Microeconomics Prelim August 2008

ANSWER KEY

Question 1.

(a). Let P the set of relevant price-wealth vectors (p, w), a subset of $\Re^{L+1}_{++}$, and denote by $\tilde{x}_j : P \to \Re_{++}$ a consumer's Walrasian demand for good j, $j = 1,\dots L$, assumed to be strictly positive and differentiable on P.

(a).1. What do we mean when we say that good j is a *necessity* for the consumer at (p, w)? Same for *luxury* and for *borderline necessity-luxury*.

Necessity: $0 < \dfrac{\partial \tilde{x}_j(p,w)}{\partial w} \dfrac{w}{\tilde{x}_j(p,w)} < 1$.

Luxury: $\dfrac{\partial \tilde{x}_j(p,w)}{\partial w} \dfrac{w}{\tilde{x}_j(p,w)} > 1$.

Borderline Necessity-Luxury: $\dfrac{\partial \tilde{x}_j(p,w)}{\partial w} \dfrac{w}{\tilde{x}_j(p,w)} = 1$.

(a).2. Show that the concepts of *luxury* and *borderline necessity-luxury* can be characterized by a property of the budget share function $b_j(p,w)$ of the good. Can you do the same with the concept of *necessity*? Explain.

The property is the sign of the derivative $\dfrac{\partial b_j(p,w)}{\partial w} \equiv \dfrac{\partial (p_j \tilde{x}_j(p,w))}{\partial w}$, which can be computed as $\dfrac{1}{w^2}\left[p_j \dfrac{\partial \tilde{x}_j(p,w)}{\partial w} - p_j \tilde{x}_j(p,w)\right] = \dfrac{p_j \tilde{x}_j(p,w)}{w^2}\left[\dfrac{w}{\tilde{x}_j(p,w)} \dfrac{\partial \tilde{x}_j(p,w)}{\partial w} - 1\right]$. Hence, $\dfrac{\partial b_j(p,w)}{\partial w} > 0 \Leftrightarrow \dfrac{w}{\tilde{x}_j(p,w)} \dfrac{\partial \tilde{x}_j(p,w)}{\partial w} > 1$, which is the definition of a luxury at (p, w). Similarly for the necessity-luxury borderline.

If good j is a necessity, then the previous equality shows that $\dfrac{\partial b_j(p,w)}{\partial w} < 0$. But if $\dfrac{\partial \tilde{x}_j(p,w)}{\partial w} < 0$, then $\dfrac{\partial b_j(p,w)}{\partial w} < 0$, but the good is not a necessity in the sense of the above definition.

For the rest of this question we consider the indirect utility function

$$v : P \to \Re : v(p,w) = \left[\frac{F(p)}{\ln(w / C(p))} + G(p)\right]^{-1}, \qquad (1)$$

where $C(p) >> 0$, and the functions C, F and G are such that $v(p, w)$ has the properties of an indirect utility function on P.

(b). For $j = 1, \ldots, L$, obtain the Walrasian demand function $\tilde{x}_j(p,w)$ and the budget share function $b_j(p,w)$ corresponding to (1).

By Roy's identity, $\tilde{x}_j(p,w) = -\dfrac{\partial v(p,w)/\partial p_j}{\partial v(p,w)/\partial w}$, which if computed from (1) yields

$$\tilde{x}_j(p,w) = -\frac{-\left[\dfrac{F(p)}{\ln(w/C(p))}+G(p)\right]^{-2}\left[\dfrac{\dfrac{\partial F}{\partial p_j}\ln\dfrac{w}{C}-\dfrac{C}{w}\left(-\dfrac{w}{C^2}\right)\dfrac{\partial C}{\partial p_j}F}{\left(\ln\dfrac{w}{C}\right)^2}+\dfrac{\partial G}{\partial p_j}\right]}{-\left[\dfrac{F(p)}{\ln(w/C(p))}+G(p)\right]^{-2}\times\dfrac{-\dfrac{C}{w}\left(\dfrac{1}{C}\right)F}{\left(\ln\dfrac{w}{C}\right)^2}}$$

$$= \frac{\dfrac{\partial F}{\partial p_j}\ln\dfrac{w}{C}-\dfrac{C}{w}\left(-\dfrac{w}{C^2}\right)\dfrac{\partial C}{\partial p_j}F+\dfrac{\partial G}{\partial p_j}\left(\ln\dfrac{w}{C}\right)^2}{\dfrac{F}{w}},$$

i. e., $$\tilde{x}_j(p,w) = \frac{w}{F}\frac{\partial F}{\partial p_j}\ln\frac{w}{C}+\frac{w}{C}\frac{\partial C}{\partial p_j}+\frac{w}{F}\frac{\partial G}{\partial p_j}\left(\ln\frac{w}{C}\right)^2. \qquad (2)$$

This in turn yields

$$b_j(p,w) \equiv \frac{p_j\tilde{x}_j}{w} = \frac{p_j}{F}\frac{\partial F}{\partial p_j}\ln\frac{w}{C}+\frac{p_j}{C}\frac{\partial C}{\partial p_j}+\frac{p_j}{F}\frac{\partial G}{\partial p_j}\left(\ln\frac{w}{C}\right)^2. \qquad (3)$$

(c). Consider first the case of (1) with $G(p) = 0$, all p. Show that if good j is a luxury at some $(\bar{p},\bar{w}) >> 0$, then it is a luxury at $(\bar{p}, w)$, for all $w > 0$.

If $G(p) = 0$, $\forall p$, then

$$b_j(p,w) \equiv \frac{p_j}{F}\frac{\partial F}{\partial p_j}\ln\frac{w}{C}+\frac{p_j}{C}\frac{\partial C}{\partial p_j}. \qquad (4)$$

As seen in (a).2, luxury at $(\overline{p},\overline{w}) \Leftrightarrow \frac{\partial b_j(\overline{p},\overline{w})}{\partial w} > 0$, which for (4) is equivalent to

$\frac{\overline{p}_j}{F}\frac{\partial F(\overline{p})}{\partial p_j} > 0$, in which case $\frac{\partial b_j(\overline{p},w)}{\partial w} > 0, \forall w > 0$.

(d). Consider now the general case of (1) where $G(p)$ is not always zero.

(d).1. Suppose that good j is a luxury at some $(\overline{p},\overline{w}) >> 0$. Does it follow that it is a luxury at $(\overline{p},w)$, for all $w > 0$? Explain.

Now $\frac{\partial b_j(\overline{p},w)}{\partial \ln(\frac{w}{C(p)})} = \frac{p_j}{F}\frac{\partial F}{\partial p_j} + \frac{p_j}{F}\frac{\partial G}{\partial p_j} 2\ln(\frac{w}{C(p)})$, which may in principle change signs as w varies, in which case good j is a luxury at some w's, but at for others.

(d).2. Suppose that all consumers in the economy have identical preferences, of the type represented by (1). Under which conditions on the functions C, F and G can the consumers' aggregate demand be a function of (only) prices and aggregate wealth? Argue clearly. Discuss the possibility of a positive representative consumer with these preferences.

Demand is a function on only prices and aggregate wealth if and only if, for $j = 1,\ldots, L$, all consumers' Engel curves for good j are affine with a common slope across consumers. From (2), this requires that, for $j = 1,\ldots, L$, $\frac{\partial F(p)}{\partial p_j} = \frac{\partial G(p)}{\partial p_j} = 0, \forall p$, i. e., $F(p)$ and $G(p)$ must be constant functions. In that case, the Gorman positive representative consumer theorem applies.

Consider for instance $F(p) = 1$ and $G(p) = 0$, i. e.,

$v(p,w) = \left[\frac{1}{\ln(w/C(p))}\right]^{-1} = \ln w - \ln C(p))$. The Cobb-Douglas direct utility function $\alpha \ln x_1 + (1-\alpha)\ln x_2$ has as indirect utility function $\alpha \ln\frac{\alpha w}{p_1} + (1-\alpha)\ln\frac{(1-\alpha)w}{p_2} =$

$\ln w - \ln\left(\left[\frac{p_1}{\alpha}\right]^{\alpha}\left[\frac{p_2}{1-\alpha}\right]^{1-\alpha}\right)$, which is of this form.

Question 2.

(a). Price-taking firm. Consider a price-taking firm with production set $Y \subset \Re^L$ and facing a strictly positive price vector *p*.

(a).1. Write the firm's profit-maximizing problem.

max $_y$ $p \cdot y$ subject to $y \in Y$, *p* given.

(a).2. Independence from normalization. Normalize all prices with good *j* as numeraire, and write the profit maximizing problem under this normalization. Show that the same solution (or set of solutions) obtains no matter which good *j* is chosen as numeraire.

In order to normalize prices with good *j* as numeraire, we divide all prices by p_j. The profit maximization problem becomes max $_y$ $(\frac{1}{p_j}p) \cdot y$ subject to $y \in Y$, $\frac{1}{p_j}p$ given. No matter which *j* we choose, this problem has the same solutions as the one in (a).1, because

$p \cdot y^0 \geq p \cdot y^1 \Leftrightarrow (\frac{1}{p_j}p) \cdot y^0 \geq (\frac{1}{p_j}p) \cdot y^1$, for any $j \in \{1,\ldots,L\}$

(b). Firm with market power. Now we specialize to two goods ($L = 2$), where good 1 is an input in the firm, and good 2 is its output. The production set of the firm is $\{(y_1, y_2) \in \Re^2 : y_1 \leq 0, y_2 \leq -y_1 / c\}$, where *c* is a positive parameter. The firm is a price setter, and faces the following demand function for its output:

$$\tilde{x}_2(p_1, p_2) = \frac{\omega p_1}{p_1 c + \frac{\alpha}{1-\alpha} p_2},$$

where the parameters ω and α satisfy $\omega > 0$ and $\alpha \in (0,1)$.

(b).1. Write the firm's profits as a function of the prices (p_1, p_2) (disregard the possibility that $y_2 < -y_1 / c$.)

We must have $y_1 = -cy_2$, and $y_2 = \tilde{x}_2(p_1, p_2)$. Hence, profits are given by

$$(p_2 - cp_1)\frac{\omega p_1}{p_1 c + \frac{\alpha}{1-\alpha} p_2}. \qquad (2.1)$$

(b).2. Write, analyze and, if possible, solve the firm's profit-maximizing problem when prices are normalized with good 1 as numeraire.

Write $\hat{p}_2 \equiv \frac{p_2}{p_1}$. Then, from (2.1), profits are given by the function

$\hat{\Pi}(\hat{p}_2) \equiv (\hat{p}_2 - c)\dfrac{\omega}{c + \frac{\alpha}{1-\alpha}\hat{p}_2}$, with derivative

$$\hat{\Pi}'(\hat{p}_2) = \frac{\omega}{c + \frac{\alpha}{1-\alpha}\hat{p}_2} + (\hat{p}_2 - c)\frac{-\frac{\alpha}{1-\alpha}\omega}{\left[c + \frac{\alpha}{1-\alpha}\hat{p}_2\right]^2} = \frac{\omega}{\left[c + \frac{\alpha}{1-\alpha}\hat{p}_2\right]^2}\left(c + \frac{\alpha}{1-\alpha}\hat{p}_2 - (\hat{p}_2 - c)\frac{\alpha}{1-\alpha}\right)$$

$$= \frac{\omega}{\left[c + \frac{\alpha}{1-\alpha}\hat{p}_2\right]^2}\left(c + c\frac{\alpha}{1-\alpha}\right) = \frac{\omega}{\left[c + \frac{\alpha}{1-\alpha}\hat{p}_2\right]^2}\left(\frac{c}{1-\alpha}\right),$$

always positive. Hence, the higher $\hat{p}_2$, the higher the profits. The maximization problem has no solution.

(b).3. Write, analyze and, if possible, solve the firm's profit-maximizing problem when prices are normalized with good 2 as numeraire.

Now write $\hat{\hat{p}}_1 \equiv \dfrac{p_1}{p_2}$, and profits are given by the function:

$\hat{\hat{\Pi}}(\hat{\hat{p}}_1) \equiv (1 - c\hat{\hat{p}}_1)\dfrac{\omega\hat{\hat{p}}_1}{c\hat{\hat{p}}_1 + \frac{\alpha}{1-\alpha}}$, with derivative $\hat{\hat{\Pi}}'(\hat{\hat{p}}_1) = -c\dfrac{\omega\hat{\hat{p}}_1}{c\hat{\hat{p}}_1 + \frac{\alpha}{1-\alpha}} + (1 - c\hat{\hat{p}}_1)\dfrac{\omega[c\hat{\hat{p}}_1 + \frac{\alpha}{1-\alpha}] - c\omega\hat{\hat{p}}_1}{\left(c\hat{\hat{p}}_1 + \frac{\alpha}{1-\alpha}\right)^2}$

$= \dfrac{\omega}{\left(c\hat{\hat{p}}_1 + \frac{\alpha}{1-\alpha}\right)^2}\left[-c\hat{\hat{p}}_1(c\hat{\hat{p}}_1 + \frac{\alpha}{1-\alpha}) + (1 - c\hat{\hat{p}}_1)(c\hat{\hat{p}}_1 + \frac{\alpha}{1-\alpha} - c\hat{\hat{p}}_1)\right]$. The bracketed term can be written

$-(c\hat{\hat{p}}_1)^2 - c\hat{\hat{p}}_1\frac{\alpha}{1-\alpha} + \frac{\alpha}{1-\alpha} - c\hat{\hat{p}}_1\frac{\alpha}{1-\alpha} = -(c\hat{\hat{p}}_1)^2 - 2c\hat{\hat{p}}_1\frac{\alpha}{1-\alpha} + \frac{\alpha}{1-\alpha}$, positive if $\hat{\hat{p}}_1 = 0$ and negative if $\hat{\hat{p}}_1 = \frac{1}{c}$. Because $\hat{\hat{\Pi}}(0) = \hat{\hat{\Pi}}(\frac{1}{c}) = 0$, this means that profits are positive somewhere in the interval $(0, \frac{1}{c})$, and since, moreover, profits are negative for $\hat{\hat{p}}_1 > \frac{1}{c}$, we know that a maximizer of profits on $[0, \frac{1}{c}]$ (there must be at least one, and any maximizer must be interior) maximizes profits on $\Re_+$.

As noted, $\hat{\hat{\Pi}}'(\hat{\hat{p}}_1) = 0 \Leftrightarrow -(c\hat{\hat{p}}_1)^2 - 2c\hat{\hat{p}}_1\frac{\alpha}{1-\alpha} + \frac{\alpha}{1-\alpha} = 0$, a quadratic equation that can be written $c^2\hat{\hat{p}}_1^{\,2} + 2c\frac{\alpha}{1-\alpha}\hat{\hat{p}}_1 - \frac{\alpha}{1-\alpha} = 0$, with roots $\hat{\hat{p}}_1 = \dfrac{-2c\frac{\alpha}{1-\alpha} \pm \sqrt{\left(2c\frac{\alpha}{1-\alpha}\right)^2 + 4c^2\frac{\alpha}{1-\alpha}}}{2c^2}$, of which only the positive root gives a positive solution, i. e.,

$$\hat{\hat{p}}_1 = \frac{-2c\frac{\alpha}{1-\alpha} + 2c\sqrt{\left(\frac{\alpha}{1-\alpha}\right)^2 + \frac{\alpha}{1-\alpha}}}{2c^2} = \frac{-\frac{\alpha}{1-\alpha} + \sqrt{\left(\frac{\alpha}{1-\alpha}\right)^2 + \frac{\alpha}{1-\alpha}}}{c}$$

. (It can be checked that this root is indeed in $(0, \frac{1}{c})$.) It follows that when prices are normalized with good 2 as numeraire. the profit maximization problem has a unique solution.

(b).4. Compare (b).2 and (b).3. Very different.

Compare with (a).2. In (b).2 and (b).3, where the firm has market power, normalization matters: the implications of profit maximization vary according to which good is used as numeraire. In (a).2, on the contrary, where the firm is a price taker, normalization does not matter.

Answer key for Question 3

(a) In a competitive equilibrium of $\mathcal{E}(u,\omega)$, agent i maximizes $V_i(m_i(x^i),(m_j(\bar{x}^j))_{j\neq i})$ under his/her budget constraint $\bar{p}x^i \leq \bar{p}\omega^i$, taking the consumption $(\bar{x}^j)_{j\neq i}$ of the other agents as given. This is equivalent to maximizing $m_i(x^i)$ under the budget constraint. Thus the maximization problem is the same in the economy $\mathcal{E}(u,\omega)$ and in the economy $\mathcal{E}^{ego}(m,\omega)$ and, since the market clearing conditions are the same, the competitive equilibria are the same.

(b) Suppose x^* is a Pareto optimal allocation of $\mathcal{E}(u,\omega)$ and suppose it is not Pareto optimal in $\mathcal{E}^{ego}(m,\omega)$. There exists a feasible allocation x such that $m_i(x^i) \geq m_i(x^{*i})$ with at least one strict inequality. Given the monotonicity assumptions on the functions V_i, $u_i(x) \geq u_i(x^*)$, with a strict inequality for the agent(s) such that $m_i(x^i) > m_i(x^{*i})$. This contradicts the optimality of x^* in $\mathcal{E}(u,\omega)$.

(c)(i) obvious

(c)(ii) $u_1(x(\alpha)) = \alpha^{\frac{2}{3}} + 0.9\,(2-\alpha)^{\frac{2}{3}}$. Thus $u_1'(x(\alpha)) = \dfrac{2}{3}\left(\alpha^{-\frac{1}{3}} - 0.9\,(2-\alpha)^{-\frac{1}{3}}\right)$.

$u_1'(x(\alpha)) > 0 \iff \alpha^{-\frac{1}{3}} > 0.9\,(2-\alpha)^{-\frac{1}{3}} \iff \alpha(1+0.9^{-3}) < 2\times 0.9^{-3}$.

Let $\overline{\alpha} = \frac{2\times 0.9^{-3}}{1+0.9^{-3}} = \frac{2}{1.73}$. If $\alpha < \overline{\alpha}$, $u_1(x(\alpha))\uparrow$, and if $\alpha > \overline{\alpha}$, $u_1(x(\alpha))\downarrow$.

The same calculation for agent 2 leads to considering $\underline{\alpha} = \frac{2}{1+0.9^{-3}} = \frac{2\times 0.73}{1.73}$. If $\alpha < \underline{\alpha}$, $u_2(x(\alpha))\uparrow$, and if $\alpha > \underline{\alpha}$, $u_2(x(\alpha))\downarrow$.

If $\alpha < \underline{\alpha}$, the allocation $x(\alpha)$, although materially efficient, is not Pareto optimal in $\mathcal{E}(u,\omega)$. Increasing α benefits both agents because the decrease in material well-being of agent 2 is more than compensated by the increase in utility due to his/her concern for the well-being of agent 1. In the same way, if α is larger than $\overline{\alpha}$, decreasing α increases the utility of both agents despite the fact that the consumption of agent 1 decreases.

(d) In the economy described in the previous question, if the endowment of one of the agents is sufficiently small relative to that of the other, the competitive equilibrium will be a materially efficient allocation (First Theorem of Welfare economics for $\mathcal{E}^{ego}(m,\omega)$) corresponding to a share of the resources α for agent 1 either close to zero or close to 1. From (c), the allocation is not Pareto optimal for $\mathcal{E}(u,\omega)$. Thus the competitive equilibria of the economies with large inequalities are not Pareto optimal, and the First Theorem of Welfare Economics does not hold for these economies.

(e) If x^* is a Pareto optimal allocation of $\mathcal{E}(u,\omega)$, it is materially efficient. By the second Theorem of Welfare Economics applied to $\mathcal{E}^{ego}(m,\omega)$, it can be obtained as a competitive equilibrium of $\mathcal{E}^{ego}(m,\omega)$ after redistribution of resources, and by (a) this is also a competitive equilibrium for $\mathcal{E}(u,\omega)$. Thus the Second Theorem of Welfare Economics holds for all economies $\mathcal{E}(u,\omega)$.

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Answer Keys for Questions 4 and 5

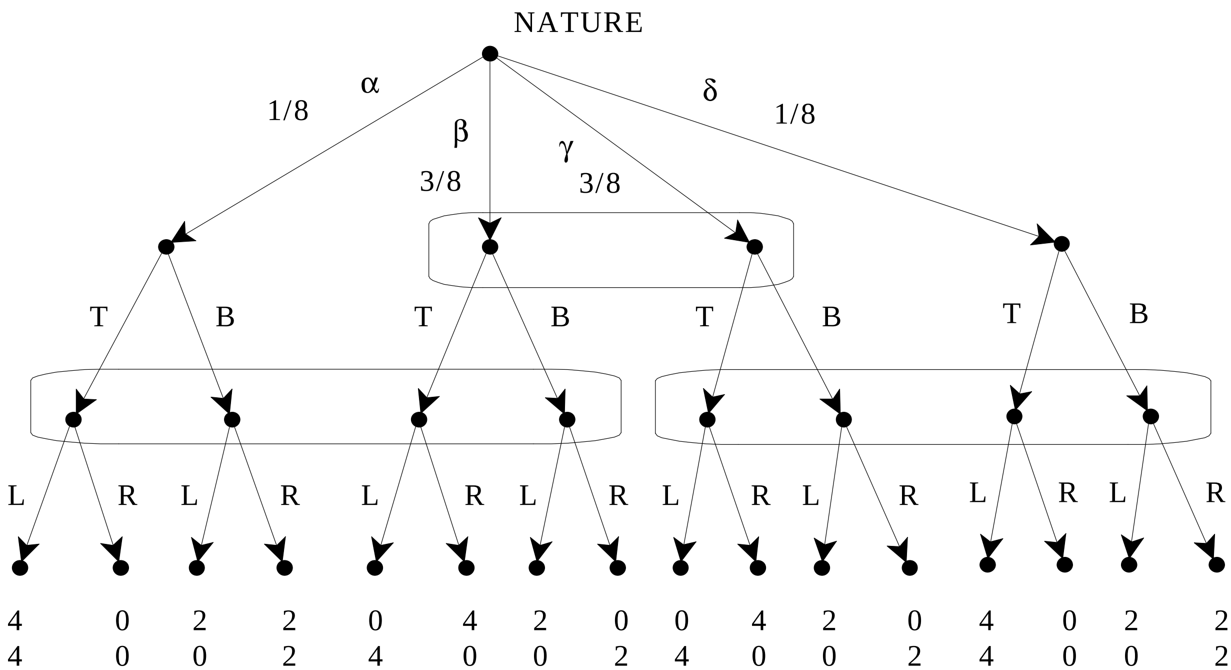
4.

(a) It is the trivial partition $\{\{\alpha,\beta,\gamma,\delta\}\}$.

(b) **(b.1)** Yes, since at every state each player knows what action he himself is taking.

(b.2) $\mathbf{RAT}_1 = \{\alpha\}$ (at α T $\rightarrow$ 4, B $\rightarrow$ 2, at β and γ T $\rightarrow$ 2, B $\rightarrow$ 1, at δ T $\rightarrow$ 0, B $\rightarrow$ 2) and $\mathbf{RAT}_2 = \{\gamma,\delta\}$ (at α and β L $\rightarrow$ 1, R $\rightarrow$ 1.5, at γ and δ L $\rightarrow$ 1, R $\rightarrow$ 1.5), so that **RAT** = $\emptyset$ and thus K_1**RAT** = K_2**RAT** = $\emptyset$.

(c) The imperfect information game is as follows:



(d) Player 1 has 8 strategies: TTT, TTB, TBT, TBB, BTT, BTB, BBT, BBB.
Player 2 has four strategies: LL, LR, RL, RR.

(e) No. Player 1 can increase his payoff by switching to TBT: $\pi_1(TTT, LL) = \frac{1}{8}4 + \frac{3}{8}0 + \frac{3}{8}0 + \frac{1}{8}4 = 1$ while $\pi_1(TBT, LL) = \frac{1}{8}4 + \frac{3}{8}2 + \frac{3}{8}2 + \frac{1}{8}4 = 2.5$.

(f) Player 1's beliefs must be $\frac{1}{2}$ at the left node and $\frac{1}{2}$ at the right node of his information set. Player 2's beliefs at his information set on the left must be: $\frac{1}{4}$ at the left-most node and $\frac{3}{4}$ at the third node from the left and his beliefs at the other information set must be $\frac{3}{4}$ at the left-most node and $\frac{1}{4}$ at the third node from the left. To prove consistency take the same behavior strategy $\begin{pmatrix} T & B \\ 1-\frac{1}{n} & \frac{1}{n} \end{pmatrix}$ at each information set of player 1 and take the limit of the corresponding beliefs obtained using Bayes' rule.

(g) By Nash's theorem, the game has at least one (possibly mixed-strategy) equilibrium. Since the game has no proper subgames, every Nash equilibrium is also subgame-perfect.

(h) No. Sequential rationality fails at player 1's information set in the middle (where, by Bayes' rule his beliefs must be $\frac{1}{2}$ on each node): player 1 would get a higher payoff by choosing T with probability 1.

5.

(a) Normalize Rachel's von Neumann-Morgenstern utility function by setting $U(z_1) = 1$ and $U(z_4) = U(z_5) = 0$. Thus the expected utility of lottery $L_2 = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ \frac{85}{100} & 0 & 0 & \frac{7}{100} & \frac{8}{100} \end{pmatrix}$ is $\frac{85}{100}$. Since Rachel is indifferent between this lottery and z_2 for sure, $U(z_2) = \frac{85}{100} = \frac{17}{20}$. The expected utility of the lottery $L_3 = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ 0 & \frac{14}{17} & 0 & \frac{2}{17} & \frac{1}{17} \end{pmatrix}$ is $\frac{14}{17}\left(\frac{85}{100}\right) = \frac{7}{10}$. Since Rachel is indifferent between this lottery and z_3 for sure, $U(z_3) = \frac{7}{10}$.

(b) **STEP 1**: determine if Rachel would accept a loan at the rate r_L. Obtaining such a loan is playing the lottery $L_L = \begin{pmatrix} z_1 & z_4 \\ 1-p_L & p_L \end{pmatrix}$ whose expected utility is $EU(L_L) = (1-p_L)1 + p_L 0 = 1 - p_L$. If this is greater than $U(z_3) = 0.7$, that is, if $p_L \leq 0.3$ Rachel will apply for the loan, otherwise she won't.

STEP 2: calculate Ross's expected profits with rate r_L. For Ross this is the lottery $\begin{pmatrix} r_L X & -X \\ 1-p_L & p_L \end{pmatrix}$ whose expected value is $X\left[r_L - p_L(1+r_L)\right] = X\left[(1+r_L)(1-p_L) - 1\right]$. This is non-negative iff $p_L \leq \dfrac{r_L}{1+r_L}$.

STEP 3: determine if Rachel would apply for a loan at the rate r_H. Obtaining a loan at the higher rate r_H is playing the lottery $L_H = \begin{pmatrix} z_2 & z_5 \\ 1-p_H & p_H \end{pmatrix}$ whose expected utility is

$EU(L_H) = (1-p_H)U(z_2) + p_H 0 = (1-p_H)\left(\frac{85}{100}\right)$. If this is less than $U(z_3) = 0.7$ Rachel would **not** accept a loan at rate r_H. This is the case if and only if $p_H > \dfrac{3}{17} = 0.176$.

Thus we have a first set of sufficient conditions:

$$\boxed{p_L \leq \min\left\{0.3, \frac{r_L}{1+r_L}\right\} \text{ and } p_H > \frac{3}{17}} \qquad (1)$$

(*in this region of the parameter space Rachel would only accept a loan at the rate* r_L *and Ross makes non-negative profits from such a loan*).

STEP 4. Now we have to consider the case where Rachel would accept either type of loan, which happens when $p_L \leq 0.3$ and $p_H \leq \dfrac{3}{17}$. In this case Ross would offer her a loan at the rate r_L if and only if

$X\left[r_L - p_L(1+r_L)\right] \geq X\left[r_H - p_H(1+r_H)\right]$ and $p_L \leq \dfrac{r_L}{1+r_L}$.

Thus we have found a second region:

$$\boxed{p_L \leq \min\left\{0.3, \frac{r_L}{1+r_L}\right\},\ p_H \leq \frac{3}{17} \text{ and } r_L - p_L(1+r_L) \geq r_H - p_H(1+r_H)} \qquad (2)$$

(*in this region both contracts are acceptable to Rachel but the low-rate contract yields a higher expected profit*).

(c) If $r_L = 10\%$, $r_H = 20\%$, $p_L = 0.10$ and $p_H = 0.15$ we are neither in region (1) (because $p_H < \dfrac{3}{17} = 0.176$) nor in region 2 because (because $p_L > \dfrac{r_L}{1+r_L} = 0.091$). For these values of the parameters Rachel would accept both loans but a higher-rate loan is more profitable for Ross and it yields a positive profit of 0.02X. Thus they would sign a loan contract at the rate r_H = 20%.

(d) First of all note that the normalized von Neumann-Morgenstern utility function of a type H is such that $U(z_1)=U(z_2)=1,\ U(z_3)=0$ and $0<U(z_4)=U(z_5)<1$. Thus the expected utility of both lottery $L_L=\begin{pmatrix} z_1 & z_4 \\ 1-q_L & q_L \end{pmatrix}$ and lottery $L_H=\begin{pmatrix} z_2 & z_5 \\ 1-q_H & q_H \end{pmatrix}$ is positive and hence greater than $U(z_3)=0$. Thus type H will apply for a loan, no matter what the rate. If both contracts are offered, then the expected utility of an r_L loan is $E(L_L) = \left(1-\frac{3}{25}\right)+\frac{3}{25}U(z_4)=\frac{22}{25}+\frac{3}{25}U(z_4)$ while the expected utility of an r_H loan is $E(L_H)$ $=\left(1-\frac{1}{5}\right)+\frac{1}{5}U(z_5)=\frac{20}{25}+\frac{5}{25}U(z_5)$. Since $0<U(z_4)=U(z_5)<1$ we have that $E(L_L) > E(L_H)$ and thus the H types would apply for an r_L loan.

If $n_L = n_H = 0$, then expected profits are zero.

If $n_L > 0$ and $n_H = 0$, then both type H and type L will apply (since $p_L < 0.3$: see Step 1 above). The bank's expected profit from a type L is $1000\frac{23}{25} - 10000\frac{2}{25} = 120$ while the expected profit from a type H is $1000\frac{22}{25} - 10000\frac{3}{25} = -320$. The probability of the loan being taken by a type $i \in \{L,H\}$ is $\frac{N_i}{N}$ where $N = N_L + N_H$. Thus the expected profit from a single loan at rate r_L is $\pi_L = 120\frac{N_L}{N} - 320\frac{N_H}{N}$ and total expected profit are $\left(120\frac{N_L}{N}-320\frac{N_H}{N}\right)n_L$

If $n_L = 0$ and $n_H > 0$, then only type H will apply (since $p_H = 0.2 > 0.176$: see Step 4 above), giving rise to the lottery $\begin{pmatrix} 2{,}000 & -10{,}000 \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$ whose expected value is -400. Thus expected profits from each one of these loans is $\pi_H = -400$ and total expected profits are $-400n_H$

If $n_L > 0$ and $n_H > 0$, then both types apply for the r_L loan and total expected profits are as in the case where $n_L > 0$ and $n_H = 0$, namely $\left(120\frac{N_L}{N}-320\frac{N_H}{N}\right)n_L$.[1]

Thus Ross optimal decision is

$$\begin{cases} \text{offer } m \text{ loans at the lower rate } r_L, \text{ if } \left(120\dfrac{N_L}{N}-320\dfrac{N_H}{N}\right)>0 \\ \text{offer no loans at all, otherwise} \end{cases}$$

(e) If $N_L = 5{,}000$ and $N_H = 1{,}000$ then $\left(120\frac{N_L}{N}-320\frac{N_H}{N}\right)=46.67$ and thus Ross will offer m loans at the lower rate r_L.

[1] To be more sophisticated, we can consider several cases. Case 1: every potential borrower is allowed to submit only one application. In this case everybody submits an application for an r_L loan and thus expected profits are as indicated above. Case 2: potential borrowers can apply to both types of loans. In this case the H types would apply for both types of loans and the L types would apply only for an r_L loan. If applications identify applicants by name, then Ross will be able to correctly sort out types and would give an r_L loan to those who only applied for that and know that an r_H loan would be given to an H type and thus expect a profit of $120n_L - 400n_H$; this expression is maximized by setting $n_H = 0$. If applications are initially anonymous, then Ross's best course of action would be to first choose n_H borrowers (knowing that they are H types) and then choose n_L borrowers from the remaining pool, for an expected profit of $\pi = n_L\left(120\dfrac{N_L}{N} - 320\dfrac{N_H - n_H}{N}\right) - 400n_H$. This expression is maximized when n_H = 0, in which case we are back to the expression given above.

(d) First of all note that the normalized von Neumann-Morgenstern utility function of a type *H* is such that $U(z_1) = U(z_2) = 1,\ U(z_3) = 0$ and $0 < U(z_4) = U(z_5) < 1$. Thus the expected utility of both lottery $L_L = \begin{pmatrix} z_1 & z_4 \\ 1-q_L & q_L \end{pmatrix}$ and lottery $L_H = \begin{pmatrix} z_2 & z_5 \\ 1-q_H & q_H \end{pmatrix}$ is positive and hence greater than $U(z_3) = 0$. Thus type H will apply for a loan, no matter what the rate. If both contracts are offered, then the expected utility of an r_L loan is $E(L_L) = \left(1-\frac{3}{25}\right)+\frac{3}{25}U(z_4) = \frac{22}{25}+\frac{3}{25}U(z_4)$ while the expected utility of an r_H loan is $E(L_H) = \left(1-\frac{1}{5}\right)+\frac{1}{5}U(z_5) = \frac{20}{25}+\frac{5}{25}U(z_5)$. Since $0 < U(z_4) = U(z_5) < 1$ we have that $E(L_L) > E(L_H)$ and thus the H types would apply for an r_L loan.

If $n_L = n_H = 0$, then expected profits are zero.

If $n_L > 0$ and $n_H = 0$, then both type *H* and type *L* will apply (since $p_L < 0.3$: see Step 1 above). The bank's expected profit from a type *L* is $1000\frac{23}{25} - 10000\frac{2}{25} = 120$ while the expected profit from a type *H* is $1000\frac{22}{25} - 10000\frac{3}{25} = -320$. The probability of the loan being taken by a type $i \in \{L, H\}$ is $\frac{N_i}{N}$ where $N = N_L + N_H$. Thus the expected profit from a single loan at rate r_L is $\pi_L = 120\frac{N_L}{N} - 320\frac{N_H}{N}$ and total expected profit are $\left(120\frac{N_L}{N} - 320\frac{N_H}{N}\right)n_L$

If $n_L = 0$ and $n_H > 0$, then only type *H* will apply (since $p_H = 0.2 > 0.176$: see Step 4 above), giving rise to the lottery $\begin{pmatrix} 2{,}000 & -10{,}000 \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$ whose expected value is -400. Thus expected profits from each one of these loans is $\pi_H = -400$ and total expected profits are $-400n_H$

If $n_L > 0$ and $n_H > 0$, then both types apply for the r_L loan and total expected profits are as in the case where $n_L > 0$ and $n_H = 0$, namely $\left(120\frac{N_L}{N} - 320\frac{N_H}{N}\right)n_L$.[1]

Thus Ross optimal decision is

$$\begin{cases} \text{offer } m \text{ loans at the lower rate } r_L, \ \text{if } \left(120\frac{N_L}{N} - 320\frac{N_H}{N}\right) > 0 \\ \text{offer no loans at all, otherwise} \end{cases}$$

(e) If $N_L = 5{,}000$ and $N_H = 1{,}000$ then $\left(120\frac{N_L}{N} - 320\frac{N_H}{N}\right) = 46.67$ and thus Ross will offer *m* loans at the lower rate r_L.

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[1] If some H types, being rationed out of an r_L loan apply for an r_H one, then Ross's expected profits become $\pi = n_L\left(120\frac{N_L}{N} - 320\frac{N_H - n_H}{N}\right) - 400n_H$. This expression is maximized when $n_H = 0$.