1 Answers to the Sept 08 macro prelim - Long Questions

1. Suppose that a representative consumer receives an endowment of a non-storable consumption good. The endowment evolves exogenously according to

$$\Delta \ln C_t = \mu + \rho \Delta \ln C_{t-1} + \sigma \varepsilon_t, \tag{1}$$

where Δ is the difference operator and ε_t is an $iid\ N(0,1)$ random variable. The consumer's preferences are

$$E_t \sum_{j=0}^{\infty} \beta^j \ln C_{t+j}.$$

(a) Solve recursively for the value of the endowment process. (I.e., write the Bellman equation and solve for the value function.)

(b) What is the marginal value of an increase in the mean growth rate of the endowment?

(c) What is the marginal value of a reduction in volatility, as measured by a decline in σ?

(d) Provide intuition about the relative magnitudes of the marginal values in (b) and (c).

ANSWER:

a. Begin by writing the endowment process as

$$\begin{aligned} \ln C_t - \ln C_{t-1} &= \mu + \rho(\ln C_{t-1} - \ln C_{t-2}) + \sigma \varepsilon_t, \\ \ln C_t &= \mu + (1+\rho)\ln C_{t-1} - \rho \ln C_{t-2} + \sigma \varepsilon_t. \end{aligned} \tag{2}$$

This can be expressed in companion form as

$$X_t = m + A X_{t-1} + \Omega \varepsilon_t, \tag{3}$$

where

$$X_t = \begin{bmatrix} \ln C_t \\ \ln C_{t-1} \end{bmatrix}, \quad m = \begin{bmatrix} \mu \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1+\rho & -\rho \\ 1 & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} \sigma \\ 0 \end{bmatrix}. \tag{4}$$

The state vector for this problem is X_t, and the Bellman equation is

$$V(X_t) = \ln C_t + \beta E_t V(X_{t+1}). \tag{5}$$

Conjecture that the value function is linear in X_t,[1]

$$V(X_t) = b_0 + b_1 X_t, \tag{6}$$

[1] This is obvious. It follows from the fact that expected utility is linear in expectations of $\ln C_{t+j}$ and that expectations of $\ln C_{t+j}$ are linear in X_t.

where b_0 is a scalar and b_1 is a row vector conformable with X_t. Substitute the guess into (5),

$$\begin{aligned} b_0 + b_1 X_t &= \ln C_t + \beta E_t(b_0 + b_1 X_{t+1}), \\ &= e_1 X_t + \beta[b_0 + b_1(m + AX_t)], \\ &= \beta(b_0 + b_1 m) + (e_1 + \beta b_1 A)X_t, \end{aligned} \tag{7}$$

where $e_1 = [1 \ \ 0]$. To solve for (b_0, b_1), equate powers of X_t on both sides of this equation.

$$\begin{aligned} b_0 &= \beta(b_0 + b_1 m), \\ b_1 &= e_1 + \beta b_1 A. \end{aligned} \tag{8}$$

Assuming the inverse exists,[2] the second condition can be solved to find

$$b_1 = e_1(I - \beta A)^{-1}. \tag{9}$$

Substitute this solution into the condition for b_0 to find

$$b_0 = \frac{e_1(I - \beta A)^{-1} m}{1 - \beta}. \tag{10}$$

Thus the value function can be expressed as

$$\begin{aligned} V(X_t) &= \frac{e_1(I - \beta A)^{-1} m}{1 - \beta} + e_1(I - \beta A)^{-1} X_t, \\ &= e_1(I - \beta A)^{-1}\left[\frac{m}{1 - \beta} + X_t\right]. \end{aligned} \tag{11}$$

For later use, define a^{ij} to be the $(i, j)th$ element of $(I - \beta A)^{-1}$. It follows that

$$e_1(I - \beta A)^{-1} = (a^{11} \quad a^{12}). \tag{12}$$

Hence the value function can also be expressed as

$$V(X_t) = \frac{a^{11}\mu}{1 - \beta} + a^{11} \ln C_t + a^{12} \ln C_{t-1}. \tag{13}$$

b. The average growth rate of consumption is $\bar{g} = \mu/(1 - \rho)$. Hence we can write the value function in terms of mean growth as

$$V(X_t) = \frac{a^{11}(1 - \rho)}{1 - \beta}\bar{g} + a^{11} \ln C_t + a^{12} \ln C_{t-1}. \tag{14}$$

Taking the derivative wrt $\bar{g}$ gives the marginal value of an increase in growth,

$$\frac{\partial V(X_t)}{\partial \bar{g}} = \frac{a^{11}(1 - \rho)}{1 - \beta} > 0. \tag{15}$$

[2] The eigenvalues of A are 1 and ρ. Since consumption growth is weakly autocorrelated in US data, $0 < \rho < 1$. Since $\beta < 1$, the eigenvalues of βA are both inside the unit circle.

c. Since $V(X_t)$ does not depend on σ, it follows that a reduction in volatility has a marginal value of $\partial V(X_t)/\partial\sigma = 0$.

d. Intuition: an increase in mean growth raises expected $\ln C$ in every future period. A reduction in volatility alters the conditional variance of future paths, but not the conditional mean. Hence it has no affect on expected utility. Roughly speaking, the effect of lower volatility is symmetric for future $\ln C$, sometimes increasing it and other times decreasing it. When taking expectations, the increases offset the decreases, leaving expected $\ln C$ unaffected.

6.Consider a variation of the Sidrauski monetary model with a constant population. Specifically, assume that the representative agent's maximize lifetime utility is given by:

$$\sum_{t=0}^{\infty} \beta^t \left[U(c_t) + V\left(\frac{M_t}{P_t}\right)\right]$$

where $U(\cdot)$ and $V(\cdot)$ are concave, twice-differentiable functions, c_t denotes consumption and M_t is money chosen in period t. Each period, agents use beginning of period nominal balances, the revenue from sales of output and a lump-sum monetary transfer to purchase consumption, investment and new money. In contrast to the Sidrauski model, both capital and money are used as inputs into the production process. Letting y_t denote output, the production function is given by:

$$y_t = \left(1 - z\left(\frac{M_t}{P_t}\right)\right) f(k_t)$$

where $z'(\cdot) < 0, z''(\cdot) > 0, z(0) = 1, \lim_{M_t/P_t \to \infty} z(M_t/P_t) = 0$. The function $f(k_t)$ has standard properties. The money supply in this economy is growing at the constant rate $\mu > 0$ and capital depreciates at the constant rate of $\delta < 1$.

(a) Derive and interpret the necessary conditions associated with the agent's maximization problem.

(b) Define a steady-state equilibrium in this economy. Contrast the effects of money growth in this setting with those obtained in the original Sidrauski model. (NOTE: It is NOT necessary to solve explicitly for $dc/d\mu$, $dk/d\mu$, etc. An intuitive argument is sufficient.)

Answer:

The maximization problem can be written as the following dynamic programming problem:

$$W(k_t, M_{t-1}) = \max \left\{ \begin{array}{c} U(c_t) + V\left(\frac{M_t}{P_t}\right) + \beta W(k_{t+1}, M_t) + \\ \lambda_t \left[f(k_t)\left(1 - z\left(\frac{M_t}{P_t}\right)\right) + \frac{M_{t-1}}{P_t} + \frac{T_t}{P_t} + k_t(1-\delta) - c_t - \frac{M_t}{P_t} - k_{t+1} \right] \end{array} \right\}$$

where the vector of states is (k_t, M_{t-1}) and T_t represents the lump-sum monetary transfer. After applying the envelope theorem, the necessary conditions are:

$$U'(c_t) = \beta \left\{ U'(c_{t+1}) \left[f'(k_{t+1}) \left(1 - z\left(\frac{M_{t+1}}{P_{t+1}}\right)\right) + 1 - \delta \right] \right\} \qquad (16)$$

$$U'(c_t)\left[f(k_t)\,z'\left(\frac{M_t}{P_t}\right)+1\right]=V'\left(\frac{M_t}{P_t}\right)+\beta U'(c_{t+1})\frac{P_t}{P_{t+1}} \tag{17}$$

Note that the marginal product of capital is affected by the quantity of real balances (eq.(16)), also, the term $f(k_t)\,z'\left(\frac{M_t}{P_t}\right)$ in the second expression reflects the marginal effect that money has on the output (capturing transacations costs). Recall that $z'(\cdot) < 0$.

A steady-state is defined by a constant level of consumption, capital and real balances that satisfy eqs. (16) and (17) and the resource constraint. In addition, a monetary steady state implies that inflation will be equal to the monetary growth rate. Define $\theta = 1/(1+\mu)$, these steady-state values are the solution to:

$$1=\beta\left[f'\left(\bar{k}\right)(1-z(\bar{m}))+1-\delta\right] \tag{18}$$

$$\frac{V'(\bar{m})}{U'(\bar{c})}=f\left(\bar{k}\right)z'(\bar{m})+1-\beta\theta \tag{19}$$

$$f\left(\bar{k}\right)(1-z(\bar{m}))=\bar{c}+\delta\bar{k} \tag{20}$$

From this it is apparent that this economy will not exhibit superneutrality: changes in θ (i.e. inversely related to the monetary growth rate) will affect the marginal product of capital and therefore the steady-state level of the capital stock. And this will influence the steady-state level of consumption. Taking the total differential of eq.(18) and rearranging yields (dropping the arguments of the functions for expositional purposes):

$$\frac{dk}{dm}=\frac{f'z'}{f''(1-z)}>0$$

Hence, real balances and capital will be positively related. This makes sense: if real balances fall, then the net MPK of capital falls. Consequently, the capital stock must fall to maintain equality of the net MPK and agents' discount factor and depreciation. We can then use this in the resource constraint, eq. (20). Taking the total differential of this expression we have:

$$\frac{dc}{dm}=\left[f'(1-z)-\delta\right]\frac{dk}{dm}-f'z'>0$$

The term in brackets is positive as can be seen from eq. (18). The assumption that $z' < 0$ and the previous result establishes that consumption real balances will be positively related. The final step is to solve for $\frac{dm}{d\theta}$. This can be done by taking the total differential of eq. (19) and then using the previous results. But things get messy algebraically and there is no general result. Typically, an increase in the monetary growth rate will cause a fall in steady-state real balances since the implied inflation tax increases. This story will hold in this economy but, as already demonstrated, capital and consumption will also fall. But it is also possible for real balances to increase if the marginal effect on production outweighs the implied inflation tax. In either situation, however, superneutrality (a feature in the basic Sidrauski setup) does not hold.

7.Consider a representative agent economy in which household preferences are given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln c_t + A \ln (1 - h_t)\right]$$

Each period, households supply labor h_t, and capital, k_t, to firms; the factor prices are denoted w_t and r_t,respectively. In addition, they choose consumption and investment every period. The depreciation rate of capital is 100%. Firms in the economy have technology given by:

$$y_t = z_t k_t^{\alpha} h_t^{1-\alpha}$$

where z_t denotes an $i.i.d.$ technology shock with $E(z_t) = 1$. Firms make input choices in order to maximize profits. Given this environment, do the following:

(a) Define a recursive competitive equilibrium.

(b) Solve for the equilibrium functions which determine consumption, capital, and labor.

(c) Suppose a new asset that entitles the owner to the stream of future consumption is introduced into this economy. That is, this asset is identical to equity whose dividend is consumption. Let q_t denote the price of this asset; solve for the equilibrium price of equity.

(d) Since capital chosen in period t, i.e. k_{t+1}, can be used to generate the future path of expected consumption, it would seem logical that $k_{t+1} = q_t$. Prove that this is not the case. Why?

(e) Suppose one used Hall's method to test the implications of the permanent income hypothesis within this economy; would the stochastic implications of the life-cycle hypothesis be supported? (Note: The analysis is simplified if all variables are in logs.)

ANSWER: The economy wide state variables are the current technology shock and the beginning of period capital stock, i.e. the pair (z_t, k_t). The individual's state variables include the economy wide state variables and their individual capital stock, i.e. the triplet (a_t, z_t, k_t) where a_t denotes the individual's capital stock. A recursive competitive equilibrium can then be defined by a set of functions: a value function that defines the household's maximization problem (I do not write this down - it is not necessary to provide this), a wage function, $w(z_t, k_t)$, a capital price function, $r(z_t, k_t)$, a set of decision rules for households: $c(a_t, z_t, k_t), h(a_t, z_t, k_t), a(a_t, z_t, k_t)$ and a corresponding set of aggregate per capita decision rules, $C(z_t, k_t), H(z_t, k_t), k(z_t, k_t)$ These functions must satisfy:

i. The household's problem (as defined by the value function.)

ii. The necessary conditions for firms which will imply that factor prices are equal to the respective factor marginal products.

iii. The consistency of individual and aggregate decisions: $c(k_t, z_t, k_t) = C(z_t, k_t), h(k_t, z_t, k_t) = H(z_t, k_t), a(k_t, z_t, k_t) = k(z_t, k_t)$. Note that market clearing implies that the individual's capital stock equals the economy-wide per-capita capital stock.

iv. The aggregate resource constraint: $z_t k_t^{\alpha} H(z_t, k_t)^{1-\alpha} = y(z_t, k_t) = C(z_t, k_t) + k(z_t, k_t)$.

b. We can solve for the competitive equilibrium by solving the corresponding social planner problem. This is given by the following dynamic programming problem (note that there is no time subscript on the expectations operator since the shocks are i.i.d.):

$$v(k_t, z_t) = \max_{(c_t, k_{t+1}, h_t)} \left[\ln c_t + A \ln(1-h_t) + \beta E\left[v(k_{t+1}, z_{t+1})\right]\right]$$

subject to the aggregate resource constraint:

$$z_t k_t^{\alpha} h_t^{1-\alpha} = y_t = c_t + k_{t+1}$$

The necessary conditions become:

$$\frac{1}{c_t} = \alpha\beta E\left[\frac{1}{c_{t+1}} z_{t+1} k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha}\right] = \alpha\beta E\left[\frac{y_{t+1}}{c_{t+1}}\right]\frac{1}{k_{t+1}}$$

or

$$\frac{k_{t+1}}{c_{t+1}} = \alpha\beta E\left[\frac{y_{t+1}}{c_{t+1}}\right] \tag{21}$$

$$\frac{A}{1-h_t} = (1-\alpha)\frac{1}{c_t} z_t k_t^{\alpha} h_t^{-\alpha}$$

We have frequently studied this economy and know that consumption and capital will be a constant fraction of output. To show this, make the conjecture that $c_t = \theta y_t, k_{t+1} = (1-\theta) y_t$ where the unknown is the term θ. Using this conjecture in eq. (21) yields $\theta = (1-\alpha\beta)$. That is

$$c_t = (1-\alpha\beta) y_t$$

$$k_{t+1} = \alpha\beta y_t$$

Using this result in eq. (21) implies that labor is constant (multiplying numerator and denominator by h_t):

$$\frac{h_t}{1-h_t} = \frac{(1-\alpha)}{A(1-\alpha\beta)}$$

c. and d. The equilibrium price of equity is determined by analyzing the associated necessary condition at market clearing quantities. The necessary condition associated with equity is:

$$q_t \frac{1}{c_t} = \beta E\left[\frac{1}{c_{t+1}}(q_{t+1} + c_{t+1})\right]$$

Given the functional form for preferences, it is reasonable to conjecture that equity will also be a constant fraction of output, i.e. $q_t = \omega y_t$ where the unknown is the parameter ω. Using this conjecture and the previous result for consumption yields $\omega = \frac{\beta(1-\alpha\beta)}{1-\beta}$ or

$$q_t = \frac{\beta(1-\alpha\beta)}{1-\beta} y_t$$

We had the earlier result that $k_{t+1} = \alpha\beta y_t$. It is easy to show that $q_t > k_{t+1}$ given that $\alpha < 1$. The reason is that owning capital entitles one to the MPK of capital - this additional output can then be consumed. It is easy to see the distinction by rewriting the relevant necessary conditions

$$equity : \frac{q_t}{c_t} = \beta E\left[\frac{q_{t+1}}{c_{t+1}} + 1\right]$$

$$capital : \frac{k_{t+1}}{c_t} = \alpha\beta E\left[\frac{k_{t+2}}{c_{t+1}} + 1\right]$$

e. First examine the implications for consumption. The solution to consumption implies

$$\begin{aligned}\ln c_{t+1} &= \ln(1-\alpha\beta) + \ln y_{t+1} = \\ &\ln(1-\alpha\beta) + \ln z_{t+1} + \alpha \ln k_{t+1} + (1-\alpha)\ln \bar{h}\end{aligned}$$

or, combining the constant terms and denoting these as A_0

$$\ln c_{t+1} = A_0 + \ln z_{t+1} + \alpha \ln k_{t+1} \tag{22}$$

But we know that $\ln k_{t+1} = \ln \alpha\beta + \ln y_t$ and $\ln c_t = \ln(1-\alpha\beta) + \ln y_t$. Combining these we have

$$\ln k_{t+1} = \ln \alpha\beta - \ln(1-\alpha\beta) + \ln c_t \tag{23}$$

Using eq. (23) in eq.(22) yields (where A_1 denotes the new constant terms):

$$\ln c_{t+1} = A_1 + \ln z_{t+1} + \alpha \ln c_t \tag{24}$$

Taking expectations (and assuming $E(\ln z_{t+1}) = 0$) yields

$$E[\ln c_{t+1}] = A_1 + \alpha \ln c_t$$

This is basically Hall's result. The greater implication is that adding additional variables into the regression of future consumption (in logs) on past consumption will not improve the prediction.