Prelim 2018 Retake Solutions (Caramp)

1 The Choice of Project

1. If effort is observable then entrepreneurs will never shirk\(^1\)

Then

- Expected return of project 1: \(p_H R\)
- Expected return of project 2: \(q_H R\)
- Denote \(R_j^i\) the return of project \(j\) for agent \(i\) in case of success. Then for any \(N\)

\[
\begin{align*}
p_H R_1^E &= I - N \\
q_H R_2^E &= I - N
\end{align*}
\]

Since \(p_H > q_H\), then

\[
p_H R_1^E = p_H R - p_H R_1^F = p_H R - (I - N) > q_H R - (I - N) = q_H R - q_H R_2^F = q_H R_2^E
\]

Which establishes that project 2 is strictly dominated by project 1 when there is no information problem.

2. For project 1 the constraints are:
   - The resource constraint:
   \[
   R = R_1^F + R_1^E
   \]
   - Financier participation constraint:
   \[
   p_H R_1^F \geq I - N
   \]
   - Entrepreneur incentive compatibility:
   \[
   p_H R_1^E \geq p_L R_1^E + B \Rightarrow R_1^E \geq \frac{B}{\Delta p}
   \]

For project 2 the constraints are:
   - The resource constraint:
   \[
   R = R_2^F + R_2^E
   \]

\(^1\)We assume that if they shirk, this is verifiable by a court and the entrepreneur can be punished so that it won’t be optimal for her to exert low effort/shirk
• Financier participation constraint:

\[ q_H R_2^F \geq I - N \]

• Entrepreneur incentive compatibility:

\[ q_H R_2^E \geq q_L R_2^E + b \Rightarrow R_2^E \geq \frac{b}{\Delta q} \]

Conditional on undertaking project 1, the contracting problem the entrepreneur solves is:

\[
\begin{align*}
\max_{\{R_1^F, R_1^E\}} & \quad p_H R_1^E \\
\text{subject to} & \quad R = R_1^F + R_1^E \\
& \quad p_H R_1^F \geq I - N \\
& \quad R_1^E \geq \frac{B}{\Delta p}
\end{align*}
\]

(1)

The objective function can be rewritten as \( p_H (R - R_1^F) \), then the incentive compatibility constraint of financiers will always bind in the optimal contract. Therefore the objective function can be rewritten as

\[ p_H R_1^E = p_H (R - R_1^F) = p_H R + (N - I) \]

which is strictly positive since the project has positive NPV. Thus, E can undertake project 1 if and only if the IC constraint is satisfied

\[ I - N \leq p_H \left( R - \frac{B}{\Delta p} \right) \Leftrightarrow N \geq I - p_H \rho_1 \equiv \bar{N}_p \]

Similar calculations show that the project is feasible for E if and only if \( N \geq I - q_H \rho_2 \equiv \bar{N}_q \).

3. Note that \( \bar{N}_q < \bar{N}_p \) since \( p_H \rho_1 < q_H \rho_2 \) by assumption.

Thus, we can partition \([0, 1]\) into three regions:

- \([0, \bar{N}_q]\) where \( N \) is not enough to invest in any project. Then the optimal policy in \([0, \bar{N}_q]\) is to not invest
- \([\bar{N}_q, \bar{N}_p]\) where \( N \) is enough only to invest in project 2. Since project 2 has positive NPV, E chooses to undertake project 2 in this region
- \([\bar{N}_p, 1]\) where \( N \) is enough to invest in project 1 and 2. The optimal choice is to invest in project 1 since by assumption \( p_H > q_H \), which implies:

\[ p_H R_1^E = p_H R - (I - N) > q_H R - (I - N) = q_H R_2^E \]

2
Project 2 is worse in terms of payoffs but better in terms of incentives. Intuitively the contract is doing 3 things: paying the financier, paying the entrepreneur and making sure incentives are aligned (prevent misbehavior). Project 2 pledgeable output is higher, which implies that is less costly for the contract to prevent misbehavior, and less "skin in the game" is needed. Hence under asymmetric information there is a role for project 2 (remember that in point 1, there was no role for project 2 since there was no information problem, hence the only dimension in which project 2 was better than project 1 was not relevant).

4. Assume that now, financiers cannot see the project E chooses.

If $N \geq \bar{N}_p$, the contract designed for project 1 imply that if project 1 is chosen the entrepreneur will behave (not shirk). The new source of potential misbehavior is choosing to pursue project 2 when funding for project 1 was provided by the financiers. But this is not privately optimal for the entrepreneur since $p_H R + (N - 1) > q_H R + (N - I)$. This means, when $N \geq \bar{N}_p$, financiers know that entrepreneurs will undertake project 1 without the need of effectively observing them.

If $N \in [\bar{N}_q, \bar{N}_p)$, by what we discussed previously financiers will only want to give funds to entrepreneurs if it is privately optimal for entrepreneurs to pursue project 2 conditional on getting the loan from financiers and they will ask for $R^*_F = \frac{1-N}{q_H}$.

If entrepreneurs have incentives to deviate and choose project 1 and shirk, then financiers will not lend to them. If E undertakes project 2 and doesn’t shirk, she gets

$$q_H \left( R - \frac{I - N}{q_H} \right)$$

If after signing the contract and getting the funding she decides to undertake project 1, she gets

$$p_H \left( R - \frac{I - N}{p_H} \right) > q_H \left( R - \frac{I - N}{q_H} \right)$$

That is, she wants to deviate and undertake project 1. However, when $N < \bar{N}_p$, we know that

$$p_H \left( R - \frac{I - N}{p_H} \right) < p_L \left( R - \frac{I - N}{p_H} \right) + B$$

hence

$$B > (p_H - p_L) \left( R - \frac{I - N}{p_H} \right) > (p_H - p_L) \left( R - \frac{I - N}{q_H} \right)$$

which implies

$$p_H \left( R - \frac{I - N}{q_H} \right) < p_L \left( R - \frac{I - N}{q_H} \right) + B$$

and therefore

$$q_H \left( R - \frac{I - N}{q_H} \right) < p_H \left( R - \frac{I - N}{q_H} \right) < p_L \left( R - \frac{I - N}{q_H} \right) + B$$

which implies that when $N \in [\bar{N}_q, \bar{N}_p)$, if E gets funding, she will undertake project 1 and shirk. But we know that in that case the project has negative NPV, so the financiers will not lend to Es with this level of wealth.
Now we assume:

\[ p_H R - c > q_H R > I \]

Hence it is more profitable to pay \( c \) and implement the project 1 rather than not paying \( c \) and implementing project 2.

Note that \( R - \frac{b}{\Delta p} \) is the pledgable income of the monitored project, \( p_H \left( R - \frac{b}{\Delta p} \right) \) is the maximum compensation that a financier can get.

There is going to be a level of \( N \) for which financing the monitored project is possible, call it \( \tilde{N} \):

\[ p_H \left( R - \frac{b}{\Delta p} \right) = I - \tilde{N} + c \]

\[ \tilde{N} = I + c - p_H \hat{\rho}_1 \]

Monitoring is useful only if \( \tilde{N} < \overline{N}_p \) (that is, it allows some “low” wealth Es to get funding for project 1). Formally:

\[ \tilde{N} < \overline{N}_p \iff p_H \left( R - \frac{b}{\Delta p} \right) - c > p_H \left( R - \frac{B}{\Delta p} \right) \]

\[ p_H \frac{B - b}{\Delta p} > c \]

Note that project 2 is going to be implemented if the required skin in the game is smaller than under project 1 monitored. Hence, we need to compare \( \tilde{N}_q = I + \frac{b}{\Delta q} - R \) with \( \tilde{N} = I + p_H \frac{b}{\Delta p} - (p_H R - c) \)

\[ \tilde{N}_q < \tilde{N} \iff I - q_H \left( R - \frac{b}{\Delta q} \right) < I - p_H \left( R - \frac{b}{\Delta q} \right) + c \]

Using \( \Delta p = \Delta q \)

\[ c > (p_H - q_H) \left( R - \frac{b}{\Delta p} \right) \]

Project 2 will be undertaken by entrepreneurs with net worth \( N \in [\tilde{N}_q, \tilde{N}] \)
2 Bubbles

1. Given prices, the problem of the agent is:

\[
\max_{\{c_{t+1}^t, m_t^t\}} \frac{(c_{t+1}^t)^{1-\sigma}}{1-\sigma} + \frac{(c_{t+1}^t)^{1-\sigma}}{1-\sigma}
\]

subject to

\[
c_t^t + p_t m_t^t \leq w_1 \quad (\lambda_t^t)
\]
\[
c_{t+1}^t \leq w_2 + p_{t+1} m_t^t \quad (\lambda_{t+1}^t)
\]

Since the utility function satisfies inada conditions, consumption will be strictly positive, hence the FOC characterizing the solution of the agents problem are:

\[
(c_t^t)^{-\sigma} = \lambda_t^t
\]
\[
(c_{t+1}^t)^{-\sigma} = \lambda_{t+1}^t
\]
\[
\lambda_t^t p_t = \lambda_{t+1}^t p_{t+1}
\]

2. An equilibrium for this economy is sequence of prices \(\{p_t\}_{t=0}^\infty\) and allocations of consumption \(\{c_t, c_{t+1}\}_{t=1}^\infty\) and money \(\{m_t\}_{t=0}^\infty\) such that:

- \(\forall \ t \geq 1\), given prices \(c_t^t, c_{t+1}^t, m_t\) solves the problem of the agent born at \(t\)
- \(c_0\) solves the problem of the initial generation
- Markets clear:

\[
M = m_t
\]
\[
c_t^t + c_{t-1}^t = w_1 + w_2
\]

Equilibrium is characterized by the following system of equations:

\[
(c_t^t)^{-\sigma} = \lambda_t^t
\]
\[
(c_{t+1}^t)^{-\sigma} = \lambda_{t+1}^t
\]
\[
\lambda_t^t p_t = \lambda_{t+1}^t p_{t+1}
\]
\[
c_t^t + p_t m_t = w_1
\]
\[
c_{t+1}^t = w_2 + p_{t+1} m_t
\]
\[
M = m_t
\]
\[
c_t^t + c_{t-1}^t = w_1 + w_2
\]

3. Before guessing, let’s work the equilibrium characterization:

Using the optimality conditions:

\[
(c_t^t)^{-\sigma} p_t = (c_{t+1}^t)^{-\sigma} p_{t+1}
\]
And the budget constraint:

\[(w_1 - p_t m_t)^{-\sigma} p_t = (w_2 + p_{t+1} m_t)^{-\sigma} p_{t+1}\]

Combined with market clearing:

\[(w_1 - p_t M)^{-\sigma} p_t = (w_2 + p_{t+1} M)^{-\sigma} p_{t+1}\]

Now, guess \(p_t = p_{t+1} = p > 0\), then:

\[(w_1 - pM)^{-\sigma} p = (w_2 + pM)^{-\sigma} p\]

\[w_1 - w_2 = 2pM\]

\[p = \frac{w_1 - w_2}{2M}\]

This verify our guess, \(p\) does not depend on \(t\) and by assumption \(w_1 > w_2\), hence \(p > 0\).

4. **Claim 1.** If \(p_t = 0\) then \(p_{t+k} = 0 \forall k \geq 1\)

**Proof.** Let \(p_t = 0\) and assume to the contrary that \(p_{t+1} > 0\). Then at today’s prices of \(p_t = 0\) we would have excess demand since we can get money for free today and then trade it for goods tomorrow. Then for \(p_t = 0\) to be an equilibrium, agent’s must believe also that \(p_{t+1} = 0\).

What is more, assume \(p_{t+k} > 0\) from \(t\) point’s of view, then it must be that \(p_{t+k-1} > 0\) by our previous argument... and iterating we would get that \(p_t > 0\). Hence \(p_t = 0\), imposes \(p_{t+k} = 0 \forall k \geq 1\).  

The previous claim says, once the bubble burst it burst forever. Hence there are two possible cases \(p_t = 0\) that implies that autarky is the only equilibrium allocation or \(p_t > 0\). We proceed to characterize the second case.

Let \(p_t > 0\), denote \(c^t_{1+t, a}, c^t_{1+t, b}\) the consumption next period when the bubble burst and continues respectively. Therefore, given the agents beliefs \(1 - \pi\) that the bubble will bursts the agent solves:

\[
\begin{align*}
\max_{\{c^t_{1+t}, c^t_{1+t, a}, m_t\}} & \quad \frac{(c^t_{1+t})^{1-\sigma}}{1-\sigma} + \pi \frac{(c^t_{1+t, b})^{1-\sigma}}{1-\sigma} + (1-\pi) \frac{(c^t_{1+t+1, a})^{1-\sigma}}{1-\sigma} \\
\text{subject to} & \quad c^t_{1+t} + p_t m_t \leq w_1 \quad (\lambda^t_1) \\
& \quad c^t_{1+t+1, b} \leq w_2 + p_{t+1} m_t \quad (\lambda^t_{t+1, b}) \\
& \quad c^t_{1+t, a} \leq w_2 \quad (\lambda^t_{t+1, a}) 
\end{align*}
\]
Inada and monotonicity imply:

\[ c^i_t = w_1 - p_t m_t \]
\[ c^i_{t+1,b} = w_2 + p_{t+1} m_t \]
\[ c^i_{t+1,a} = w_2 \]

Therefore we can rewrite the agent’s program as:

\[
\max_{\{m_t\}} \frac{(w_1 - p_t m_t)^{1-\sigma}}{1 - \sigma} + \pi \frac{(w_2 + p_{t+1} m_t)^{1-\sigma}}{1 - \sigma} + (1 - \pi) \frac{(w_2)^{1-\sigma}}{1 - \sigma} \tag{6}
\]

\[
p_t (w_1 - p_t m_t)^{-\sigma} = \pi p_{t+1} (w_2 + p_{t+1} m_t)^{-\sigma} \tag{7}
\]

Guess \( p_t = p_{t+1} = p \), then using market clearing \( m_t = M \) we get:

\[
(w_1 - pM)^{-\sigma} = \pi (w_2 + pM)^{-\sigma}
\]

\[
(w_2 + pM) = \pi \frac{i}{\pi} (w_1 - pM)
\]

\[
p = \frac{\pi \frac{i}{\pi} w_1 - w_2}{(\pi \frac{i}{\pi} + 1)M}
\]

\[
\frac{\Delta p}{\Delta \pi} = \frac{\pi \pi \frac{i}{\pi} (w_1 - w_2)}{\sigma (\pi \frac{i}{\pi} + 1)^2 M} > 0
\]

Where we used \( w_1 > w_2 \)

5. With probability \( \pi_t \) the bubble survives, from the previous point we know that the bubble is increasing in the belief of the bubble not bursting. Since prices are a function of the beliefs of the newborn generation, we write \( p_{t+1}(\hat{\pi}) \) to denote the next period price of the bubble when beliefs of the \( t + 1 \) generation are \( \hat{\pi} \). Assume that there is some set of beliefs \( \Pi \) and from the contemporaneous period the belief that next period generation will have beliefs \( \hat{\pi} \) is \( q(\hat{\pi}) \) where \( \sum_{\hat{\pi} \in \Pi} q(\hat{\pi}) = 1 \)

\[
\max_{\{c^i_{t+1,1}(\hat{\pi}), m_t\}} \ln c^i_t + \pi_t \sum_{\hat{\pi} \in \Pi} q(\hat{\pi}) \ln c^i_{t+1}(\hat{\pi})
\]

subject to

\[
c^i_t + p_t m_t \leq w_1
\]

\[
c^i_{t+1}(\hat{\pi}) \leq p_{t+1}(\hat{\pi}) m_t \quad \forall \hat{\pi} \in \Pi
\]
As noted we can rewrite the problem as:

\[
\max_{(m_t)} \ln(w_t - p_t m_t) + \pi_t \sum_{\pi \in \Pi} q(\pi) \ln(p_{t+1}(\pi) m_t)
\]

(9)

FOC

\[
p_t(w_t - p_t m_t)^{-1} = \pi_t \sum_{\pi \in \Pi} q(\pi) p_{t+1}(\pi) (p_{t+1}(\pi) m_t)^{-1}
\]

(10)

Market Clearing:

\[
p_t(w_t - p_t M)^{-1} = \pi_t \sum_{\pi \in \Pi} q(\pi) p_{t+1}(\pi) (p_{t+1}(\pi) M)^{-1}
\]

Using \(\sum_{\pi \in \Pi} q(\pi) = 1\)

\[
p_t = \pi_t \frac{w_t}{M} - \pi_t \frac{p_t M}{M}
\]

\[
p_t = \frac{\pi_t}{1 + \pi_t} \frac{w_t}{M}
\]

Note \(\frac{\partial p_t}{\partial \pi_t} = \frac{1}{(1 + \pi_t)^2} \frac{w_t}{M} > 0\), hence when \(\pi_t\) is high/low prices today are high/low. The only driver of the price fluctuations are beliefs.

Let’s consider two levels of beliefs optimists: \(\pi^H\) and pessimists \(\pi^L\). \(\pi^H = \pi^L + \Delta \pi\) where \(\Delta \pi > 0\) and \(\pi^H \leq 1\)

Consider any arbitrary sequence of beliefs \((\pi_1, \ldots, \pi_t, \ldots)\). Whenever \(\pi_t = \pi^H\) we have:

- Equilibrium prices \(p_H = \frac{\pi_H}{1 + \pi_H} \frac{w_t}{M}\)

- Consumption allocations of generation \(t\) given by:

\[
c_{t,H} = w_t - p_H M = \frac{1}{1 + \pi_H} w_t
\]

Now it’s key the beliefs of generation \(t + 1\) that are going to buy the money:

- If \(\pi_{t+1} = \pi^H\)

\[
c_{t+1,H} = \frac{\pi_H}{1 + \pi_H} w_t
\]

- If \(\pi_{t+1} = \pi^L\)

\[
c_{t+1,L} = p_L M = \frac{\pi_L}{1 + \pi_L} w_t
\]

Now markets at \(t + 1\) will clear since
If generation $t + 1$ have beliefs $\pi^H$

$$c_{t+1, H} + c_{t+1, H} = \frac{\pi_H}{1 + \pi_H} w_1 + \frac{1}{1 + \pi_H} w_1 = w_1$$

If generation $t + 1$ have beliefs $\pi^L$

$$c_{t+1, H} + c_{t+1, L} = \frac{\pi_L}{1 + \pi_L} w_1 + \frac{1}{1 + \pi_L} w_1 = w_1$$

Analogous argument show us that if $\pi_t = \pi^L$ we get individual rationality and aggregate consistency.

$$c_{t+1, H} + c_{t+1, H} = \frac{\pi_H}{1 + \pi_H} w_1 + \frac{1}{1 + \pi_H} w_1 = w_1$$