PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

Directions: Answer all questions. Feel free to impose additional structure on the problems below, but please state your assumptions clearly. Point totals for each question are given in parentheses.

1. (10) Briefly discuss the following statements (keep your answers short and concise):
   
   (a) Provide an intuitive - but concise - explanation for how the existence and uniqueness of the value function as defined by the Bellman equation associated with the standard growth model was established. In your answer, be sure to identify the metric space used in the analysis.

   (b) Within the context of the representative agent consumption-based capital asset pricing model, discuss the factors that affect the equilibrium level of the yield on risk-free, one-period bonds.

2. (20) Consider a simple, representative agent RBC model in which output, $y_t$, is produced via a standard Cobb-Douglas production function:

   $$y_t = z_t k_t^{1-\alpha} h_t$$

   where $k_t$ denotes beginning-of-period capital, $h_t$ is labor, and $z_t$ is an i.i.d. technology shock. The depreciation rate of capital is 100%. In each period, agents make consumption and labor decisions in order to maximize lifetime expected utility:

   $$E_0 \left[ \sum_{t=0}^{\infty} \beta^t U (c_t, h_t) \right]$$

   Within this environment, consider two variations defined by the functional form for $U (\cdot)$.

   (a) In Economy A, agents have preferences given by:

   $$U (c_t, h_t) = \ln c_t - \frac{1}{2} h_t^2$$

   In this economy, do the following

   i. Express the maximization problem as social planner problem and write down the associated Bellman equation.

   ii. Solve for the equilibrium policy functions describing consumption, investment and labor.

   (b) In Economy B, agents have preferences given by:

   $$U (c_t, h_t) = \ln \left( c_t - \frac{1}{2} h_t^2 \right)$$

   i. Express the maximization problem as social planner problem and write down the associated Bellman equation.

   ii. Solve for the equilibrium policy functions describing consumption, investment and labor.

   (c) Compare the equilibrium behavior in both economies and provide an explanation for the differences.
3. (20) Consider a variation of the Sidrauski monetary model with a constant population. Specifically, assume that the representative agent’s maximize lifetime utility is given by:

\[
\sum_{t=0}^{\infty} \beta^t \left[ U(c_t) + V\left(\frac{M_t}{P_t}\right) \right]
\]

where \(U(\cdot)\) and \(V(\cdot)\) are concave, twice-differentiable functions, \(c_t\) denotes consumption and \(M_t\) is money chosen in period \(t\). Each period, agents use beginning of period nominal balances, the revenue from sales of output and a lump-sum monetary transfer to purchase consumption, investment and new money. In contrast to the Sidrauski model, both capital and money are used as inputs into the production process. Letting \(y_t\) denote output, the production function is given by:

\[
y_t = \left(1 - z\left(\frac{M_t}{P_t}\right)\right) f\left(k_t\right)
\]

where \(z'(\cdot) < 0, z''(\cdot) > 0, z(0) = 1, \lim_{M_t/P_t \to \infty} z(M_t/P_t) = 0\). The function \(f(k_t)\) has standard properties.

The money supply in this economy is growing at the constant rate \(\mu > 0\) and capital depreciates at the constant rate of \(\delta < 1\).

(a) Derive and interpret the necessary conditions associated with the agent’s maximization problem.

(b) Define a steady-state equilibrium in this economy. Use the equations defining the steady-state equilibrium to establish the relationship between steady-state consumption and real balances (i.e. \(dc/d\bar{m}\)) and steady-state capital and real balances (i.e. \(d\bar{k}/d\bar{m}\)) where \((\bar{c}, \bar{m}, \bar{k})\) denote steady-state values.

(c) Now consider the relationship between \(\bar{m}\) and the growth rate of money \(\mu\). Does this economy exhibit superneutrality? Provide an intuitive explanation for your answer (Do NOT try to establish an analytic result as this gets messy and will depend on functional forms.)
4. (20) Consider a version of the continuous time Mortensen-Pissarides labor market model with match-specific productivity. Labor force is normalized to 1, and there is a large measure of firms that can enter the market and search for workers. A firm can enter the labor market with exactly one vacancy, and the total measure of vacancies $v$ will be determined endogenously by free entry. A matching function, $m = m(u, v)$, brings together unemployed workers and vacant firms; $m$ is increasing in both arguments and exhibits CRS. As is standard, let $\theta \equiv v/u$ denote the market tightness.

Unlike the baseline model, here not all meetings result in a match. When a firm and an unemployed worker first meet, they draw a match-specific productivity $x$ from a cdf $F(x)$, with support in the set $[0, 1]$. The random draws of $x$ are iid across matches and time. Upon observing the specific realization of $x$, the firm and worker decide whether they will form a productive match, which can produce $x$ units of the numeraire good per unit of time. Alternatively (if the realization of $x$ is too low), they may decide that it is not worth forming a match. In this case, the worker stays unemployed, and the firm stays in the large pool of firms that can potentially search for workers. If a match is indeed formed, the two parties negotiate over the wage, which will be contingent on the match-specific productivity, $x$. In the negotiation process, let $\beta \in (0, 1)$ denote the worker’s bargaining power. Crucially, a productive match keeps its idiosyncratic productivity $x$ for as long as it is active/alive. Existing productive matches are terminated at an exogenous Poisson rate, $\lambda > 0$.

To close the model, we will make a few more standard assumptions. While a firm is searching for a worker it has to pay a recruiting cost, $c > 0$, per unit of time. All agents discount future at the rate $r > 0$, and all unemployed workers enjoy a benefit $z > 0$ per unit of time. We further impose that $z \leq \int_0^1 xdF(x)$. Throughout this question focus on steady states. Also, start by taking as given that when firms and workers first meet, they decide whether to form a match based on a “reservation rule”. More precisely, assume that there exists a unique $R \in (0, 1)$, such that a productive match will be formed if and only if $x \geq R$. Of course, $R$ is an endogenous variable which, eventually, will be characterized.

(a) Describe the Beveridge curve (BC) for this economy. Does the equilibrium relationship between $u$ and $\theta$ depend on any other endogenous variables here?

(b) Describe the value functions for a worker who is unemployed ($U$), and for a worker who is actively working in a match with idiosyncratic productivity $x$ ($W(x)$).

(c) Describe the value functions for a vacant firm ($V$), and for a firm actively matched with a worker, when the idiosyncratic productivity of that match is $x$ ($J(x)$).

(d) Exploiting part (c) and free entry (i.e., the fact that $V = 0$ in any equilibrium), provide an equilibrium condition that $J(x)$ must satisfy for all $x \geq R$.

(e) For any $x \geq R$, describe the wage curve (WC) equation, $w(x)$. The right-hand side should contain only parameters and endogenous variables (not value functions).

(f) Use your findings in part (e) to provide a formula for $J(x)$, $x \geq R$, that does not contain the term $w(x)$.

Since here we have 4 equilibrium objects ($u, \theta, w(x), R$), we also need 4 equilibrium conditions. We already have the BC and the WC. The remaining two are just the job creation (JC) and job destruction (JD) conditions. In what follows, to simplify the analysis, feel free to assume that $x$ is distributed uniformly in $[0, 1]$.

(g) Derive the JC condition for this economy, by substituting your formula for $J(x)$, from part (f), into the equilibrium condition that you reported in part (d).

(h) Derive the JD condition for this economy. (Hint: Evaluate your formula for $J(x)$, from part (f), at the value where jobs are “destructed” (or not worth keeping), just like we did in the standard model with endogenous destruction in class.)

(i) Summarize the 4 steady state equilibrium conditions, and discuss in as much detail as you can the existence and uniqueness of equilibrium.
5. Consider an economy that consists of two islands, \( i = \{1, 2\} \). Each island has a large population of infinitely-lived, identical agents, normalized to the unit. There is a unique consumption good, say, coconuts, which is not storable across periods. Although within each island agents have identical preferences over consumption, across islands there is a difference: Agents in island 2 are more patient. More precisely, the lifetime utility for the typical agent in island \( i \) is given by

\[
U_i(\{c_i^t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta_i^t \ln(c_i^t)
\]

where \( \beta_i \in (0, 1) \), for all \( i \), and \( \beta_2 > \beta_1 \).

Due to weather conditions in this economy, island 1 has a production of \( e > 0 \) units of coconuts in even periods and zero otherwise, and island 2 has a production of \( e \) units of coconuts in odd periods and zero otherwise. Agents cannot do anything to boost this production, but they can trade coconuts, so that the consumption of the typical agent in island \( i \), in period \( t \), is not necessarily equal to the production of coconuts on that island in that period (which may very well be zero). Assume that shipping coconuts across islands is costless.

(a) Describe the Arrow-Debreu equilibrium (ADE) allocations in this economy using Negishi’s method.

(b) Describe the ADE prices in this economy.

(c) Plot the equilibrium allocation for the typical agent in island \( i \), i.e., \( \{\hat{c}_i^t\}_{t=0}^{\infty}, i = \{1, 2\} \), against \( t \). Is there any period \( t \) in which \( \hat{c}_1^t = \hat{c}_2^t \)? If yes, please provide a closed form solution for that value of \( t \).
6. Consider the standard growth model in discrete time. There is a large number of identical households (normalized to 1). Each household wants to maximize life-time discounted utility

\[ U(\{c_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t (\ln c_t + \gamma \ln c_{t-1}), \quad \gamma > 0, \]

that is, households preferences are characterized by “habit persistence”. Each household has an initial capital stock \(x_0\) at time 0, and one unit of productive time in each period, that can be devoted to work. Final output is produced using capital and labor services,

\[ y_t = F(k_t, n_t) = k_t^a n_t^{1-a}. \]

This technology is owned by firms whose number will be determined in equilibrium. Output can be consumed (\(c_t\)) or invested (\(i_t\)). We assume that households own the capital stock (so they make the investment decision) and rent out capital services to the firms. We also assume that the capital stock \((x_t)\) fully depreciates at the end of a given period, i.e. \(\delta = 1\). Finally, it is assumed that households own the firms, i.e. they are claimants to the firms’ profits.

(a) In this economy, why is it a good idea to describe the AD equilibrium capital stock allocation by solving the (easier) Social Planner’s Problem?

(b) Fully characterize (i.e. find a closed form solution for) the equilibrium allocation of the capital stock. (Hint: Derive the Euler equation, and “guess and verify” a policy rule of the form \(k_{t+1} = g k_t\), where \(g\) is an unknown to be determined.)

(c) What is the capital stock equal to as \(t \to \infty\)? What is the ADE value of the rental rate of capital and the rental rate of labor as \(t \to \infty\)?

(d) Express the ADE price of the consumption good in any period \(T\) as a function of parameters of the model and the sequence of capital stock up to period \(T\).