University of California, Davis          Date: September 3, 2009
Department of Economics                 Time: 5 hours
Macroeconomics                          Reading Time: 20 minutes

<u>Answers to Salyer's Questions</u>

**Question 2** Consider a representative agent, optimal growth model with no population growth in which agents have logarithmic preferences and assume that agents' discount factor ($\beta$) is equal to 0.97. If the economy is growing at 3%, what will the equilibrium one-period real interest rate be in this economy? (All interest rates and growth rates are expressed on an annual basis.) Suppose consumption uncertainty is introduced into this economy. How will this affect the (average) equilibrium real interest rate?

ANSWER: The Euler equation associated with bond purchases is:

$$1 = \beta \left(1 + g_{t+1}\right)^{-1} R_t$$

where $g_t$ is consumption growth and $R_t$ is the gross interest rate. Taking logs yields

$$0 = \ln \beta - g_{t+1} + r_t$$

where $r_t$ is the net interest rate. Rearranging yields: $r_t = g_t - \ln \beta$. The numbers given in the problem imply that $r_t \approx 6\%$. (Note that $\ln 0.97 \approx \ln \left(\frac{1}{1.03}\right) \approx 0.03$.) If there was consumption uncertainty, the average real interest rate would be lower as consumption uncertainty lowers the certainty equivalent of next period's consumption making current consumption relatively more abundant. Hence the price of current consumption relative to future consumption, i.e. the real interest rate, falls.

**Question 3** Let $\tilde{r}_{t+1}$ denote the realized gross one-period return on a risky asset purchased at time $t$ and $r_{f,t}$ denote the gross one-period return on a risk-free asset (also purchased at time $t$). Assuming that consumption growth and returns are serially uncorrelated, then the risk premium associated with the risky asset can be written as:

$$E\left(\tilde{r}_{t+1}\right) - r_{f,t} = -\beta r_{f,t} \left[\rho\left(m_{t+1}, \tilde{r}_{t+1}\right) \sigma\left(m_{t+1}\right) \sigma\left(\tilde{r}_{t+1}\right)\right]$$

where $\beta$ is the discount factor, $m_t$ represents agents' stochastic discount factor and $\rho\left(\cdot\right)$ and $\sigma\left(\cdot\right)$ represent correlation and standard deviation respectively. Derive the above expression and discuss its implications for the equity premium puzzle.

ANSWER:The Euler equations associated with the risky and riskless assets are:

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\begin{align}
1 &= \beta E\left(m_{t+1}\tilde{r}_{t+1}\right) \tag{1}\\
1 &= \beta E\left(m_{t+1}\right) r_{f,t} \tag{2}
\end{align}
$$

Using the definition of covariance and eq.(2), eq.(1) can be written as:

$$E\left(\tilde{r}_{t+1}\right) - r_{f,t} = -\frac{Cov\left(m_{t+1}, \tilde{r}_{t+1}\right)}{E\left(m_{t+1}\right)} = -\beta r_{f,t} Cov\left(m_{t+1}, \tilde{r}_{t+1}\right)$$

Using the definition of correlation, we have the desired result:

$$E\left(\tilde{r}_{t+1}\right) - r_{f,t} = -\beta r_{f,t} \left[\rho\left(m_{t+1}, \tilde{r}_{t+1}\right) \sigma\left(m_{t+1}\right) \sigma\left(\tilde{r}_{t+1}\right)\right]$$

This expression demonstrates that a negative correlation between investor's stochastic discount factor and the return on the risky asset is required for a positive risk premium - this is the defining aspect of the CCAPM. It also demonstrates why the basic CCAPM has trouble matching quantitatively the observed equity premium: the standard deviation of the stochastic discount factor is too small when standard preferences are used.

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**Question 6** Consider the following simple RBC model with capital adjustment costs in which output is a function of beginning of period capital and an *i.i.d.* stochastic technology shock:

$$y_t = z_t k_t^\alpha; \quad \alpha < 1$$

The representative agent has standard time-separable logarithmic preferences over current and future consumption and maximizes expected discounted utility:

$$E_0 \left[ \sum_{t=0}^{\infty} \beta^t \ln c_t \right]; \quad 0 < \beta < 1$$

The law of motion for capital is given by:

$$k_{t+1} = k_t (1 - \delta) + F(i_t, k_t)$$

where the function $F(i_t, k_t)$ represents the adjustment costs associated with investment. Given this environment, do the following:

a. Express the agent's maximization problem as a dynamic programming problem. Interpret the resulting necessary conditions. In particular, show that the shadow price of capital, denote this as $q_t$, is related to the adjustment cost function, $F(\cdot)$. (Hint: It is recommended that you set up the law of motion as an additional constraint in the Lagrangian.)

b. To put more structure on the model, assume that the adjustment cost function takes the form:

$$F(i_t, k_t) = \left[ 1 - S \left( \frac{i_t}{k_t} \right) \right] i_t$$

It is assumed that $S(\delta) = S'(\delta) = 0$. What is the rationale for this assumption? What is the implication for $q_t$ in the steady-state?

ANSWER: The maximization problem can be expressed as the following dynamic programming problem:

$$V(k_t, z_t) = \max_{(c_t, k_{t+1}, i_t)} \left[ \begin{array}{c} \ln c_t + \beta E\left[ V(k_{t+1}, z_{t+1}) \right] \\ + \lambda_t \left( z_t k_t^\alpha - c_t - i_t \right) \\ + \lambda_t q_t \left( k_t (1 - \delta) + F(i_t, k_t) - k_{t+1} \right) \end{array} \right]$$

Note that the law of motion for capital has two associated Lagrange multipliers: $q_t$ is used to measure the resource costs of additional capital (i.e. the shadow price) and then this is weighted by $\lambda_t$ in order to obtain the shadow price in terms of utility. The necessary conditions are (after using the envelope theorem):

$$\frac{1}{c_t} = \lambda_t \tag{3}$$

$$-\lambda_t + \lambda_t q_t \frac{\partial F(i_t, k_t)}{\partial i_t} = 0 \tag{4}$$

$$q_t \lambda_t = \beta E \left\{ \lambda_{t+1} \left[ \alpha z_{t+1} k_{t+1}^{\alpha-1} + q_{t+1} \left( 1 - \delta + \frac{\partial F(i_{t+1}, k_{t+1})}{\partial k_{t+1}} \right) \right] \right\} \tag{5}$$

The first necessary condition is standard and states that the utility shadow price of additional output is equal to the marginal utility of consumption. In order to interpret the second and third expressions, note that if there were no adjustment costs, then $\frac{\partial F(i_t, k_t)}{\partial i_t} = 1$ (i.e. additional investment yields new capital on a one-for-one basis) and $\frac{\partial F(i_{t+1}, k_{t+1})}{\partial k_{t+1}} = 0$. Then it is clear from eq.(4) that, in the standard case, $q_t = 1$. With adjustment costs, additional investment produces less than one unit of capital so $q_t > 1$. The Euler

equation associated with capital (eq. (5)) is also standard when there are no adjustment costs. With adjustment costs, the marginal costs of additional capital (the left hand side) is equal to the expected marginal gain. These are the sum of the marginal product of capital and the value of the capital stock next period. This, in turn is the sum of undepreciated capital and the effects that additional capital have on adjustment costs.

For part (b) of the question, first note that the functional form implies:

$$\frac{\partial F\left(i_t, k_t\right)}{\partial i_t} = 1 - S\left(\frac{i_t}{k_t}\right) - i_t S'\left(\frac{i_t}{k_t}\right)\frac{1}{k_t}$$

$$\frac{\partial F\left(i_t, k_t\right)}{\partial k_t} = S'\left(\frac{i_t}{k_t}\right)\left(\frac{i_t}{k_t}\right)^2$$

Since, in steady-state, the capital stock is not changing, the restrictions on the functional form imply that capital adjustment costs do not affect the necessary conditions for the economy. In steady state, $\left(\frac{\bar{i}}{k}\right) = \delta$. Then the restrictions on $S\left(\cdot\right)$ imply $\frac{\partial F(i_t, k_t)}{\partial i_t} = 1$ and $\frac{\partial F(i_t, k_t)}{\partial k_t} = 0$ so we are back to the standard setting with the implication that $\bar{q} = 1$.

**Question 7** An economy is populated by identical agents with expected lifetime utility given by:

$$E_0\left[\sum_{t=0}^{\infty} \beta^t \ln c_t\right]$$

Each period, agents rent their beginning-of-period capital stock $(k_t)$ to (identical) firms (the rental rate of capital is denoted $r_t$) and also supply inelastically one unit of labor to firms (the wage rate is denoted $w_t$). The income generated by these factor supplies are used to acquire consumption $(c_t)$ and new capital. The depreciation rate of capital is assumed to be 100%. Firms choose labor and capital every period in order to maximize profits where output is given by the technology:

$$y_t = z_t k_t^{\alpha} h_t^{1-\alpha}$$

where the law of motion for $z_t$ is $z_t = z_{t-1}^{\rho}\varepsilon_t$ where $0 < \rho < 1$ and $\varepsilon_t$ is an *i.i.d.* innovation. Assume that the unconditional mean of both $z_t$ and $\varepsilon_t$ is 1. Given this scenario, answer the following:

a. Write down the firm's and household's maximization problems and derive and interpret the associated necessary conditions. Define a recursive competitive equilibrium for this economy.

b. Rather than solve directly for the competitive equilibrium, one can instead solve an associated social planner problem. Express the relevant social planner problem and derive the associated necessary conditions. Define the solution to the social planner problem (Do not actually solve the model.)

c. One way to solve the social planner problem would be to log-linearize the expressions characterizing an equilibrium. Do this for the model described in part (b) and present the resulting set of linear expectational difference equations in matrix form. Describe how one would use this system to find the solution to the model; be precise in your description and identify the form that the solution will take.

ANSWER:

$(a)$ The firm's maximization problem is a static profit maximization problem:

$$\max_{k_t, h_t}\left[z_t k_t^{\alpha} h_t^{1-\alpha} - w_t l_t - r_t k_t\right]$$

This leads to the familiar condition that factors are paid their marginal products and implies factor demands: $l_t = l(z_t, k_t)$ and $k_t = k(z_t, k_t)$. For households, their maximization problem can be expressed as the following dynamic programming problem:

$$V(a_t, k_t, z_t) = \max_{c_t, k_{t+1}} \left[ \ln c_t + \beta E_t \left( V(a_{t+1}, k_{t+1}, z_{t+1}) \right) \right]$$

subject to the budget constraint: $w_t + r_t a_t = c_t + a_{t+1}$ where $a_t$ denotes the household's beginning of period capital stock. For this problem to be well defined, it is assumed that agents know the distribution for the technology shocks and also the functions describing the factor prices: $w_t = w(z_t, k_t)$, $r_t = r(z_t, k_t)$. In addition, it is assumed that agents know the law of motion of the aggregate capital stock: $k_{t+1} = G(k_t, z_t)$. (Note that labor is not a choice variable for households so the supply of labor is 1.) This produces policy functions for consumption and capital: $c_t = c(a_t, k_t, z_t)$, $a_{t+1} = a(a_t, k_t, z_t)$.

A recursive equilibrium is defined by

1. A value function as defined above and the policy functions noted above that define factor demand and household choices. (The factor price functions are implied by the factor marginal products.)

2. Markets clear: $l(z_t, k_t) = 1$, $k(z_t, k_t) = k_t$, $c(k_t, k_t, z_t) + a(k_t, k_t, z_t) = z_t k_t^\alpha$ (note that $a_t = k_t$ due to the representative agent assumption).

3. Expectations are rational: $a(k_t, k_t, z_t) = G(k_t, z_t)$.

The necessary condition associated with the household maximization problem is the Euler equation:

$$c_t^{-1} = \beta E_t \left[ c_{t+1}^{-1} r_{t+1} \right]$$

$(b)$ The social planner problem can be written as:

$$V(k_t, z_t) = \max_{c_t, k_{t+1}} \left[ \ln c_t + \beta E_t \left( V(k_{t+1}, z_{t+1}) \right) \right]$$
$$\text{subject to } z_t k_t^\alpha = c_t + k_{t+1}$$

with the associated Euler equation:

$$c_t^{-1} = \alpha \beta E_t \left[ c_{t+1}^{-1} z_{t+1} k_{t+1}^{\alpha-1} \right]$$

Note that the social planner problem and the recursive competitive equilibrium will have the same solution given that $r_{t+1} = \alpha z_{t+1} k_{t+1}^{\alpha-1}$.

$(c)$ The equations characterizing the solution to the social planner problem are:

$$
\begin{aligned}
c_t^{-1} &= \alpha \beta E_t \left[ c_{t+1}^{-1} z_{t+1} k_{t+1}^{\alpha-1} \right] \\
z_t k_t^\alpha &= c_t + k_{t+1} \\
z_t &= z_{t-1}^\rho \varepsilon_t
\end{aligned}
$$

Log linearizing these around the steady-state yields the following system of equations:

$$
\begin{pmatrix} -1 & 0 & 0 \\ -\frac{\bar{c}}{\bar{k}} & \alpha & 1 \\ 0 & 0 & \rho \end{pmatrix}
\begin{pmatrix} \hat{c}_t \\ \hat{k}_t \\ \hat{z}_t \end{pmatrix}
=
\begin{pmatrix} -1 & 1 & 1 \\ 0 & \bar{k}^{1-\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \\ \hat{z}_{t+1} \end{pmatrix}
+
\begin{pmatrix} \varepsilon_{c,t+1} \\ 0 \\ \hat{\varepsilon}_{t+1} \end{pmatrix}
$$

where the last terms represent the forecast errors. This can be written as $\mathbf{A}\mathbf{u}_t = \mathbf{B}\mathbf{u}_{t+1} + \varepsilon_{t+1}$. A solution procedure (as described in 200E) is to pre-multiply both sides by $\mathbf{A}$ and then decompose the resulting matrix as: $\mathbf{A}^{-1}\mathbf{B} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$ where $\mathbf{Q}$ is a matrix whose columns are the eigenvectors $\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{\Lambda}$ is a

diagonal matrix with the associated eigenvalues on the diagonal. This yields the decoupled set of equations $\mathbf{Q}^{-1}\mathbf{u}_t = \mathbf{\Lambda}\mathbf{Q}^{-1}\mathbf{u}_{t+1}$ (ignoring the expectational errors). Given that the dimension of the dynamic system is 3 and we have 2 pre-determined variables, we require that one eigenvalue be less than one for a regular equilibrium. This eigenvalue determines which row of the $\mathbf{Q}^{-1}$ matrix defines the policy function for consumption since stability requires $\mathbf{Q}^{-1}_{\lambda_i<1}\mathbf{u}_t = 0$ for that row. The solution is therefore given by two linear policy functions: one for consumption and one for capital. The latter is obtained by replacing $\hat{c}_t$ in the second row of the the $\mathbf{A}$ matrix with the consumption policy function.