1 Answers to Macro Prelim, Q3,Q4, Q7 June 2009

3. True or False: An implication of the Hansen indivisible labor RBC model is that consumption and labor productivity will have the same time series properties.

   ANSWER: True. Equilibrium is characterized by equality of agents marginal rate of substitution between consumption and leisure and the marginal product of labor. Assuming standard Cobb-Douglas production and utility given by: $\ln c_t + A(1-h_t)$ (where the parameter $A > 0$), this condition is:
$$c_t = \frac{(1-\alpha)}{A}\frac{y_t}{h_t}$$
   Hence consumption and labor productivity (y_t/h_t) will have the same time series characteristics.

4. Consider an exchange economy populated by identical agents that trade equity shares, z_t, defined as title to the endowment process. (That is, this is the same asset priced in the Lucas tree model.) Denote the price of equity as q_t. Agents also trade one-period bonds which cost p_t units of consumption in period t and return 1 unit of consumption in the following period. In addition to these assets, a one-period forward contract on bonds is traded. In this contract, agents agree at time t to pay ϕ_t units of consumption in period $t+1$ for the promise of one unit of consumption to be received in period $t+2$. The endowment, x_t, is stochastic and varies over the interval $(\underline{x}, \overline{x})$; furthermore, x_t is assumed to be independently and identically distributed. Given this environment, agents choose a sequence of consumption and assets in order to maximize
$$E_0\left[\sum_{t=0}^{\infty}\beta^t U(c_t)\right]$$
   (a) Formulate the agent's problem as a dynamic programming problem. Be explicit in identifying the state and control variables.

   (b) Derive and interpret the necessary conditions which characterize the solution to this maximization problem.

   (c) Define a recursive competitive equilibrium in this economy.

   (d) Prove that equilibrium bond and equity prices are positively correlated with the endowment while the price of the forward contract is constant. Explain these results.

ANSWER: The dynamic programming problem associated with the question is:

$$
\begin{aligned}
& V\left(z_{t-1}, b_{t-1}, f_{t-1}, f_{t-2}; x_t\right) \\
&= \max_{\left(c_t, z_t, b_t, f_t\right)} \left\{U\left(c_t\right) + \beta E V\left(z_t, b_t, f_t, f_{t-1}; x_{t+1}\right)\right\} \\
& + \lambda_t\left[z_{t-1}\left(q_t + x_t\right) + b_{t-1} + f_{t-2} - c_t - z_t q_t - p_t b_t - \phi_{t-1} f_{t-1}\right]
\end{aligned} \tag{1}
$$

The arguments of the value function denote the state variables. Note that prices have been suppressed (for simplicity) since, in equilibrium, these are a function of the aggregate state variable, x_t. Also note that, since the endowment is distributed as i.i.d., there is no time subscript on the expectations operator. The first order conditions associated with bonds and equity have been derived numerous times, so I will not do that here. Focusing on the forward contract, we have the derivative:

$$
\beta E\left[\frac{\partial V\left(z_t, b_t, f_t, f_{t-1}; x_{t+1}\right)}{\partial f_t}\right] = 0 \tag{2}
$$

This derivative is the change in utility due to forward conracts purchased in the previous period. Applying the envelope theorem to eq.(1) yields:

$$
\frac{\partial V\left(z_{t-1}, b_{t-1}, f_{t-1}, f_{t-2}; x_t\right)}{\partial f_{t-1}} = \beta E\left[\frac{\partial V\left(z_t, b_t, f_t, f_{t-1}; x_{t+1}\right)}{\partial f_{t-1}}\right] - \lambda_t \phi_{t-1} \tag{3}
$$

Note that the derivative on the RHS of the above expression represents the change in utility due to forward contracts purchased 2 periods ago. We need to apply the envelope theorem to that by taking the following derivative

$$
\frac{\partial V\left(z_{t-1}, b_{t-1}, f_{t-1}, f_{t-2}; x_t\right)}{\partial f_{t-2}} = \lambda_t \tag{4}
$$

Updating eq.(4) and using this in eq. (3) and then updating that expression to use in eq. (2) we have:

$$
\beta E\left\{\beta E\left[U'_{t+2}\right] - U'_{t+1}\phi_t\right\} = 0
$$

(where we have used the condtion that $\lambda_t = U'_t$). Or, using the law of iterated expectations:

$$
\beta E\left[U'_{t+2}\right] = E\left[U'_{t+1}\right]\phi_t \tag{5}
$$

Eq.(5) has the standard MC=MB interpretation: the RHS represents the expected utility loss from buying the contract at time t while the LHS represents the expected utility gain from the return in period $t+2$. At an optimum, these must be equal. The necessary conditions associated with equity and bonds are:

$$
q_t = \frac{\beta E\left[U'_{t+1}\left(q_{t+1} + x_{t+1}\right)\right]}{U'_t}
$$

$$p_t = \frac{\beta E\left[U'_{t+1}\right]}{U'_t}$$

c. Many answers confused the recursive structure of equity prices with the definition of a recursive equilibrium. Simply because the equity price expression can recursively be solved forward to yield the expression:

$$q_t = \frac{\sum_{i=1}^{\infty} \beta^i E\left[U'_{t+i}\left(q_{t+i} + x_{t+i}\right)\right]}{U'_t}$$

has very little to do with the definition of equilibrium. A *recursive competitive equilibrium* is defined by four functions: the functions defining the prices of the three assets, $q(x), p(x), \phi(x)$, and a value function $V(z_{t-1}, b_{t-1}, f_{t-1}, f_{t-2}; x_t)$, such that, (i) given $q(x), p(x), \phi(x)$, $V(z_{t-1}, b_{t-1}, f_{t-1}, f_{t-2}; x_t)$ solves the consumer's maximization problem and (ii) markets clear. The rational expectations assumption in this context is that the price functions agents use to solve their maximization problem are the same implied by market clearing.

d. The assumption that the endowment is i.i.d. implies the numerators in the prices for equity and bonds are constant. Hence, this establishes that both prices are increasing in the endowment. The price of the forward contract, in contrast, will be constant since it is determined by the ratio of two forecasts both of which are invariant over time.

7. Consider a representative agent, exchange economy similar to that studied by Mehra and Prescott. Specifically, it is assumed that the endowment, x_t, grows stochastically as given by $x_{t+1} = \lambda_{t+1} x_t$ where the growth rate, λ_t,is assumed to be independently and identically distributed. Agents maximize lifetime expected utility:

$$E_0\left[\sum_{t=0}^{\infty} \beta^t \left(c_t - hc_{t-1}\right)^{1-\gamma}\right]$$

where $\beta \in (0,1)$ and $0 \leq h < 1$. In this economy, agents trade one-period bonds that cost p_t units of consumption in period t and return one unit of consumption in period $t+1$. Given this environment, do the following:

(a) Define a recursive competitive equilibrium and derive the necessary condition associated with optimal bond purchases.

(b) Suppose that $h = 0$. Characterize the behavior of equilibrium bond prices? How does a value of $h > 0$ affect the volatility of bond prices? Discuss the implications that these results have for the assumption of habit persistence to help resolve some asset pricing puzzles.

ANSWER: The aggregate state vector for this problem is $s_t = (x_t, \lambda_t)$ while for the individual, previous period bond holdings and last period's consumption are also state variables. (One could also include the bond price as a state variable but since, in equilibrium, this will be a function of s_t it is not necessary.) Consequently, the maximization problem can be expressed as the following dynamic programming problem:

$$V\left(b_{t-1}, c_{t-1}; s_t\right) = \max \left\{ \begin{array}{c} \left(c_t - hc_{t-1}\right)^{1-\gamma} + \beta E\left[V\left(b_t, c_t; s_{t+1}\right)\right] \\ +\mu_t\left[x_t + b_{t-1} - c_t - p_t b_t\right] \end{array} \right\} \tag{6}$$

A recursive equilibrium will be defined by two functions: $V\left(b_{t-1}, c_{t-1}; s_t\right)$ as defined in eq.(6) (with agents taking the bond price as given) and a bond price function: $p\left(x_t, \lambda_t\right)$ that is consistent with market clearing. This implies that the equilibrium bond price function is defined by the necessary condition associated with bond purchases when this is evaluated at market clearing quantities: $c_t = x_t, b_t = 0$. This necessary condition is:

$$p\left(x_t, \lambda_t\right) \tilde{\mu}_t = \beta E\left[\tilde{\mu}_{t+1}\right]$$

where $\tilde{\mu}_t = \left[\left(x_t - hx_{t-1}\right)^{-\gamma} - h\beta E\left[\left(x_{t+1} - hx_t\right)^{-\gamma}\right]\right]$. Given the process for the endowment and the assumption of isoelastic preferences, the bond price will be homogeneous of degree one in x_t. It is straightforward to establish this by subsitutiting out for future values of x_t in terms of x_{t-1} and the realized path of growth rates: $x_t = x_{t-1}\lambda_t$ and $x_{t+1} = x_{t-1}\lambda_t\lambda_{t+1}$ etc. Then the condition defining the bond price becomes:

$$p\left(\lambda_t\right)\left[\left(\lambda_t - h\right)^{-\gamma} - h\beta\lambda_t^{-\gamma}\theta\right] = \beta\lambda_t^{-\gamma}\theta\left[1 - h\beta\theta\right]$$

where $\theta = E\left(\lambda_{t+1}^{-\gamma}\right)$ which is a constant due to the *i.i.d.* assumption. Rearraging this yields:

$$p\left(\lambda_t\right)\left[\frac{\lambda_t^{\gamma}}{\left(\lambda_t - h\right)^{\gamma}} - h\beta\theta\right] = \beta\theta\left[1 - h\beta\theta\right]$$

Note that the RHS is a constant. If $h = 0$, then this expression simplifies to $p\left(\lambda_t\right) = \beta\theta$ so that bond prices are constant. Since today's growth provides no information about future consumption growth, then the relative price of current consumption to future consumption will be constant. Clearly, when $h > 0$,bond prices will not be constant. It is easy to show that $p'\left(\lambda_t\right) < 0$ (when $\lambda > h$) and that the volatility of bond prices is increasing in h.

The implication for asset pricing is that habit persistence increases the volatility of agents' stochastic discount factors for a given volatility of consumption growth. This helps to resolve the equity premium puzzle since it will increase the covariance between agents' stochastic discount factor and the return on equity. However, on the negative side, habit persitence produces volatility of interest rates greater than seen in the data.