Prelim 2018 Solutions (Caramp)

Fire sales and investment opportunities

1. Consumers have access to a storage technology that has a gross rate of return of 1, which implies that the gross rate of return on risk-free bonds cannot be below 1. Moreover, they are endowed with a 'large' amount of the consumption good in \(t = 1\). In this context, 'large' implies that consumers have more endowment than the entrepreneurs would want to borrow even at the lowest possible interest rate. Therefore, at least some consumers will end up saving in the storage. Competition implies that the equilibrium interest rate on risk-free bonds cannot be higher than 1. Therefore, \(r = 0\).

2. The problem of an entrepreneur is given by

\[
\max_{c_2, \ k \geq 0, b_2} E[c_2]
\]

s.t. \(q_1 k \leq N_1 + b_2\)
\(c_2 + b_2 \leq Ak\)

Monotonicity of preferences implies that both constraints are satisfied with equality. Note that as long as \(b_2 \leq Ak\), \(b_2\) is risk free (we can make sure that this is the case later). The first constraint implies that

\[
b_2 = q_1 k - N_1
\]

Introducing this expression in the second constraint and isolating \(c_2\) we get

\[
c_2 = Ak - q_1 k + N_1
\]

Replacing in the objective function

\[
\max_{k \geq 0} E[(A - q_1)k + N_1] = (A - q_1)k + N_1
\]

where the last equality holds because \(A, q_1, k\) and \(N_1\) are all known in \(t = 1\). The FOC of this problem with respect to \(k\) is

\[
A - q_1 \leq 0
\]

Hence, the entrepreneurs’ demand for capital is given by
\[ k = \begin{cases} 
0 & \text{if } q_1 > A \\
x \in [0, \infty) & \text{if } q_1 = A \\
\infty & \text{if } q_1 < A
\end{cases} \]

Let’s turn to the consumers’ problem. A consumer solves

\[
\max_{\tilde{c}_2, k, \tilde{s} \geq 0, b_2} E[\tilde{c}_2] \\
\text{s.t.} \quad b_2 + \tilde{s} \leq e_1 + q_1(\overline{k} - \tilde{k}) \\
\tilde{c}_2 \leq c_2 + G(\tilde{k}) + b_2 + \tilde{s}
\]

where \( \tilde{s} \) denotes savings in the storage technology. Similar monotonicity arguments as before imply that the constraints are satisfied with equality in an optimum. Following similar steps as before simplify the consumer’s problem to

\[
\max_{k \geq 0} G(\overline{k}) - q_1 \tilde{k} + q_1 \overline{k} + e_1
\]

The FOC with respect to \( \tilde{k} \) is

\[ q_1 = G'(\overline{k}) \]

From the entrepreneur’s demand for capital it should be clear that the equilibrium price cannot be less than \( A \). Suppose \( q_1 > A \). In that case, \( k = 0 \) and \( \tilde{k} = \overline{k} \). But \( G'(\overline{k}) < A \), which contradicts the consumers’ FOC. Therefore, in equilibrium we have \( q_1 = A, A = G'(\overline{k} - k^*) \) and \( k = k^* \).

3. The entrepreneur’s problem is now given by

\[
\max_{c_2, k \geq 0} E[c_2] \\
\text{s.t.} \quad q_1 k \leq N_1 \\
\quad c_2 \leq Ak
\]

Trivially,

\[ k = \frac{N_1}{q_1} \]

On the other hand, the problem of the consumers hasn’t changed. Hence, the equilibrium is given by \( k = \frac{N_1}{q_1} \) and
\[ q_1 = G' \left( \bar{k} - \frac{N_1}{q_1} \right) \]

Since \( G''(\cdot) < 0 \), the RHS is decreasing in \( q_1 \), hence the intersection is unique. It is immediate to see that \( q_1 \) is increasing in \( N_1 \): as \( N_1 \) increases, the entrepreneurs’ demand for capital increases and the amount of capital that the consumers hold decreases, increasing their marginal product and hence the equilibrium price.

Finally, the utility of the entrepreneur is \( c_2 = Ak = A \frac{N_1}{q_1} \).

4. The problem of an entrepreneur is given by

\[
\begin{align*}
    \max_{c_2, k_1, k_2(a) \geq 0, b_1, b_2(a)} & \quad E[c_2] \\
    \text{s.t.} & \quad q_0 k_1 \leq N_0 + b_1 \\
    & \quad q_1(a) k_2(a) + b_1 \leq (q_1(a) + a) k_1 + b_2(a) \\
    & \quad c_2(a) + b_2(a) \leq Ak_2(a)
\end{align*}
\]

Following similar steps as before, we can simplify the problem to

\[
\max E[(A - q_1(a)) k_2(a) + (q_1(a) + a - q_0) k_1 + N_0]
\]

Taking FOCs, we get

\[
\begin{align*}
    (k_1) : & \quad E[q_1(a) + a] - q_0 \leq 0 \\
    (k_2(a)) : & \quad A - q_1(a) \leq 0
\end{align*}
\]

By now it should be clear that an equilibrium requires positive and finite entrepreneurs’ demand for capital, hence \( q_1(a) = A \) and \( q_0 = 2A \).

The problem of a consumer is

\[
\begin{align*}
    \max_{\tilde{c}_2, \tilde{k}_1, \tilde{k}_2(a) \geq 0, b_1, b_2(a)} & \quad E[\tilde{c}_2] \\
    \text{s.t.} & \quad \tilde{s}_1 + b_1 \leq c_0 + q_0 (\bar{k} - \tilde{k}_1) \\
    & \quad \tilde{s}_2(a) + b_2(a) \leq c_1 + G(\tilde{k}_1) + q_1(a)(\tilde{k}_1 - \tilde{k}_2(a)) + b_1 + \tilde{s}_1 \\
    & \quad \tilde{c}_2(a) \leq c_2 + G(\tilde{k}_2(a)) + \tilde{s}_2(a) + b_2(a)
\end{align*}
\]

Similar steps as above lead to the following simplified problem
\[
\max E[G(\tilde{k}_2(a)) + e_1 + G(\tilde{k}_1) + q_1(a)(\tilde{k}_1 - \tilde{k}_2(a)) + e_0 + q_0(\tilde{k} - \tilde{k}_1)]
\]

The FOCs are

\[
(\tilde{k}_1) : \quad G'(\tilde{k}_1) + E[q_1(a)] = q_0
\]

\[
(\tilde{k}_2(a)) : \quad G'(\tilde{k}_2(a)) = q_1(a)
\]

Using that \(q_1(a) = A\) and \(q_0 = 2A\), we get that \(k_1 = k_2 = k^*\) and

\[
G'(\tilde{k} - k^*) = A
\]

5. We know from question 3. that \(q_1\) is increasing in \(N_1\). Since \(N_1\) is now increasing in \(a\), \(q_1(a)\) is increasing in \(a\).

6. The entrepreneurs’ problem is

\[
\max_{c_2, k_1, k_2(a) \geq 0, b_1} \quad E[c_2]
\]

s.t. \(q_0k_1 \leq N_0 + b_1\)

\(q_1(a)k_2(a) + b_1 \leq (q_1(a) + a)k_1\)

\(c_2(a) \leq Ak_2(a)\)

From question 3. we know that the utility of the entrepreneurs is given by \(A \frac{N_1}{q_1}\). Using that \(N_1(a) = (q_1(a) + a)k_1 - b_1\) and \(b_1 = q_0k_1 - N_0\), the entrepreneur’s problem simplifies to

\[
\max E\left[ A \frac{(q_1(a) + a)k_1 - q_0k_1 + N_0}{q_1(a)} \right]
\]

The FOC is

\[
E\left[ A \frac{(q_1(a) + a) - q_0}{q_1(a)} \right] = 0
\]

in an interior solution. Hence

\[
q_0 = \frac{E\left[ \frac{A}{q_1(a)} (q_1(a) + a) \right]}{E\left[ \frac{A}{q_1(a)} \right]}
\]
7. Using the hint, we have

\[ E \left[ \frac{A}{q_1(a)} (q_1(a) + a) \right] = \text{cov} \left( \frac{A}{q_1(a)}, q_1(a) + a \right) + E \left[ \frac{A}{q_1(a)} \right] E[q_1(a) + a] \]

Thus

\[ q_0 = E[q_1(a) + a] + \frac{\text{cov} \left( \frac{A}{q_1(a)}, q_1(a) + a \right)}{E \left[ \frac{A}{q_1(a)} \right]} \]

It is immediate to see that the covariance is negative, hence

\[ q_0 \leq E[a + q_1(a)] \]

Why is the price of capital in \( t = 0 \) less than the fundamental value of capital, even though the entrepreneur is not constraint in \( t = 0 \)? The reason is that capital is a relatively bad investment opportunity in \( t = 0 \). When \( a \) is low, capital pays little (both in terms of dividend \( a \) and resale price \( q_1(a) \)). But when \( a \) is low, \( q_1(a) \) is also low, so the return of having capital in \( t = 2 \) is high, i.e. \( \frac{A}{q_1(a)} \) is high. Hence, the return of holding capital in \( t = 1 \) is negatively correlated with the entrepreneur’s investment opportunities (high marginal value of funds), so the entrepreneurs choose to hold less capital to reduce their risk, driving the equilibrium price down.

**Heterogeneous Effects of Precautionary Savings**

1. The problem of a farmer is

\[
\max \ E \left[ \sum_{t=0}^{\infty} -\beta^t \exp(-\gamma c_t) \right] \\
\text{s.t. } c_t + k_{t+1} + a_{t+1} \leq y_t + (1 + rK - \delta) k_t + (1 + r) a_t \\
\lim_{t \to \infty} \frac{a_t}{(1 + \gamma)^t} \geq 0
\]

Since farmers can produce capital at a cost of 1, the equilibrium price cannot be higher than 1. If the price was lower than 1, no farmer would produce capital and eventually the supply of capital would go to zero. Hence, the price of capital must be equal to 1.

Let \( \lambda_t \) be the Lagrange multiplier associated to the budget constraint. The FOCs are
\[(c_t) : \quad \beta^t \exp(-\gamma c_t) = \lambda_t \]
\[(k_{t+1}) : \quad E_t[(1 + r^K - \delta)\lambda_{t+1}] \leq \lambda_t \]
\[(a_{t+1}) : \quad E_t[(1 + r)\lambda_{t+1}] = \lambda_t \]

Since some farmers have to hold the capital in equilibrium, they need to be indifferent between holding capital or bonds. Hence

\[r = r^K - \delta.\]

2. The farmers’ intratemporal Euler equation is

\[\exp(-\gamma c_t) = \beta(1 + r)E_t[\exp(-\gamma c_{t+1})] \]

or

\[1 = \beta(1 + r)E_t[\exp(-\gamma(c_{t+1} - c_t))] \]

Let’s use the guess. We have

\[c_{t+1} - c_t = c + r\left(\frac{a_{t+1} + k_{t+1}}{y_t + (1 + r)(k_t + a_t) - c_t}\right) + \frac{r}{1 + r}y_{t+1} - c_t \]
\[= c + r(y_t + (1 + r)(k_t + a_t) - c_t) + \frac{r}{1 + r}y_{t+1} - c_t \]
\[= c + r(y_t + (1 + r)(k_t + a_t)) + \frac{r}{1 + r}y_{t+1} - (1 + r)c_t \]
\[= c + r(y_t + (1 + r)(k_t + a_t)) + \frac{r}{1 + r}y_{t+1} - (1 + r)\left(c + r(a_t + k_t) + \frac{r}{1 + r}y_t\right) \]
\[= -r\sigma + \frac{r}{1 + r}y_{t+1} \]

Hence, the FOC is

\[1 = \beta(1 + r)E_t[\exp(\gamma \sigma - \frac{\gamma r}{1 + r}y_{t+1})] \]

or

\[1 = \beta(1 + r)\exp\left(\frac{\gamma r}{1 + r}\sigma + \frac{1}{2} \left(\frac{\gamma r}{1 + r}\sigma^2\right)\right) \]

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After some algebra, we get

$$\zeta = -\frac{1}{\gamma r} \log[\beta(1 + r)] + \frac{\gamma r}{1 + r} - \frac{1}{2} \frac{\gamma r}{(1 + r)^2} \sigma^2$$

3. We have

$$C^F = \int c_f df = \zeta + r \int_{=0}^{=K^F} a_f df + \int_{=K^F} k_f df + \frac{r}{1 + r} \gamma = \zeta + rK^F + \frac{r}{1 + r} \gamma.$$ Integrating the budget constraint, we get

$$C^F + K^F = \gamma + (1 + r)K^F$$

For both expressions to hold, we need

$$\zeta + rK^F + \frac{r}{1 + r} \gamma = \gamma + rK^F$$

or

$$\frac{\gamma}{1 + r} = \zeta = -\frac{1}{\gamma r} \log[\beta(1 + r)] + \frac{\gamma r}{1 + r} - \frac{1}{2} \frac{\gamma r}{(1 + r)^2} \sigma^2$$

and hence

$$\log[\beta(1 + r)] = -\frac{1}{2} \left( \frac{\gamma r}{1 + r} \right)^2 \sigma^2 < 0$$

which implies

$$\beta(1 + r) < 1$$

4. The problem of a worker is

$$\max E_t \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right] \text{ s.t. } c_t + k_{t+1} \leq w_t + (1 + r)k_t$$
5. Let $\lambda_t$ be the Lagrange multiplier associated to the budget constraint. The FOCs of the worker’s problem are

\[
(c_t): \quad \beta^t u'(c_t) = \lambda_t \\
(k_{t+1}): \quad E_t[(1 + r)\lambda_{t+1}] \leq \lambda_t
\]

If $c_t$ constant, then $\lambda_t = \beta^t$, and the intertemporal Euler equation simplifies to

\[
\beta(1 + r) \leq 1
\]

Since $\beta(1 + r) < 1$ in equilibrium, $k^W = 0$.

Hence, putting together the problem of the farmers and workers, we can conclude that there exists an equilibrium with constant aggregate farmers consumption and capital holdings, constant individual workers’ consumption, workers hold zero capital, and the interest rate and wage are constant.

6. From question 3., one can see that if $\sigma^2$ increases, $r$ decreases. Moreover,

\[
\frac{\partial c}{\partial r} = -\frac{\gamma}{(1 + r)^2} < 0
\]

(since $c = \frac{\gamma}{1 + r}$), hence $c(a, k, y)$ increases for all $(a, k, y)$ and therefore $C^F(K^F)$ increases for all $K^F$. From the budget constraint

\[
C^F = \gamma + rK^F
\]

Since $r$ decreases with the increase in $\sigma^2$, it must be true that $K^F$ increases. But if $K^F$ increases, the wage $w$ increases, and hence the welfare of workers increases.