Duration of a Bond

Consider a pure discount bond with a face value of F and a maturity of N. Then the price of the bond is:

$$P_d = \frac{F}{(1+i)^N}$$

The elasticity of the bond price with respect to the interest rate is found by first taking logs and then taking the derivative:

$$\ln P_d = \ln F - N\ln(1+i)$$

Taking the derivative yields:

$$\frac{d\ln P_d}{di} = -N\frac{1}{(1+i)}$$

Identifying the derivative of the log as a percentage change (why is this appropriate?) and using discrete notation we have the following

$$\%\Delta P_d = -N\frac{\Delta i}{(1+i)}$$

Hence, the maturity of the pure discount bond directly determines the volatility of bond prices due to interest rate changes.

We use this insight to calculate the volatility of the price of any bond by first constructing the bond's duration. In calculating the duration of a bond, we convert the maturity of a given bond to the maturity of an equivalent zero coupon bond. Then the above formula determines the elasticity.

As an example, consider a coupon bond that has a coupon rate of r, a face (or par) value of F, and a maturity of N. The coupon payments are, therefore, rF; assume that these are received annually. Then the price of the bond is:

$$P_c = \frac{rF}{(1+i)} + \frac{rF}{(1+i)^2} + \frac{rF}{(1+i)^3} + \ldots + \frac{F(1+r)}{(1+i)^N}$$

Note that the value of each of the payments is equivalent to the price of a pure discount bond that promises a return of rF in period t (for $t < N$); for the final payment, its value equivalent to the price of a pure discount bond that pays out $F(1+r)$ dollars N periods from now.

Hence, we can construct a weighted average of the maturities of these different payments where the weights are the fraction of the total value of the payments, i.e. the price of the bond, P_c; this weighted average is the duration of the bond. For this example, it is calculated as:

$$D = \frac{\frac{rF}{(1+i)}}{P_c}(1) + \frac{\frac{rF}{(1+i)^2}}{P_c}(2) + \frac{\frac{rF}{(1+i)^3}}{P_c}(3) + \ldots + \frac{\frac{F(1+r)}{(1+i)^N}}{P_c}(N)$$

With the duration calculated, the elasticity of the bond price is given by:

$$\%\Delta P_c = -D\frac{\Delta i}{(1+i)}$$

In general, the formula for the duration of a bond is

$$D = \sum_{t=1}^{N} \frac{\frac{C_t}{(1+i)^t}}{P_b}(t)$$

where C_t denotes the payment received in period t and P_b is the price of the bond.

Here is a numerical example. Suppose we have following information: A two-year coupon bond withh semi-annual payments has a face (i.e. par) value of $1000 and a coupon rate of 10%. The current two-year interest rate is 8%. With this information, answer the following:

a. what is the price of the bond?
b. what is the duration of the bond?
c. Calculate the change in the bond if the interest rate falls to 6% (using the elasticity equation).

a. The bond pricing formula is:

$$P_c = \frac{50}{(1.04)} + \frac{50}{(1.04)^2} + \frac{50}{(1.04)^3} + \frac{1050}{(1.04)^4} = 1036.30$$

b. The duration of the bond is:

$$D = \frac{48.80}{1036.30}(1) + \frac{46.23}{1036.30}(2) + \frac{44.45}{1036.30}(3) + \frac{897.54}{1036.30}(4) = 3.73$$

Note that a period defined above is 6 months - so to convert to annual terms (since interest rates are quoted in annual terms) we divide by two. So the annual duration figure is

$$D = \frac{3.73}{2} = 1.87 \ \mathit{years}$$

c. Using the elasticity formula:

$$\%\Delta P_c = -1.87\frac{(-0.02)}{1.08} = 0.035$$

hence the change in the bond price is

$$\Delta P_c = (0.035)(1036.30) = 36.27$$