

ANSWER KEY, PRELIM MACROECONOMICS, 200D

Winter quarter, 2008/2009.

Question 1 Consider the sequence of function $\{v_n(x)\}_{n=0}^\infty$ defined by

$$v_{n+1}(x) = \max_{x' \in [0, x]} \{u(x - x') + \beta v_n(x')\} \quad (1)$$

with $v_0(x) = 0$. Under which conditions on u and β do we know that $v(x) = \lim_{n \rightarrow \infty} \{v_n(x)\}_{n=0}^\infty$ exists, is *unique*, *continuous*? Explain briefly.

Answer If u is bounded and continuous, then v_1 is continuous and bounded too (by the Theorem of the maximum; notice that the feasibility correspondence is continuous). Repeating the same argument, v_n is continuous and bounded. If $\beta \in (0, 1)$ then (1) is a contraction map (by Blackwell's sufficient conditions), and $\{v_n\}$ is a Cauchy sequence. Since the space of bounded continuous functions is complete, $\{v_n\}$ converges to a unique limit point which is also bounded and continuous.

Question 4 In the first problem-set of the course we considered the following “income-fluctuation problem”

$$V = \max_{\{c_t(y^t), a_{t+1}(y^t)\}_{t=0}^\infty} E_0 \sum_{t=0}^\infty \beta^t u(c_t(y^t)) \quad (2)$$

$$\text{subject to} \quad c_t(y^t) + a_{t+1}(y^t) = (1 + r)a_t(y^{t-1}) + y_t \quad (3)$$

where y_t is the individual's stochastic endowment in period t , and $y^t = (y_0, y_1, \dots, y_t)$ is the corresponding history. Assume $\beta(1 + r) = 1$.

(i) Let $y_{t+1} = \rho y_t + \varepsilon_t$, where ε_t is a zero mean random shock. What is the Bellman equation corresponding to (2) – (3) above? (Feel free to use the expectations operator instead of summing/integrating over possible events.)

(ii) In the problem-set, we showed that if $u(c) = ac - \frac{1}{2}bc^2$, the solution took the form

$$c_t = \frac{r}{1 + r} [a_t(1 + r) + E_t \sum_{s=0}^\infty \frac{1}{(1 + r)^s} y_{t+s}] \quad (4)$$

That is, consumption in period t equals the annuity value of total assets plus “permanent income” (a term coined by Milton Friedman).

Find the consumption policy function that satisfies your Bellman equation in (ii), and show that your (recursive) solution coincides with (4) (Hint: Use the

(recursive) Euler equation to derive the policy rule – the value function is very difficult to recover.)

- (iii) What is the marginal propensity to consume, $\frac{\partial c_t}{\partial y_t}$? How does the MPC change with the parameter ρ ? Interpret.

Answer

- (i) The Bellman equation is given by

$$v(a, y) = \max_{a'} \{u(a(1+r) + y - a') + \beta E[v(a', \rho y + \varepsilon)]\} \quad (5)$$

- (ii) Guess that $c = Aa + By + C$. From the first order conditions, using the envelope theorem, we have that $E[c'] = c$, and thus that $Aa' + B\rho y + C = c$. Using the budget constraint we know that $a' = (1+r)a + y - c$. Plugging in and solving for coefficients we get that $A = r$, $B = \frac{r}{1+r-\rho}$ and $C = 0$. Thus,

$$c = \frac{r}{1+r} \left[a(1+r) + \frac{1+r}{1+r-\rho} y \right] \quad (6)$$

Since $\frac{1+r}{1+r-\rho} y_t = E_t \sum_{s=0}^{\infty} \frac{1}{(1+r)^s} y_{t+s}$, we are done.

- (iii) The marginal propensity to consume is equal to $\frac{r}{1+r-\rho}$. The parameter ρ dictates how a change in current income transfers to a change permanent income. If $\rho = 1$, a unit increase in current income leads to a $\frac{1+r}{r}$ increase in permanent income, and the MPC is one (since the individual will consume the annuity value, $\frac{r}{1+r}$, of the permanent income increase). If ρ is zero, current income affect permanent income one-to-one, and the MPC is very small.

Question 5 Consider the following sequence problem

$$V(a_0, w_0) = \max_{\{c_t\}_{t=0}^{\infty}} \sum \beta^t u(c_t) \quad (7)$$

$$\text{subject to} \quad a_{t+1} + c_t = (1+r_t)a_t + w_t, \quad t = 0, 1, \dots \quad (8)$$

- (i) Assume that $r_t = r(x_t)$, $w_t = w(x_t)$, and that $x_{t+1} = (1-\rho)\bar{x} + \rho x_t$. What is the Bellman equation corresponding to the above sequence problem?
- (ii) Now assume for simplicity that $r_t = r$, (a constant), but that $w_{t+1} = \alpha w_t + \beta w_{t-2}$. What is the relevant (the smallest sufficient) state vector?
- (iii) What is your answer to (ii) if $w_{t+1} = \sum_{i=0}^N \gamma^i w_{t-i}$?
- (iv) How does your answer to (iii) change if $N = \infty$?

Answer

- (i) The Bellman equation is given by

$$v(a, x) = \max_{a'} \{u(a(1+r(x)) + w(x)) + \beta v(a', (1-\rho)\bar{x} + \rho x)\} \quad (9)$$

- (ii) The relevant state vector is (w, w_{-1}, w_{-2}) .

(iii) The answer is $(w, w_{-1}, w_{-2}, \dots, w_{-N})$.

(iv) Then a sufficient state variable is w . To see this, note that $w_{t+1} = \sum_{i=0}^{\infty} \gamma^i w_{t-i} = w_t + \gamma \sum_{i=0}^{\infty} \gamma^i w_{t-1-i} = (1 + \gamma)w_t$.