibrium. Consider first Type 1 workers. From part (b) we know that  $t = 2.5$  maximizes  $u_1(m, t)$  subject to  $m = 8 + 2t$  for every  $t$  (thus in particular for  $t < 9$ ); if a type 1 person chose  $t \geq 9$ , then his utility would be  $u_1(8 + 2.5t, t) = 180 + 25t - 4t^2$ , which is decreasing in  $t$  in the range  $t \geq 9$ ; thus the highest value in this range is  $t = 9$  with a corresponding utility of 81, which is less than the utility of choosing  $t = 2.5$ . Consider now Type 2 workers. For them,  $u_2(m, t)$  subject to  $m = 8 + 2.5t$  is decreasing in the range  $t \geq 9$  [we know this from part (b)], so that the optimal value of  $t$  in this range is  $t = 9$  with a corresponding utility of 243; on the other hand, for  $t < 9$ ,  $u_2(m, t)$  with  $m = 8 + 2t$  is maximized at  $t = 5$  with a utility of  $u_2 = 230$ .

(c.3) In this case ( $t^* = 11$ ) there is no separating equilibrium. By the same argument as in part (c.2), for a type 2 the optimal value of  $t$  in the range  $t \geq 11$  is  $t = 11$  with a corresponding utility of  $u_2 = 213$ ; thus a type 2 would prefer “masquerading as a type 1 person” by choosing  $t = 5$  (see part (c.2)) obtaining a utility of  $u_2 = 230$  and would therefore not be paid according to his true productivity.

(d) (d.1) We have that  $m(t) = \begin{cases} 8+2t & \text{if } t < 4 \\ 8+(2.5-0.5q)t & \text{if } t \geq 4 \end{cases}$ . Consider first type 1. We know from part

(b) that the maximum of  $u_1(m, t)$  subject to  $m = 8 + 2t$  in the range  $t < 4$  is achieved  $t = 2.5$  with  $u_1 = 205$ . On the other hand, replacing  $m$  with  $8 + (2.5 - 0.5q)t$  in  $u_1(m, t)$  yields the function  $U_1(t, q) = 180 + 25t - 5qt - 4t^2$  which is strictly decreasing in  $t$  in the range  $t \geq 4$  (for any  $q \in [0, 1]$ ).  $U_1(4, q) \geq 205$  if and only if  $q \leq 0.55$ . Thus type will choose  $\begin{cases} t=4 & \text{if } q < 0.55 \\ t=2.5 & \text{if } q > 0.55 \end{cases}$  and be indifferent between  $t = 2.5$  and  $t = 4$  if  $q = 0.55$ . Now consider type 2. Replacing  $m$  with  $8 + 2t$  in  $u_2(m, t)$  yields the function  $180 + 20t - 2t^2$  which is strictly increasing in the range  $t < 4$ ; at  $t = 4$  the value of this function is 228. Replacing  $m$  with  $8 + (2.5 - 0.5q)t$  in  $u_2(m, t)$  yields the function  $U_2(t, q) = 180 + 25t - 5qt - 2t^2$ . Now  $U_2(4, q) = 248 - 20q > 228$  for all  $q < 1$  so that choosing  $t = 4$  is better for type 2 than choosing  $t < 4$ . However,

$\left. \frac{\partial U_2(t, q)}{\partial t} \right|_{t=4} = 9 - 5q > 0$  for all  $q \in [0, 1]$  and thus type 2 would not choose  $t = 4$ . The optimal

choice of type 2 is given by the solution to  $\frac{\partial U_2(t, q)}{\partial t} = 0$  which is  $t(q) = \frac{25 - 5q}{4}$ .

(d.2) Since there is no value of  $q$  at which both types choose the same value of  $t$ , there is no pooling equilibrium.