ANSWER KEY

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PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

Directions: Answer all questions. Feel free to impose additional structure on the problems below, but please state your assumptions clearly. Point totals for each question are given in parentheses.

1. (10) In his highly acclaimed book, *Capital in the Twenty-First Century*, Thomas Piketty claims that, if the rate of return on capital exceeds the per-capita growth rate of the economy, then capital's share of income will grow. Is this statement consistent with the Ramsey-Cass-Koopmans optimal growth model as studied in class?

   ANSWER: The optimal growth model is characterized by two equations: the law of motion of capital and the Keynes-Ramsey condition. In addition, our analysis was done following that of the Solow model in which we had assumed a Cobb-Douglas production function which is consistent with constant factor shares. In particular, the model was solved under the maintained assumption of a constant share of capital. Then the Keynes-Ramsey condtion is the relevant condition for this question and states that:

   $$r = \delta + \gamma g$$

   where r is the rate of return on capital, δ is the depreciation rate, γ is the elasticity of marginal utility (i.e. relative risk aversion or the inverse of the intertermporal elasiticity of substitution), and g is the rate of per-capita consumption growth which is equal to per-capita output growth (equal to consumption growth). Clearly for $\delta > 0$ and $\gamma \geq 1$ then the rate of return on capital will exceed g. Hence $r > g$ is consistent with constant factor shares in the Cass-Koopmans model. (Clearly if γ is sufficiently small, the $r < g$ is possible but this too is consistent with a constant capital share by construction.) Piketty must be using a different model than what we studied in class - critically, his model assumes some form of heterogeneity.

2. (20) An economy is populated by identical, infinitely-lived agents (there is no population growth) that maximize the present discounted value of lifetime utility given by

$$\sum_{t=0}^{\infty} \beta^t \left[\ln c_t + A\left(1 - h_t - s_t\right)\right]; \; \beta \in (0,1), \; A > 0$$

where c_t denotes consumption, h_t is time spent in work activity and s_t is time spent in acquiring consumption goods, i.e. s_t denotes shopping time. (Note that the above functional form implies that utility is linear in leisure.) It is assumed that shopping time is an increasing function of consumption but a decreasing function of real balances. That is, holding money reduces shopping time. It is assumed that this function is linear in the level consumption and the log of real balances:

$$s_t = c_t - \ln\left(\frac{M_t}{P_t}\right)$$

where M_t is money chosen at time t, and P_t is the nominal price level. At the beginning of the period, agents hold money from the previous period (M_{t-1}) and also receive new money from the government which is distributed as a lump sum transfer. Agents produce output (y_t) using a logarithmic production function with labor as the only input: $y_t = \ln h_t$. This is sold and the proceeds, along with the beginning-of-period money and monetary transfer, are used to purchase consumption and new money. The money supply is growing at the constant rate μ implying $\bar{M}_{t+1} = \bar{M}_t (1 + \mu)$ where $\bar{M}$ is used to denote the aggregate money stock. It is assumed that $\mu > 0$. Given this environment, do the following:

(a) Express the agent's maximization problem as a dynamic programming problem and derive and interpret the associated necessary conditions.

(b) Define a steady-state equilibrium in this economy.

(c) How do changes in μ affect steady-state consumption, labor, leisure, and real balances? In particular, what is the relationship between money growth and utility (in steady-state)?

(d) Suppose that one period nominal bonds are introduced into this economy. These bonds cost \$1 at time t and return \$ $(1 + n_t)$ in period $t + 1$. Determine the steady-state nominal interest rate in this economy. (You can work directly from the Euler equation associated with one-period nominal bonds. You do NOT need to write down the dynamic programming problem.)

(e) Now suppose that the monetary transfer in this economy is distributed as an interest payment on beginning-of-period money holdings with the interest rate equal to the nominal interest rate determined in part (d). Would this affect the relationship between money growth (μ), steady-state real balances and utility as described in part (c)? Explain. (Again: you may work directly from the Euler equation for real balances in this new setting; it is NOT required to write down the new dynamic programming problem.)

ANSWER: The state variable in this economy is beginning-of-period real balances so that the Bellman equation (written as a Lagrangian) is:

$$V\left(\frac{M_{t-1}}{P_t}\right) = \max_{c_t, M_t, h_t} \left\{ \begin{array}{c} \ln c_t + A\left(1 - h_t - c_t + \ln\left(\frac{M_t}{P_t}\right)\right) + \beta V\left(\frac{M_t}{P_{t+1}}\right) + \\ \lambda_t \left(\frac{M_{t-1}}{P_t} + \frac{T_t}{P_t} + \ln h_t - c_t - \frac{M_t}{P_t}\right) \end{array} \right\}$$

where T_t is the lump-sum monetary transfer and I have used the function for s_t to express that in terms of consumption and real balances. The necessary conditions (after applying the Envelope Theorem) are:

$$\lambda_t = \frac{1}{c_t} - A \tag{1}$$

$$\lambda_t \frac{1}{P_t} = A \frac{1}{M_t} + \beta \lambda_{t+1} \frac{1}{P_{t+1}} \tag{2}$$

$$A = \lambda_t \frac{1}{h_t} \tag{3}$$

The first condtion (eq.(1) shows that the marginal utility of wealth (i.e. the Lagrange multiplier) is not equal to the MU of consumption because of the shopping time costs (represented by the constant MU of leisure, A). Eq. (2) is the Euler eq. associated with real balances and has the traditional MC=MB condition. Note that the first term on the RHS shows the utility gain from real balances which reduces shopping time. Eq. (3) is the labor-leisure trade-off where the MRS between leisure and wealth is equal to the MPL.

A steady-state in this economy is characterized by constant labor (h_{ss}) and real balances (m_{ss}) such that the necessary conditions are satisfied when evaluated at these quantities and markets clear. Note that this implies steady state consumption is given by $c_{ss} = \ln h_{ss}$ while $M_t = M_{t-1} + T_t = M_{t-1}(1+\mu)$. Furthermore, constant real balances imply $(P_{t+1}/P_t) = (1+\mu)$. To solve for steady-state, note that eqs. (1) and (3) imply that steady-state labor is not affected by money growth. That is, solving for h_{ss} (and using the goods market equilibrium condition) yields:

$$\ln h_{ss}(1+h_{ss}) = \frac{1}{A}$$

Note that this result implies that the Lagrange multiplier in steady-state, λ_{ss}, is also independent of the monetary growth rate (see eq. (3)).

Turning to real balances, using eq. (2) yields:

$$m_{ss} = \frac{A}{\lambda_{ss}}\left(1 - \frac{\beta}{1+\mu}\right)^{-1} \tag{4}$$

where λ_{ss} is the steady-state level of the Lagrange multiplier which, as noted above, is independent of μ. Note that, consequently, the right-hand side is decreasing in μ : since the monetary growth rate is equal to the inflation rate in steady-state, increases in μ result in a higher inflation tax on money (the only asset). Hence, the real value of money falls which, from the shopping time function, means agents spend more time in shopping activities. Since h_{ss} is constant (implying c_{ss} is constant), this means less time is spent in leisure. This carries the further implication that utility falls. So money is not superneutral in this economy despite the fact that consumption is not affected by money growth.

If one period nominal bonds as described are introduced into this economy, the associated Euler equation is:

$$\lambda_t \frac{1}{P_t} = \beta \lambda_{t+1} \frac{(1+n_t)}{P_{t+1}}$$

Evaluating this at steady-state implies:

$$(1+n_{ss})^{-1} = \beta \frac{P_t}{P_{t+1}} = \beta(1+\mu)^{-1} \tag{5}$$

Note that this is just the Fisher relationship (where the steady-state real interest rate is determined by β).

If money is distributed as an interest rate on money holdings, the Euler equation associated with money becomes (modifying eq. (2):

$$\lambda_t \frac{1}{P_t} = A\frac{1}{M_t} + \beta \lambda_{t+1} \frac{(1+n_t)}{P_{t+1}} \tag{6}$$

In steady-state, the implication is that:

$$m_{ss} = \frac{A}{\lambda_{ss}}\left(1 - \frac{\beta(1+n_{ss})}{1+\mu}\right)^{-1} = \frac{A}{\lambda_{ss}}(1-1)^{-1}$$

The right-hand side is infinite implying an infinite level of real balances - we have attained the optimal quantity of money. (I am sidestepping important issues of whether this equilibrium is well-defined.) The important point is that when new money is issued as an interest payment (determined by the Fisher relationship), real balances are not affected by money growth and the economy is at the Pareto optimum. Note: As can be seen from eq. (5) this implies that the money given to households in the form of an interest payment implies their money balances are growing at a rate greater than μ. Hence, there must be lump sum taxes so that the growth of the aggregate money stock in equilibrium is indeed μ. That is $T_t = M_{t-1}(1+\mu)\frac{\beta-1}{\beta}$. It was not necessary to discuss this aspect of equilibrium in your answer.

3. (20) Consider a standard RBC framework with Cobb-Douglas technology and a 100% capital depreciation rate. Specifically, output is given by:

$$y_t = k_t^{\alpha} h_t^{1-\alpha}$$

Uncertainty is introduced into the economy via preference shocks which affect the disutility of working. The one-period utility function has the following functional form:

$$U(c_t, h_t) = \ln\left(c_t - \theta_t \frac{h_t^{1+\psi}}{1+\psi}\right)$$

where c_t is consumption and h_t is labor with θ_t the stochastic shock to preferences. It is assumed that θ_t is distributed *i.i.d.* with *c.d.f.* given by $G(\theta)$ and *p.d.f.* given by $g(\theta)$. Each period agents choose labor, consumption and capital in order to maximize expected lifetime utility:

$$E_0\left[\sum_{t=0}^{\infty} \beta^t U(c_t, h_t)\right]$$

(a) Given this environment, define a recursive competitive equilibrium for this economy.

(b) Solve the model as a Social Planner problem. In particular, determine the functions that define equilibrium consumption, labor and investment. (Hint: First examine the conditions describing the labor market and then use these insights to solve for the goods market equilibrium.)

(c) Describe the business cycle properties implied by this model. Which are consistent with observation? Many RBC modelers prefer these preferences to those found in a Hansen-Rogerson type indivisible labor model. Why?

ANSWER: The aggregate state vector for this problem is $s_t = (k_t, \theta_t)$ while, for the individual, there is the additional state variable of the individual capital stock. A recursive competitive equilibrium is defined by a value function given by:

$$V(a_t, s_t) = \max_{c_t, h_t, a_{t+1}} \left\{ \begin{array}{c} \ln\left(c_t - \theta_t \frac{h_t^{1+\psi}}{1+\psi}\right) + \beta E\left[V(a_{t+1}, s_{t+1})\right] \\ +\lambda_t \left(w_t h_t + r_t a_t - c_t - a_{t+1}\right) \end{array} \right\} \tag{7}$$

where E denotes the expectation operator and no time subscript is present since θ_t is *i.i.d.* In the budget constraint, w_t and r_t denote the prices of labor and capital. It is assumed these factor prices are functions of s_t and agents know these functions. The policy functions for optimal labor, consumption and capital also define the RCE and, when evaluated at the equilibrium condition that $a_t = k_t$, must be consistent with equilibrium in the goods market. A definition of an RCE along these lines would be sufficient.

We can appeal to the second welfare theorem and solve for the equivalent social planner problem which is characterized by the Bellman equation:

$$V(s_t) = \max_{c_t, h_t, k_{t+1}} \left\{ \begin{array}{c} \ln\left(c_t - \theta_t \frac{h_t^{1+\psi}}{1+\psi}\right) + \beta E\left[V(s_{t+1})\right] \\ +\lambda_t \left(k_t^{\alpha} h_t^{1-\alpha} - c_t - k_{t+1}\right) \end{array} \right\}$$

The first order conditions are (after using the Envelope Theorem):

$$\theta_t h_t^{\psi} = (1-\alpha)\, k_t^{\alpha} h_t^{-\alpha} \tag{8}$$

$$\frac{1}{\left(c_t - \theta_t \frac{h_t^{1+\psi}}{1+\psi}\right)} = \alpha\beta E\left[\frac{1}{\left(c_{t+1} - \theta_{t+1}\frac{h_{t+1}^{1+\psi}}{1+\psi}\right)} k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha}\right] \tag{9}$$

The first expression is the necessary condition associated with the labor-leisure tradeoff. Note that an immediate implication is that labor supply depends only on the MPL. That is, there is no income effect on labor supply with these preferences. Also note that eq.(8) also provides the optimal labor policy function since it can be solved for h_t as a function of the state variables (k_t, θ_t). That is, the optimal policy function for labor is:

$$h_t = \left[\frac{1-\alpha}{\theta_t} k_t^{\alpha}\right]^{\frac{1}{\psi+\alpha}}$$

Note that this function implies that, ceteris paribus, labor supply is decreasing in θ_t which makes sense since the shock increases the disutility from working.

To solve for the policy functions defining consumption and capital, first multiply both sides of eq. (8) by h_t to yield

$$h_t^{1+\psi} = \frac{(1-\alpha)}{\theta_t} k_t^{\alpha} h_t^{1-\alpha} = \frac{(1-\alpha)}{\theta_t} y_t \tag{10}$$

Use this in eq. (9) :

$$\frac{k_{t+1}}{\left(c_t - \frac{(1-\alpha)}{1+\psi} y_t\right)} = \alpha\beta E\left[\frac{1}{\left(c_{t+1} - \frac{(1-\alpha)}{1+\psi} y_{t+1}\right)} y_{t+1}\right] \tag{11}$$

Note that the preference shock does not enter this expression explicitly - only implicitly through the behavior in output. This suggests that our common assumption that capital is a constant fraction of output might work. Define this fraction as κ and using this conjecture yields:

$$\kappa = \alpha\beta$$

Hence the conjecture is verified. This implies the policy functions are:

$$\begin{aligned} k_{t+1} &= \alpha\beta k_t^{\alpha}\left[\frac{1-\alpha}{\theta_t} k_t^{\alpha}\right]^{\frac{1-\alpha}{\psi+\alpha}} \\ c_t &= [1-\alpha\beta]\, k_t^{\alpha}\left[\frac{1-\alpha}{\theta_t} k_t^{\alpha}\right]^{\frac{1-\alpha}{\psi+\alpha}} \end{aligned}$$

The implication of this result is that, as seen in the data, consumption, labor, and investment are procyclical. But, contrary to what is observed, this model implies that, in percentage terms (i.e express all variables in logs), consumption and investment have the same volatility as output since they are linear in output. As can be seen by taking logs of eq. (10), the volatility of labor relative to GDP is determined by the parameter ψ.

A desirable feature of these preferences, alluded to earlier, is that there is no income effect in labor supply. As a consequence, there is a stronger amplification effect in the model due to labor's response. One can prove this by considering the analog to eq.(8) in the Hansen-Rogerson model - but I leave that to the reader.

Question 4 Answer Key

a) The value functions are as follows:

$$rV_0 = z + \lambda_0 \left[\int_{\underline{w}}^{\overline{w}} \max\{V_0, V_1(x)\} dF(x) - V_0 \right], \tag{1}$$

$$rV_1(w) = w + \lambda_1 \left[\int_{\underline{w}}^{\overline{w}} \max\{V_1(w), V_1(x)\} dF(x) - V_1(w) \right] + \delta[V_0 - V_1(w)]. \tag{2}$$

b) First, get rid of the maximum operator in (2) by writing:

$$rV_1(w) = w + \lambda_1 \left[\int_{\underline{w}}^{w} V_1(w) dF(x) + \int_{w}^{\bar{w}} V_1(x) dF(x) - V_1(w) \right] + \delta[V_0 - V_1(w)].$$

Next, apply total differentiation with respect to w. To do that you will need to apply the Leibniz rule, which we used many times in class. After some algebra you will find that

$$V_1'(w) = \frac{1}{r + \delta + \lambda_1 [1 - F(w)]}. \tag{3}$$

This expression is strictly positive because $F(w)$ is just a probability, hence, all the terms in the expression are strictly positive objects.

c) Given that $V_1'(w) > 0$, and that V_0 is a constant, it is trivial to argue that the worker will have a reservation wage, i.e. there will exist a unique R, satisfying $V_1(R) = V_0$, such that $V_1(w) > V_0$ if and only if $w > R$, so the worker will only accept offers that exceed R.

d) Use the fact that $V_1(R) = V_0$ to equate the left-hand sides of (1) and (2) (evaluated at $w = R$). After some manipulations we obtain

$$R = z + (\lambda_0 - \lambda_1) \int_R^{\bar{w}} [V_1(x) - V_0] F'(x) dx, \tag{4}$$

hence, the parameters a_i are simply given by $a_1 = 1$ and $a_2 = \lambda_0 - \lambda_1$.

e) As I explained in the hint, here the trick is to define $[V_1(x) - V_0]$ as $G(x)$ and apply integration by parts. This will yield

$$R = z + (\lambda_0 - \lambda_1) \left[V_1(\bar{w}) - V_0 - \int_R^{\bar{w}} V_1'(x) F(x) dx \right]. \tag{5}$$

But it is also easy to see that

$$\int_R^{\bar{w}} V_1'(x) dx = V_1(\bar{w}) - V_0.$$

Using this last fact into (5), allows us to write

$$R = z + (\lambda_0 - \lambda_1)\left[\int_R^{\bar{w}} V_1'(x)[1 - F(x)]dx\right].$$

Finally, recall that in part (b) we found a closed form solution for V_1'. Exploiting that formula we can arrive at the final expression which relates R only to the model's parameters. This expression is:

$$R = z + (\lambda_0 - \lambda_1)\left[\int_R^{\bar{w}} \frac{1 - F(x)}{r + \delta + \lambda_1[1 - F(x)]}dx\right].$$

f) Of course, if $\lambda_0 = \lambda_1$, it is easy to see that $R = z$.

g) The measure of agents moving into unemployment is given by $(1 - u)\delta$. The measure of agents moving out of unemployment is given by $u\lambda_0[1 - F(R)]$. Hence, the steady state rate of unemployment is

$$u = \frac{\delta}{\delta + \lambda_0[1 - F(R)]}.$$

Question 5 a) Let ϕ denote the value of money in the current period. The value function for a buyer is

$$W^B(m) = \max_{X,H,\hat{m}} \{U(X) - H + \beta V(\hat{m})\}$$
$$\text{s.t. } X + \phi\hat{m} = H + \phi(m + \mu M),$$

It can be easily verified that, at the optimum, $X = X^*$. Using this fact and replacing H from the budget constraint into W^B yields

$$W^B(m) = U(X^*) - X^* + \phi(m + \mu M) + \max_{\hat{m}} \{-\phi\hat{m} + \beta V(\hat{m})\}. \tag{6}$$

As is standard in this model, the optimal choice of the agent does not depend on the current state (due to quasi-linearity), and so the value function is linear. Also, we can collect all the terms in (6) that do not depend on the state variable and write

$$W^B(m) = \varphi m + \Lambda, \tag{7}$$

where the definition of Λ is obvious.

For the seller things are even simpler, since this agent will never want to buy any money in the Walrasian market. It is very easy to show that

$$W^S(m) = \varphi m + \Lambda^S,$$

where $\Lambda^S \equiv U(X^*) - X^*$.

b) Suppose that a buyer makes the offer. Then she will choose q, d in order to maximize her surplus subject to the participation constraint of the seller. i.e. she will give the seller enough surplus so that she doesn't walk away. Formally, consider a meeting between a buyer who carries m units of money, and a seller who, of course, carries nothing. We need to

$$\max_{d,q} \left\{u(q) + W^B(m - d) - W^B(m)\right\}$$
$$\text{s.t. } -q + W^S(d) - W^S(0) = 0,$$

and subject to $d \leq m$. Using the linearity of these functions the problem simplifies to

$$\max_{d,q} \{u(q) - \phi d\}$$
$$\text{s.t. } -q + \phi d = 0,$$
$$d \leq m.$$

It follows, that the agents should always set $q = q^*$ as long as the buyer carries enough money to buy that amount. Hence, $q(m) = \min\{\phi m, q^*\}$, $d(m) = \min\{m, q^*/\phi\}$.

Now, if it was the seller making the TIOLI offer, we would solve

$$\max_{d,q} \{-q + \phi d\}$$
$$\text{s.t. } u(q) - \phi d = 0,$$
$$d \leq m.$$

Once again, we should have $q = q^*$ as long as the buyer carries enough money to buy that amount, but now (that the seller has all the power) that amount is more expensive. Here, $q(m) = \min\{\tilde{q}(m), q^*\}$, $d(m) = \min\{m, u(q^*)/\phi\}$, where $\tilde{q} \equiv \{q : u(q) = \phi m\}$.

c) As I expliained in the hint, here it is much faster to just guess the objective. Recall that in the baseline model, we had

$$J(\hat{m}) = (-\phi + \beta\hat{\phi})\hat{m} + \beta\sigma\left[u(q(\hat{m})) - \hat{\phi}d(\hat{m})\right].$$

Now the buyer will obtain a benefit only if she gets lucky and turns out to be the one who makes the TIOLI offer (with probability θ), but if this the case, she takes all the surplus. Hence, here the objective is

$$J(\hat{m}) = (-\phi + \beta\hat{\phi})\hat{m} + \beta\sigma\theta\left[u(\hat{\phi}\hat{m}) - \hat{\phi}\hat{m}\right],$$

where I have already exploited the fact that no buyer will ever bring an amount of money greater than the one she needs in order to buy q^* (in simple words, we focus on the binding branch of the bargaining solution).

d) First, take the FOC to obtain

$$\phi = \beta\hat{\phi}\left\{1 + \sigma\theta[u'(\hat{\phi}\hat{m}) - 1]\right\}.$$

We can replace the term $\hat{\phi}\hat{m}$ with z (for real balances), and we can also do the standard trick that helps us get rid of the $\phi, \hat{\phi}$ terms (i.e. multiply both sides with $\hat{M}$). This yields

$$\frac{1 + \mu - \beta}{\beta} = \sigma\theta[u'(z) - 1].$$

But also, recall the Fisher equation, according to which, $1 + i = (1 + r)(1 + \pi) = (1 + \mu)/\beta$. Using this last fact, we can obtain the very simple demand function

$$i = \sigma\theta[u'(z) - 1]. \tag{8}$$

e) Yes, a monetary equilibrium always exists (for finite μ or i), because of the assumption that $u'(0) \to \infty$. This is different than in the model we saw in class with proportional bargaining. In that model, we needed to put an upper bound on inflation to guarantee existence of monetary equilibrium.

f) This was a bit of a tricly question. Equation (8) uniquely describes the equilibrium value of real balances for any value of i. However, the quantity of good that will be bought in the DM depends on the state of the world, i.e. it depends on who makes the TIOLI offer. Hence, strictly speaking, there exist two equilibrium q's, call them q_1 and q_2, where q_1 denotes the quantity of day good that is purchased in the event that the buyer makes the TIOLI offer, and q_2 denotes the quantity of day good that is purchased in the event that the seller makes the TIOLI offer.

By imposing the steady-state equilibrium conditions on the bargaining solution, it follows that $q_1 = z$, and $q_2 \equiv \{q : u(q) = z\}$. Of course, we have $q_1 > q_2$, which makes sense: for any given z that the buyer brought to the bilateral meeting, she will purchase less goods if the seller is making the offer because the good will be more expensive in that case.

It is easy to check that as $i \to 0$, we have $q_1 = z = q^*$, and $q_2 = u^{-1}(q^*)$.

Question 6 a) The Planners problem is indeed much easier, because one only needs to describe the allocations, and not the prices of all commodities (which are infinite sequences). What allows us to use this technique here, is the fact that in this environment both Welfare Theorems hold. So we know that the competitive allocation and the Planners allocation will coincide. After characterizing the Planners allocation, we can construct the whole competitive equilibrium, like we did in class.

b) Using any technique you like, you can arrive at the following Euler condition (which is necessary for the dynamic maximization):

$$\frac{1}{k_t^a - k_{t+1}} = \frac{a\beta k_{t+1}^{a-1}}{k_{t+1}^a - k_{t+2}}.$$

If you impose the guess I gave you as a hint, and after a little bit of algebra, you will find that

$$k_{t+1} = gk_t^a = a\beta k_t^a.$$

Notice that this term is in $(0,1)$. This result simply means that each period agents should invest a part equal to g of the output and eat the remaining $1-g$.

Moreover, since we know that in equilibrium $k_t = x_t$, and we also know x_0, we can fully characterize the whole capital stock allocation. In particular, for all $T > 0$, we have

$$k_T = (a\beta)^{1+a+\ldots+a^{T-1}} \left(x_0^a\right)^T.$$

c) Regardless of the initial condition, this economy will always converge to a steady state. To find it, take the limit as $T \to \infty$ in the equation above. We get

$$k^* = \lim_{T\to\infty} k_T = (a\beta)^{\frac{1}{1-a}}.$$

Regarding the rental rate of capital in the long run, we know that $r_t = F_K(k_t, 1)$. Therefore, as $t \to \infty$, we have $r^* = 1/\beta$. Similarly, $w_t = F_N(k_t, 1)$. Therefore, as $t \to \infty$, we have $w^* = (1-a)(a\beta)^{a/(1-a)}$.