

# Freedom as Control

Itai Sher\*

April 18, 2016

## Abstract

I present a model of freedom as *control*. Control is measured by the preferences of a decision-maker, or judge, who values flexibility and is *neutral* towards outcomes ex ante. Formally, I explore the consequences of adding a neutrality axiom to the Dekel, Lipman and Rustichini (2001) axioms for preference for flexibility. I characterize the consensus of all neutral judges about which choice situations embody more freedom. The theory extends the freedom ranking literature to situations where agents have imperfect control, as modeled by choices among lotteries. In a voting context, the consensus of neutral judges coincides with Banzhaf power.

---

\*Department of Economics, University of California, San Diego.  
email: itaisher@gmail.com

# 1 Introduction

Freedom is a complex and important value. Freedom is one of the central virtues that has been attributed to market exchange, and it has been a central concern of political philosophy. Freedom is linked to wealth: Expansion of wealth may be thought of as promoting freedom. As wealth increases – either the wealth of an individual or the wealth of society as a whole – it is often the case that the scope for control increases as well. But as Sen (2001) has emphasized, the expansion of freedom goes beyond the expansion of wealth.

This paper relates two conceptions of freedom: *neutral freedom* and *freedom as control*. In particular, I am concerned with these conceptions as they apply to the assessment of freedom in decisions and institutions. This question may be viewed as an evaluative question or as a non-evaluative question of measurement. This paper treats the question as a question of measurement with evaluative presuppositions.<sup>1,2</sup> This is discussed further below.

The idea behind neutral freedom is that, in assessing freedom, we should – at least to some extent – remain neutral about the value of the alternatives.<sup>3</sup> The idea behind freedom as control is that, in assessing freedom, we should be primarily concerned with how much control the decision maker has over the outcome. These two ideas are intimately related.

The intuitive idea behind freedom as control is that there is a distinction between the value of what I can achieve in virtue of being free and the value of being free per se. Freedom as control attempts to capture the latter. This idea, while intuitively appealing, is difficult to grasp and potentially problematic. How do we measure the control that an agent has when confronted with a menu of options?

This paper suggests a way of thinking that links the ideas of neutral freedom and freedom as control. The idea is that we measure the control

---

<sup>1</sup>Not only does the measurement of freedom have evaluative presuppositions, but the specific value judgments we make affect how we measure freedom.

<sup>2</sup>For an opposing view, see Carter (1999).

<sup>3</sup>To say that we should be neutral about the value of alternatives is not to say that no value judgments are involved. See below and Section A. Neutrality is usually conditioned on certain normative presuppositions.

inherent in a menu by the value that the menu would have to an agent who does not yet know what she will want to choose from the menu *and whose uncertainty is symmetric with respect to different options*. This way of thinking is especially interesting when we think about imperfect control. That is to say: the menu does not contain options that allow the agent to exactly determine the outcome, but only to influence the outcome. This is modeled by assuming that the menu contains lotteries over the outcomes.

For interpretive purposes, it is useful to think not in terms of a single agent who evaluates menus at one date and then chooses from a menu at a later date, but in terms of two, potentially distinct, agents: a *judge* who evaluates the menu today, and a *decision-maker* who chooses from the menu tomorrow. A little more formally:

1. A benevolent judge decides to adopt a neutral stance toward a set of alternatives. This neutrality is represented as symmetric uncertainty about the value alternatives will have to a decision-maker who will choose among them tomorrow.
2. The judge evaluates menus of lotteries over the alternatives today by his current expectation of the value that the menus will have to a decision-maker tomorrow given that the decision-maker will choose optimally tomorrow.

Several notes are in order. First, the judge may or may not be the decision-maker. This leaves open several interpretations: The assessment of freedom may reflect a social judgment about freedom, or a personal judgment of an agent about her own freedom. I have written as if there is only one decision-maker but, under a variant of the first social interpretation, there may be many possible decision-makers with different tastes. Tomorrow one of these decision makers will be chosen at random. In this case, the judge implicitly makes interpersonal comparisons among the decision-makers subject to the constraint that the judge's attitude today toward the different *alternatives* – but not necessarily the decision-makers – is symmetric.

Second, the first step involves a value judgment. The judge *decides* to adopt a neutral stance. Such a decision may be justified in some contexts,

such as when evaluating the basic pursuits that individuals choose to undertake, when each pursuit is attractive but they are difficult to compare. It may be unjustified in other cases when there are asymmetries among the alternatives that clearly make one alternative superior to the other. The value judgment is that for the purpose at hand, no alternative should be privileged over any other. Treating the alternatives neutrally in this way allows us to develop a measure that does not depend asymmetrically on the value of any specific alternative. This is what allows us to capture the notion of control. It is also what links neutrality and control.

Third, ultimately, I treat freedom in an axiomatic manner. The above story is a representation of the axioms. The axioms are based on axioms for preference for flexibility due to Dekel, Lipman and Rustichini (2001).<sup>4</sup> Arrow (1995) suggested that freedom be modeled in terms of preference for flexibility, but he did not make the link to neutrality. Dekel et al. (2001) interpret the axioms as applying to the *preferences* of a decision-maker over menus under a conception of preferences such that preferences can ultimately be reduced to choice behavior. I view the evaluation of freedom as a normative exercise, and I interpret the axioms as applying primarily to the *judgments* of someone making assessments of *freedom*. I argue that the axioms are also justified from this point of view. I also add a neutrality axiom that imposes the neutral attitude adopted in step 1 above. Much of the analysis concerns the consequences of this neutrality axiom. I do not just explore the attitudes of a single neutral judge, but also the *consensus* of all neutral judges, each of whom takes a different but symmetric view of the uncertainty about the value of alternatives.

Why is neutrality important? Why should judgments of freedom not be sensitive to what one can achieve with one's freedom? Often, judgements of freedom should be weighed according to the substantive nature of what one is able to achieve in such a way that we should not be neutral towards outcomes. However, there are also important cases where we would like to withhold judgment in assessing freedom. One of the goals of the state is to promote the freedom of its citizens. In pursuing this goal, it is natural to think that

---

<sup>4</sup>A previous axiomatization for deterministic menus was presented by Kreps (1979).

the state ought to be neutral with regard to citizens' choices, at least in many cases. For example, I ought to be able to choose my goals in life, my religious beliefs, what I say, and which candidates I vote for. The state should not adopt a view that induces it to attempt to influence these decisions, nor in general should the state base its decisions on the promotion of goals or doctrines that are too partisan. The doctrine of political liberalism takes this stance (see, e.g., Rawls (2005)). Dworkin (1985) describes liberalism as the doctrine that "political decisions must be, so far as possible, independent of any particular theory of the good life, or of what gives value to life. Since the citizens of a society differ in their conceptions, the government does not treat them as equals if it prefers one conception to another, either because one is held by a more numerous or more powerful group." The issues involved go beyond the assessment of freedom, but the motivation for neutrality in such assessment is similar.<sup>5</sup>

Paradigmatic arguments for neutrality apply to the political process. In expressing our political opinions and in voting for candidates, it is viewed as important that we should be able to support whatever candidates and positions we wish. When they do not conflict with rights and constitutional principles, the formal electoral process should not treat positions differently on the basis of their content. The formal voting method should not build in a bias against specific candidates on the basis of their views. Ballot access should not be based on the content of the issue on which we vote. The freedom of political expression should not be conditional on the views expressed. This paper applies the measure of freedom it develops to voting institutions (see Sections 5.1 and 6) and also to the impact of information on freedom (see Section 5.2). Whereas the freedom ranking literature (see Barbera, Bossert and Pattanaik (2004)) focuses on menus that allow the agent to determine outcomes, the current paper studies stochastic choice in which the outcome is only imperfectly controlled. This allows me to link my freedom measure to Banzhaf power in the case of voting (see Propositions 6 and 15) and to

---

<sup>5</sup>Interestingly, in the essay quoted above, Dworkin expresses skepticism about the possibility of measuring liberty, writing, "we do not have a concept of liberty that is quantifiable ... ." (Dworkin (1985), p. 189.)

the value of information – and specifically Blackwell’s order (see Proposition 7).

The outline of the paper is as follows. Section 2 presents the preference for flexibility axioms, and argues that they are compelling for freedom. Section 3 introduces a neutrality axiom and explores some of its consequences for ranking choice situations. A key point is that the combination of neutrality and the preference for flexibility axioms do not pin down the ranking over menus. This is significant because it is often thought that neutrality and related concepts such as impartiality, equal treatment, and universalizability in and of themselves determine normative judgments. While these concepts may impose constraints, they often leave much room for additional judgments; this paper illustrates this point in a precise setting. Section 4 provides some characterizations, and specifically presents an order – the *grading order* – that captures the consensus of all neutral judges who value flexibility. The grading order embodies precisely what is *implied* by the axioms, as opposed to what is merely consistent with them, as in a standard representation theorem. Section 5 applies the grading order to two situations described above: voting institutions and information. Section 6 shows how the neutrality axiom can be weakened to accommodate the possibility that not all alternatives are equally good substitutes for one another.

The concept of neutrality that is employed here is philosophically and technically contentious. Appendix A discusses philosophical objections both to the concept of neutrality and to the particular formalization of it found here. Appendix B provides proofs of propositions and discusses technical issues.

In closing, I mention a few distinctions that are important when discussing freedom. The philosophical literature on freedom contains a distinction between between specific and non-specific freedom, which is characterized by Carter (1999) as follows, “When we think of freedom in a specific way, we have in mind the freedom to do a specific thing or set of things. When we think of freedom in a non-specific way, we have in mind freedom as a quantitative attribute – as something an agent has more or less of in an overall sense – without concentrating on any one specific thing that the agent is free

to do.” Neutral freedom and freedom as control are in some ways related to non-specific freedom, but I am hesitant to associate them with non-specific freedom as developed by Carter because Carter attaches to non-specific freedom many philosophical theses that do not apply to the notions proposed here. Next is the distinction between measurement of freedom and evaluation of freedom. Does this paper attempt to measure or value freedom? The freedom orders that I study attempt to measure freedom, but I also assume that the more freedom one has, the more valuable one’s choice set is with respect to freedom. For this reason, the orders I study can be thought of as both measuring and valuing freedom. There is an analogy to the measurement of welfare: A measurement of welfare can also give its value.<sup>6</sup> Nor does the fact that the measure is neutral toward alternatives mean that it does not presuppose or support any value judgments. It is one thing to adopt a stance of neutrality towards alternatives, and it is another not to make any value judgements, such as, e.g., that it is appropriate to be neutral about the alternatives at hand, or that more control is better. Of course, as freedom is a complex value, we are only measuring and valuing one aspect of freedom.

A final distinction is that between the intrinsic and the instrumental value of freedom. I write about this distinction at length in Sher (2015), which employs a closely related, but distinct model.<sup>7</sup> Concerning the model of the current paper, the preference for flexibility story suggests an instrumental assessment of freedom, but this must be qualified by the neutrality axiom, which imposes a neutral stance that is not grounded in instrumental considerations.

---

<sup>6</sup>In a broader context where there are values other than freedom, and we are trying to measure the contribution of freedom to overall value, there might appear to be more of an argument for separating the value of freedom from its quantity. However, even in such a setting, the “quantity” that enters into the overall evaluation may be influenced in its measurement by considerations of value; this is analogous to the influence of one agent’s utility on a social welfare function; the measurement of the individual’s utility is itself value-laden.

<sup>7</sup>The two papers use related tools. They focus on different but overlapping aspects of freedom. It would be an interesting task to integrate the models and considerations of the papers, but this must wait for a future occasion.

## 2 The Preference for Flexibility Framework

My framework builds on the axiomatization of preference for flexibility due to Dekel et al. (2001), and a variant due to Kochov (2007) that relaxes the completeness axiom. Whereas Dekel et al. (2001) view their axioms as characterizing the preferences of an agent who values flexibility, I argue that these axioms should apply also when evaluating the freedom inherent in a decision. The evaluation does not necessarily represent the preferences of the agent who chooses from the menu, but may reflect the judgement of an “ethical observer” or it may be a summary of social attitudes about the value of freedom. The evaluation may apply to the freedom of a particular agent, or alternatively to *any* agent who might face various choices. For instance, the judgement that a prison sentence deprives an agent of much of his freedom does not apply to a particular agent, but rather applies generically to any agent receiving such a sentence.

### 2.1 Menus

Let  $Z = \{1, \dots, n\}$  be a finite set of outcomes.  $\Delta(Z)$  is the set of lotteries on  $Z$ . A lottery is expressed as  $\beta = (\beta_i : i \in Z)$ , where  $\beta_i$  is the probability of outcome  $i$ . I sometimes express a lottery as a list of probabilities. For example, when  $n = 3$ , I may write  $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$  for the lottery that puts probability  $\frac{1}{2}$  on outcome 1,  $\frac{1}{3}$  on outcome 2, and  $\frac{1}{6}$  on outcome 3. A **menu**  $M$  is a closed subset of  $\Delta(Z)$ .  $\mathcal{M}$  is the set of all menus. Endow  $\mathcal{M}$  with the metric topology generated by the Hausdorff distance. We will study an order  $\preceq$  on  $\mathcal{M}$  representing the freedom that different menus allow.

Intuitively,  $M$  is a set of alternatives faced by an agent or population of agents. We would like to evaluate the freedom of choice inherent in such a menu.  $M \preceq M'$  will mean that  $M'$  allows at least as much freedom as  $M$ . In conceptualizing freedom is this way, I follow the freedom ranking literature. (For surveys, see Barbera et al. (2004) and Dowding and van Hees (2009).) I depart from most of this literature by considering menus of lotteries rather

than deterministic menus.<sup>8</sup> This is essential insofar as, in reality, we tend to control outcomes only imperfectly; by our choices, we can bring it about that certain outcomes that we care about are more or less likely, but we can rarely, if ever, *guarantee* a certain outcome. This is essential to the applications to voting institutions and information presented in Section 5.

## 2.2 Order Axioms

First, I consider two potential axioms characterizing the nature of the order  $\preceq$ :

**Axiom 1** (*preorder*)  $\preceq$  is transitive and reflexive.

**Axiom 2** (*weak order*)  $\preceq$  is transitive and complete.

As completeness implies reflexivity, the difference between these axioms amounts to the question of whether every pair of menus is comparable. If the purpose of our axioms had been to characterize choice over menus, then completeness would be a relatively more natural, since for any pair of menus,  $M$  and  $M'$ , an agent might be faced with the choice between them, and in such a situation may have no choice but to choose.<sup>9</sup> Our goal, however, is to rank menus in terms of the freedom that they provide, and it seems perfectly legitimate that in some cases we will not be able to decide which menu provides more freedom. Indeed allowing incomparability is essential to the main result of this paper (see Section 4.2.2).

## 2.3 The DLR Axioms

I present variants of the Dekel, Lipman and Rustichini (2001) axioms for preference for flexibility.<sup>10</sup> I argue that these axioms are appropriate also for

---

<sup>8</sup>Sher (2015) is another paper that studies freedom in a menus of lotteries framework.

<sup>9</sup>Even in a choice context, there are compelling grounds for rejecting completeness.

<sup>10</sup>The monotonicity axiom is the same as in Dekel, Lipman and Rustichini (2001), but the continuity and independence axioms differ. The independence axiom of Dekel, Lipman and Rustichini (2001) differs in that it (i) is formulated with respect to strict preference and (ii) contains only the  $\Rightarrow$  direction of Axiom 5 below. The formulations presented here match those of Kochov (2007) and Galaabaatar (2010), who study preference for flexibility without completeness.

a freedom order.

**Axiom 3 (*Strong Continuity*)** For any convergent sequences  $\{M_i\}, \{N_i\}$  in  $\mathcal{M}$ ,  $(M_i \rightarrow M$  and  $N_i \rightarrow N$  and  $[\forall i, M_i \preceq N_i]) \Rightarrow M \preceq N$ .

This axiom, while natural, should be viewed as a technical assumption.

**Axiom 4 (*monotonicity*)**  $\forall M, N \in \mathcal{M}, M \subseteq N \Rightarrow M \preceq N$ .

In our setting, this axiom means that adding alternatives increases freedom.

For any menus  $M, L \in \mathcal{M}$  and  $\lambda \in [0, 1]$ , define

$$\lambda M + (1 - \lambda)L := \{\lambda\beta + (1 - \lambda)\gamma : \beta \in M, \gamma \in L\}. \quad (1)$$

Thus we can “mix” two menus by mixing every pair of selections from the two menus.

**Axiom 5 (*Independence*)**

$$\forall \lambda \in (0, 1), \forall M, N \in \mathcal{M}, \quad M \preceq N \Leftrightarrow \lambda M + (1 - \lambda)L \preceq \lambda N + (1 - \lambda)L.$$

I now provide a justification for the independence axiom as an axiom for evaluating the freedom inherent in choice situations. Broadly, the structure of the justification is similar to the one given by Dekel et al. (2001). However, Dekel et al. (2001) argue that an optimizing agent who evaluates menus ex ante will obey independence; in contrast, I argue that the evaluation of choice situation in terms of the freedom they provide should obey independence.

Let the  $(\lambda, M, L)$ -**random menu scenario (RMS)** be the situation where with probability  $\lambda$ , an agent faces menu  $M$  and with probability  $1 - \lambda$ , the agent faces menu  $L$ . A **strategy** in the  $(\lambda, M, L)$ -RMS is a function  $\sigma : \{M, L\} \rightarrow \Delta(Z)$  such that  $\sigma(M) \in M$  and  $\sigma(L) \in L$ . So a strategy specifies which lottery the agent will choose if she faces  $M$  and which lottery she will choose if she faces  $L$ . The argument for independence consists of two premises: (i) RMS  $(\lambda_1, M_1, L_1)$  provides more freedom than RMS  $(\lambda_2, M_2, L_2)$  if and only if menu  $\lambda_1 M_1 + (1 - \lambda_1)L_1$  provides more freedom than menu  $\lambda_2 M_2 + (1 - \lambda_2)L_2$ . Supporting this premise is the observation that the set of lotteries the agent can generate in the  $(\lambda, M, L)$ -RMS by varying her strategy

is precisely  $\lambda M + (1 - \lambda)L$ .<sup>11</sup> (ii) The comparison of freedom in the  $(\lambda, M, L)$  and  $(\lambda, N, L)$  RMS's should be determined by the situation in which these two scenarios differ; that is, comparing these scenarios amounts to comparing  $M$  and  $N$ , for when menu  $L$  materializes, the agent's freedom is the same in  $(\lambda, M, L)$  as in  $(\lambda, N, L)$ . For example, define a *prison scenario*  $X$  to be: With probability  $\frac{1}{3}$ , I will go to prison, and with probability  $\frac{2}{3}$ , I will go to and live in country  $X$ . We represent this as a certain RMS. Then, intuitively, my freedom is greater in prison scenario  $A$  than in prison scenario  $B$  if and only if my freedom is greater in country  $A$  than in country  $B$ . Putting (i) and (ii) together justifies independence.

## 2.4 The DLR and Kochov Representation Theorems

Call an order  $\preceq$  satisfying Axioms 1 and 3-5 a **freedom preorder** and an order satisfying Axioms 2-5 a **freedom weak order**. Let  $U = \mathbb{R}^Z$  be the set of utility functions on  $Z$ . For any  $u = (u_z : z \in Z) \in U$ , the  $z$ -component of  $u$ ,  $u_z$ , is the utility of outcome  $z$ .  $\Delta^*(U)$  is the set of probability measures on  $U$  with compact support.

**Proposition 1** (*Dekel, Lipman, Rustichini 2001, Kochov 2007*)<sup>12</sup>  $\preceq$  is a freedom preorder if and only if there exists a closed and convex set of probability measures  $P \subseteq \Delta^*(U)$  such that:

$$M \preceq M' \Leftrightarrow \left[ \int_U \max_{\beta \in M} (u \cdot \beta) \mu(du) \leq \int_U \max_{\beta \in M'} (u \cdot \beta) \mu(du), \forall \mu \in P \right], \quad (2)$$

$$\forall M, M' \in \mathcal{M}.$$

---

<sup>11</sup>This argument assumes that the agent uses a pure strategy in the RMS. A similar argument applies if we allow the agent to randomize both in the RMS and when facing a menu.

<sup>12</sup>A structurally transparent proof of the first statement in Proposition 1 based on a result of Dubra, Maccheroni and Ok (2004) is due to Galaabaatar (2010) and can be found at [http://economics.uwo.ca/about\\_us/Workshops/theory\\_docs/galaabaatar\\_apr14.pdf](http://economics.uwo.ca/about_us/Workshops/theory_docs/galaabaatar_apr14.pdf) Kochov's original proof can be found at <http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.164.647&rep=rep1&type=pdf> The second statement in Proposition 1 differs from Dekel et al. (2001) in minor ways: (i) Dekel et al. (2001) use a different continuity axiom. However, as observed by Kochov (2007), these axioms are equivalent for complete preferences. (ii) Dekel et al. (2001) use a weaker independence axiom.

$\preceq$  is a freedom weak order if and only if  $\preceq$  satisfies (2) for some singleton set  $P = \{\mu\}$  (with  $\mu \in \Delta^*(U)$ )

The first statement assuming a preorder is due to Kochov (2007), and the second assuming a weak order to Dekel et al. (2001). I now interpret this result. Suppose we accept that the axioms for a freedom weak order are compelling for freedom. Then Proposition 1 tells us that we can construct  $\preceq$  according to the following scenario: There exists some (hypothetical) agent whose utility function is chosen according to some probability measure  $\mu$ . At date 0, the agent does not yet know her utility function. She knows that at date 1 her utility function will be chosen according to  $\mu$ . She must evaluate menus  $M$  at date 0 on the assumption that at date 1, she will choose optimally from  $M$  according to whichever date 1 utility function materializes. She therefore evaluates menus ex ante according to their date 1 expected indirect utility. Arrow (1995) proposed that freedom should be understood as analogous to preference for flexibility: We do not know today what we will want tomorrow, and the value of freedom is precisely that it allows us to adapt our choices to whatever information and contingencies will arise at the time of choice.

It is important to note, however, that nothing we have assumed so far formally implies that the hypothetical agent guaranteed by Proposition 1 – if we assume the weak order axiom – coincides with the actual agent facing the choice. Indeed, one consistent interpretation of the axioms is as characterizing the freedom inherent in a choice set independently of which agent faces the choice set. Different agents who face the choice set may have different preferences. In this case, the hypothetical “as if” agent guaranteed by the proposition may not coincide with any of the actual agent who faces the menu. The hypothetical agent may represent a judge who aggregates the distribution of preferences in the population (see Sugden (1998)) or some normatively motivated re-weighting of this distribution. Suppose next that we think that the completeness axiom is too restrictive: The freedom inherent in some pairs of decision problems is incomparable. Suppose however that we believe that the other freedom axioms are compelling so that we have a freedom preorder. Then Proposition 1 tells us that there is a collection of

hypothetical agents, each of whom value flexibility, such that menu  $M$  offers more freedom than menu  $M'$  if it is consensus among these agents that menu  $M$  offers more valuable flexibility than menu  $M'$ .

In the sequel, the following terminology will be useful. A **representation** of a weak order  $\preceq$  on  $\mathcal{M}$  is a function  $\nu : \mathcal{M} \rightarrow \mathbb{R}$  such that  $M_1 \preceq M_2 \Leftrightarrow \nu(M_1) \leq \nu(M_2)$ ,  $\forall M_1, M_2 \in \mathcal{M}$ . A representation  $\nu$  of weak order  $\preceq$  is a **cardinal representation** if:<sup>13</sup>

$$\begin{aligned} \nu(\alpha M_1 + (1 - \alpha)M_2) &= \alpha\nu(M_1) + (1 - \alpha)\nu(M_2), \\ \forall M_1, M_2 \in \mathcal{M}, \forall \alpha \in [0, 1]. \end{aligned} \tag{3}$$

Consider a probability measure  $p$  over menus that selects menu  $M_1$  with probability  $\alpha$  and  $M_2$  with probability  $1 - \alpha$ . If  $\nu$  is a cardinal representation, then the mixed menu  $\alpha M_1 + (1 - \alpha)M_2$  is equal to the expected value of  $\nu$  with respect to  $p$ . If both  $\nu$  and  $\nu'$  are cardinal representations of  $\preceq$ , then  $\nu'$  is a positive affine transformation of  $\nu$ . Proposition 1 implies that: *If  $\preceq$  is a freedom weak order, then  $\preceq$  has a cardinal representation*, because if  $\nu(M) = \int_U \max_{\beta \in M}(u \cdot \beta) \mu(du)$ , then  $\nu$  satisfies (3).

### 3 Neutrality

This section introduces a neutrality axiom. A central contribution of this paper is to study the consequences of this axiom in the preference for flexibility framework.

I start with a **motivating example**.  $D_1$  and  $D_2$  are decision problems. In  $D_1$ , the agent has no choice: Outcome 1 is implemented. In  $D_2$ , with probability  $\alpha$ , the agent chooses an alternative from  $\{1, 2\}$ , and with probability  $1 - \alpha$ , outcome 2 is implemented. Let  $\delta_1$  and  $\delta_2$  be, respectively, the degenerate lotteries that put probability 1 on outcomes 1 and 2. Then  $D_1$  and  $D_2$  correspond, respectively, to the menus  $M_1 = \{\delta_1\}$  and

---

<sup>13</sup>Functions  $\nu$  satisfying (3) are often referred to as *linear* functions, or more properly, *affine* functions. I refer to representations  $\nu$  satisfying (3) as cardinal representations to distinguish them from their monotone transformations which remain representations of the given preferences but fail to satisfy (3).

$M_2 = \alpha \{\delta_1, \delta_2\} + (1 - \alpha) \{\delta_2\} = \{\alpha\delta_1 + (1 - \alpha)\delta_2, \delta_2\}$ . An agent with preference for flexibility may prefer  $D_1$  to  $D_2$ . Such an agent evaluates menus at date 0 and chooses from the menu at date 1 (see Section 2.4). Proposition 1 places no constraint on date 0 uncertainty as represented by  $\mu$ . So, an agent who, at date 0, is certain of her date 1 preferences, is a limiting degenerate case of an agent with preference for flexibility. At date 0, if the agent *knows* that at date 1, she will prefer outcome 1 to outcome 2, she will prefer  $D_1$  to  $D_2$ . However,  $D_1$  allows *no* scope for choice, so that intuitively,  $D_2$  grants the agent more freedom than  $D_1$ . The disparity is especially large when  $\alpha$  is close to 1. So, at least without further restrictions on date 0 beliefs, the preference for flexibility model does not provide an adequate measure of freedom as control. Ideally, we would like to separate the *scope for choice* in a menu from the *idiosyncratic preferences* for one alternative as opposed to another. Below, I will present an axiom that attempts to achieve this separation.

### 3.1 The Neutrality Axiom

A **permutation**  $\pi$  of  $Z$  is a bijection  $\pi : Z \rightarrow Z$ . Thus, a permutation assigns to every outcome  $z \in Z$  some other (or possibly the same) outcome  $z' \in Z$ . Let  $\Pi$  be the set of permutations on  $Z$ . For each  $\beta \in \Delta(Z)$ , and  $\pi \in \Pi$ , define  $\beta^\pi \in \Delta(Z)$  by:

$$\beta_z^\pi = \beta_{\pi(z)}, \quad \forall \pi \in \Pi, \forall z \in Z.$$

So  $\beta^\pi$  is the lottery over outcomes that results from permuting the probability of outcomes according to  $\pi$ . For example if  $Z = \{1, 2, 3\}$ ,  $\beta = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ , and  $\pi(1) = 2, \pi(2) = 3$ , and  $\pi(3) = 1$ , then  $\beta^\pi = (\frac{1}{3}, \frac{1}{6}, \frac{1}{2})$ . The menu  $M^\pi$  is formed by permuting each outcome contained in  $M$  according to  $\pi$ . That is:

$$M^\pi = \{\beta^\pi : \beta \in M\}.$$

A neutral measure of freedom is derived by adding the following axiom to the preference for flexibility framework:

**Axiom 6 (Neutrality)**  $M \sim M^\pi, \quad \forall M \in \mathcal{M}, \forall \pi \in \Pi.$

The axiom asserts indifference between a menu and any permutation of that menu. In other words, the evaluation of the menu is blind to the identity of the particular items that feature in the menu's lotteries, and so depends only on the "structure" of the menu. The evaluation is nonjudgmental in that it treats all outcomes as equivalent. The axiom implies that outcomes are viewed not only as equally good, but also that for all outcomes  $x, y$ , and  $z$ , no outcome  $x$  is viewed as a better substitute for  $y$  than for  $z$ . A relaxation of this aspect of the axiom is considered in Section 6. Call an order  $\preceq$  satisfying Axioms 1 and 3-6 a **neutral freedom preorder (NFP)** and an order  $\preceq$  satisfying Axioms 2-6 a **neutral freedom weak order (NFW)**.

### 3.2 Disagreements among Neutral Freedom Orders

I now show that neutrality does not pin down the ranking of menus in terms of freedom, and explore the degree to which different neutral freedom orders can disagree.

For any outcome  $z \in Z = \{1, \dots, n\}$ , let  $\delta_z$  be the degenerate lottery that selects outcome  $z$  with probability 1. Let  $\mathcal{M}_d$  be the set of menus that contain only such degenerate lotteries. A menu  $M \in \mathcal{M}_d$  is a **deterministic menu**. For any deterministic menu  $M$ ,  $|M|$  is the **cardinality** of  $M$ , or, in other words, the number of items in  $M$ . In general, for any set  $S$ ,  $|S|$  will be the cardinality of  $S$ .

**Proposition 2 (Ordinal Properties of NFP's)**

1. Any NFP  $\preceq$  induces a complete order on the set of deterministic menus; that is:  $M \preceq M'$  or  $M' \preceq M, \quad \forall M, M' \in \mathcal{M}_d.$
2. Let  $\preceq$  be an NFP. Then for all deterministic menus  $M_1$  and  $M_2$ :  
 $|M_1| \leq |M_2| \Rightarrow M_1 \preceq M_2.$
3. For all  $m \leq n$ , there exists an NFP  $\preceq$  such that for all  $M_1, M_2 \in \mathcal{M}_d$ ,  
(i)  $|M_1| < |M_2| \leq m \Rightarrow M_1 \prec M_2$ , and (ii)  $|M_1| = m < |M_2| \Rightarrow M_1 \sim M_2.$

Part 1 shows that every NFP completely orders the deterministic menus. This is a bit surprising as NFP's do not generally completely order the set of all menus, and there is nothing explicit in the axiomatization of an NFP about completeness. Part 2 shows that NFP's always *weakly* prefer larger deterministic menus to smaller ones. So NFP's are always indifferent between any pair of menus of the same cardinality. But NFP's need not coincide with the cardinality order for deterministic menus: The incremental value of an additional item may go to zero after some threshold (Part 3).

We now turn to cardinal properties of NFW's. Define  $[i] := \{1, 2, \dots, i\}$ . So  $[n] = \{1, \dots, n\}$ . Define:

$$V = \left\{ v \in \mathbb{R}^n : \sum_{j=0}^i (-1)^{i-j} \binom{i}{j} \Delta v(n-j) \geq 0, \quad \text{for } i = 0, \dots, n-2 \right\}. \quad (4)$$

where for any  $v = (v(i) : i \in [n]) \in \mathbb{R}^n$ ,  $\Delta v(i) := v(i) - v(i-1)$  for  $i = 2, \dots, n$ . So  $V$  can be thought of as a set of functions on  $[n]$  satisfying certain linear inequalities. When  $n = 3$ ,  $V$  imposes two linear inequalities, which can be written as:

$$0 \leq \Delta v(3) \leq \Delta v(2). \quad (5)$$

In the Appendix, I show that this relation can be generalized to arbitrary  $n$ :

$$\forall v \in V, \quad 0 \leq \Delta v(n) \text{ and } \Delta v(i+1) \leq \Delta v(i), \quad \forall i \in \{2, \dots, n-1\}. \quad (6)$$

(6) combines the condition that  $v$  is monotonically increasing with a discrete concavity assumption. Monotonicity and discrete concavity do not exhaust the content of  $V$ . When  $n = 4$ ,  $V$  imposes the inequalities: (i)  $0 \leq \Delta v(4) \leq \Delta v(3)$  and (ii)  $2\Delta v(3) \leq \Delta v(4) + \Delta v(2)$ . If  $v(1) = 0, v(2) = 1$ , and  $v(3) = v(4) = 2$ ,  $v$  satisfies (6) but not (ii).

**Proposition 3 (Cardinal Properties of NFW's)** *Let  $\nu$  be a cardinal representation of an NFW  $\preceq$ . Then:*

$$\exists v \in V, \quad \forall M \in \mathcal{M}_d, \quad \nu(M) = v(|M|). \quad (7)$$

*Conversely, if  $\nu : \mathcal{M}_d \rightarrow \mathbb{R}$  satisfies (7), then  $\nu$  can be extended to a function  $\nu : \mathcal{M} \rightarrow \mathbb{R}$ , such that  $\nu$  is a cardinal representation of an NFW  $\preceq$ .*

The proof appeals in important ways to results due to Nehring (1999), as is explained in the Appendix.<sup>14</sup> Whereas Proposition 2 shows that ordinally there is a strong relation between NFP's and ranking choice sets by cardinality (as axiomatized by Pattanaik and Xu (1990)), Proposition 3 shows that, cardinally, the story is more involved: Although different alternatives enter symmetrically into the value function, the incremental value of an alternative may be diminishing in the number of alternatives; I elaborate what this means below. So alternatives may substitute for one another in interesting ways as a function of the size of the choice set.

Proposition 3 brings a consequence of Proposition 2 into sharper relief. Part 3 of Proposition 2 showed that the same comparison of menus can be a strict preference for one NFP and an indifference for another. For stochastic menus, Proposition 3 implies that it can happen that one NFW strictly prefers menu  $M_1$  to menu  $M_2$ , while another NFW strictly prefers menu  $M_2$  to menu  $M_1$ . I now elaborate.

Define a **deterministic menu lottery (DML)** to be a vector of the form  $(p_S : S \subseteq Z, S \neq \emptyset)$  with  $p_S \geq 0, \forall$  nonempty  $S \subseteq Z$ , and  $\sum_{S \subseteq Z: S \neq \emptyset} p_S = 1$ . A DML is a probability measure over deterministic menus that gives probability  $p_S$  to the deterministic menu  $\{\delta_z : z \in S\}$ . For any DML  $p$ , define  $M^p$  to be

---

<sup>14</sup>The inequalities in  $V$  amount to total submodularity – which plays a central role in Nehring's analysis – for value functions on deterministic menus that depend only cardinality of the menu.

the menu induced by  $p$ :

$$\begin{aligned} M^p &= \sum_{S \subseteq Z: Z \neq \emptyset} p_S \{\delta_z : z \in S\} \\ &= \left\{ \sum_{S \subseteq Z: Z \neq \emptyset} p_S \delta_{z^S} : z^S \in S, \forall \text{ nonempty } S \subseteq Z \right\}. \end{aligned} \quad (8)$$

Not all menus of lotteries can be generated by DML's: When  $n = 2$ , there does not exist DML  $p$  such that  $M^p = \left\{ \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{2}{3}\right) \right\}$ . Next consider a special class of menus. For  $i \in Z$ , define  $[\delta_i] := \{\delta_1, \delta_2, \dots, \delta_i\}$  to be deterministic menu corresponding to  $[i]$ . A DML  $p$  is **canonical** if  $\sum_{i=1}^n p_{[i]} = 1$ ; that is, only menus of the form  $[\delta_i]$  are assigned positive probability. A canonical DML is abbreviated as a **CDML**. A CDML  $p$  can be written as  $p = (p_i : i \in Z)$ , where we write  $p_i$  instead of  $p_{[i]}$  for the probability of  $[\delta_i]$ .  $\mathcal{D}$  is the set of CDML's. In the case of a CDML, (8) reduces to:

$$M^p = \sum_{i=1}^n p_i [\delta_i] = \left\{ \sum_{i=1}^n p_i \delta_{z^i} : z^i \in [i], \forall i \in Z \right\}.$$

To provide an example, if  $n = 3$ , and  $p = (p_1, p_2, p_3) = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$ , then:

$$\begin{aligned} M^p &= \frac{1}{2} \{\delta_1\} + \frac{1}{3} \{\delta_1, \delta_2\} + \frac{1}{6} \{\delta_1, \delta_2, \delta_3\} \\ &= \left\{ \begin{array}{l} \frac{1}{2}\delta_1 + \frac{1}{3}\delta_1 + \frac{1}{6}\delta_1, \frac{1}{2}\delta_1 + \frac{1}{3}\delta_1 + \frac{1}{6}\delta_2, \frac{1}{2}\delta_1 + \frac{1}{3}\delta_1 + \frac{1}{6}\delta_3, \\ \frac{1}{2}\delta_1 + \frac{1}{3}\delta_2 + \frac{1}{6}\delta_1, \frac{1}{2}\delta_1 + \frac{1}{3}\delta_2 + \frac{1}{6}\delta_2, \frac{1}{2}\delta_1 + \frac{1}{3}\delta_2 + \frac{1}{6}\delta_3 \end{array} \right\} \\ &= \left\{ (1, 0, 0), \left(\frac{5}{6}, \frac{1}{6}, 0\right), \left(\frac{5}{6}, 0, \frac{1}{6}\right), \left(\frac{2}{3}, \frac{1}{3}, 0\right), \left(\frac{1}{2}, \frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right) \right\}. \end{aligned}$$

The following observation is useful:

**Observation 1** *Let  $\preceq$  be an NFW. For every DML  $p$ , let us define the CDML  $\tilde{p}$  by  $\tilde{p}_i = \sum_{S \subseteq Z: |S|=i} p_S, \forall i \in [n]$ . Then  $M^p \sim M^{\tilde{p}}$ .*

This justifies focusing on CDML's: All that matters in assessing a DML is the induced distribution over the cardinality of deterministic menus. The following corollary of Proposition 3 makes explicit how neutrality, in and of

itself, does not pin down preferences. Even if one is neutral, one must still bring further judgments to bear if one is to completely rank choice problems in terms of the freedom that they allow.

**Corollary 1 (*Scope for Disagreement among NFW's*)**

1. For every NFW  $\preceq$ , there exists  $v \in V$ ,

$$M^p \preceq M^q \Leftrightarrow \sum_{i=1}^n v(i)p_i \leq \sum_{i=1}^n v(i)q_i, \quad \forall p, q \in \mathcal{D}. \quad (9)$$

2. Conversely, for every  $v \in V$ , there exists a NFW  $\preceq$  satisfying (9).

3. (***Dimension of Disagreement***)

No two distinct elements of  $\tilde{V} = \{v \in V : v(1) = 0, v(n) = 1\}$  rank CDML's in the same way. Moreover,  $\tilde{V}$  has dimension  $n - 2$ .

Consider the case where  $n = 3$ . Then membership in  $V$  simplifies to (5), and membership in the normalized set  $\tilde{V}$  reduces to satisfying (i)  $v(1) = 0$ , (ii)  $\frac{1}{2} \leq v(2) \leq 1$ , and (iii)  $v(3) = 1$ . Corollary 1 implies that all nontrivial NFW's prefer the menu  $[\delta_2]$  to  $\frac{2}{3}[\delta_1] + \frac{1}{3}[\delta_3]$ . Corollary 1 also implies that there are distinct NFW's that differ in their comparisons of menus: When  $v(2) \in [\frac{1}{2}, \frac{2}{3})$ ,  $\frac{1}{3}[\delta_1] + \frac{2}{3}[\delta_3]$  is preferred to  $[\delta_2]$ ; when  $v(2) \in (\frac{2}{3}, 1]$ ,  $[\delta_2]$  is preferred to  $\frac{1}{3}[\delta_1] + \frac{2}{3}[\delta_3]$ . For each of the two incompatible preferences over these two menus, there is a 1-dimensional set of functions  $v$  such that the corresponding NFW's induce that preference; under the normalization  $v(1) = 0, v(3) = 1$ , no two of these  $v$ 's induce the same preference over all CDML's. The 1-dimensionality is a consequence of Part 3 of Corollary 1, where, note that when  $n = 3$ ,  $n - 2 = 1$ . Part 3 implies that there will be a similar prevalence of disagreement among NFW's when  $n > 3$ . To summarize, NFW's agree in ranking larger deterministic choice sets as (weakly) better; but NFW's disagree in that, for lotteries over deterministic menus, they display different risk attitudes about the size of the menu.

A final point: One might think we can substitute (7) for Axiom 6. This is not true: The Appendix shows that there are weak orders satisfying Axioms 2-5 that have a cardinal representation satisfying (7) but fail to satisfy

the neutrality axiom (Axiom 6). This is because (7) captures the cardinal consequences of neutrality for menus arising out of DML's but not all menus of lotteries can arise in this way.

## 4 Characterizations

I now present a representation theorem for NFP's (Section 4.1) and, more importantly, a characterization of the *consequences* of the axioms for NFP's (Section 4.2).

### 4.1 A Representation Theorem

For any  $u \in U$  and  $\pi \in \Pi$ , define  $u^\pi \in U$  by:

$$u_z^\pi := u_{\pi(z)}, \quad \forall z \in Z.$$

$u^\pi$  is the permutation of  $u$  that interchanges the roles of  $\pi(z)$  and  $z$ . The definition can be extended from individual utility functions to sets of utility functions: For any  $S \subseteq U$  and  $\pi \in \Pi$ , define  $S^\pi := \{u^\pi : u \in S\}$ . A probability measure  $\mu \in \Delta(U)$  is **symmetric** if for all  $\pi \in \Pi$  and all measurable  $S \subseteq U$ ,  $\mu(S) = \mu(S^\pi)$ . So a symmetric probability measure treats all permutations of a utility function equivalently. Let  $\Delta_{\text{sym}}^*(U)$  be the set of all symmetric probability measures in  $\Delta^*(U)$ .

**Proposition 4**  *$\preceq$  is an NFP if and only if  $\preceq$  satisfies (2) for some closed and convex subset  $P$  of  $\Delta_{\text{sym}}^*(U)$ .  $\preceq$  is an NFW if and only if  $\preceq$  satisfies (2) for some singleton set  $P = \{\mu\}$  with  $\mu \in \Delta_{\text{sym}}^*(U)$ .*

In comparison to the representation theorem for preference for flexibility (Proposition 1), this result adds neutrality to the axiom set and adds the condition that the probability measures are symmetric to the representation. The preference for flexibility multi-utility representation theorem (Proposition 1) can be interpreted as the consensus of a collection of agents behind a “veil of ignorance” which leaves them uncertain about their preferences. In

that incarnation, it is an imperfect veil, as the agents may still retain, behind the veil, some bias as to which outcomes are likely to be preferred. Adding the neutrality axiom, Proposition 4 perfects the veil, so that all outcomes are treated symmetrically.

## 4.2 The Consequences of the Axioms

### 4.2.1 The Grading Order

This section presents a preorder on menus, the *grading order*, that is used to characterize the consequences of the axioms in Theorem 1 below. Let  $\hat{G}$  be a set of *grades*. Heuristically, the reader may think of these as analogous to letter grades given in a class, A, B, C, etc. The grades can be thought of as possible evaluations of the alternatives in  $Z$ . Assume that  $|\hat{G}| = |Z|$ , so that there are as many grades as there are alternatives. A **grading scheme** is a way of evaluating alternatives. Formally, a grading scheme is a bijection  $g : Z \rightarrow \hat{G}$  that assigns a distinct grade to each item.  $\mathcal{G}$  is the set of all grading schemes. Observe that there are  $n!$  grading schemes in  $\mathcal{G}$ .  $f(\beta, g, x) := \beta_{g^{-1}(x)}$  is the probability that lottery  $\beta$  will select the outcome that leads to grade  $x$  given grading scheme  $g$ . I now introduce a hypothetical **grading procedure** that can be applied to any menu  $M$  in  $\mathcal{M}$ .

#### Grading Procedure for $M$

1. Nature selects a grading scheme  $g$  from  $\mathcal{G}$  with uniform probability.
2. A grader, having observed the chosen grading scheme  $g$ , randomizes over lotteries in  $M$ .

In step 1, each grading scheme is selected with probability  $\frac{1}{n!}$ . The grader's behavior in this procedure is determined by a **randomized strategy**  $\sigma : \mathcal{G} \rightarrow \Delta(M)$ .<sup>15</sup> I write  $\sigma^g$  – instead of  $\sigma(g)$  – for the probability measure over  $M$  that the grader uses conditional on  $g$ . If the grader uses  $\sigma$ , then conditional on Nature selecting grading scheme  $g$ , the probability that grade  $x$  will be

---

<sup>15</sup>The sigma-algebra on  $M$  is the relative sigma-algebra on  $M$  induced by the Borel sigma-algebra on  $\Delta(Z)$ .

chosen is  $\int_M f(\beta, g, x) \sigma^g(d\beta)$ . Let us now look at the situation before Nature has chosen the grade distribution: The ex ante probability that grade  $x$  will be chosen in the grading procedure is  $\gamma_x^\sigma := \frac{1}{n!} \sum_{g \in \mathcal{G}} \int_M f(\beta, g, x) \sigma^g(d\beta)$ . The **grade distribution induced by  $\sigma$**  is  $\gamma^\sigma := (\gamma_x^\sigma : x \in \hat{G})$ . In other words,  $\gamma^\sigma$  is the (ex ante) probability distribution over grades that will result if the grader uses strategy  $\sigma$  in the grading procedure. Let  $\Sigma(M)$  be the set of all possible randomized strategies  $\sigma : \mathcal{G} \rightarrow \Delta(M)$  in the grading procedure for  $M$ . Then the set of possible ex ante grade distributions that the grader can generate by varying her strategy is:

$$\{\gamma^\sigma : \sigma \in \Sigma(M)\}. \quad (10)$$

For a collection  $\{S_i\}$  of sets in a Euclidean space  $E$ , the *Minkowski sum*  $\sum_i S_i$  is the set of sums of selections from the sets:  $\sum_i S_i = \{\sum_i s_i : s_i \in S_i, \forall i\}$ . For  $S \subseteq E$  and real number  $\alpha$ , define  $\alpha S := \{\alpha s : s \in S\}$ .  $\text{co}(S)$  is the convex hull of  $S$ .

While conceptually, grades and alternatives are distinct, since  $|\hat{G}| = |Z|$ , mathematically, we may assume, without loss of generality, that  $\hat{G} = Z$ , so that alternatives double as grades. Under this assumption, it is easy to see – I show this in the Appendix – that expression (10) simplifies to

$$G(M) := \text{co} \left[ \frac{1}{n!} \sum_{\pi \in \Pi} M^\pi \right]. \quad (11)$$

$G(M)$  is the *set of ex ante grade distributions that the grader can generate in the grading procedure for  $M$* . Because grades have been formally identified with alternatives,  $G(M)$  is also a set of lotteries over alternatives in  $Z$ . Specifically,  $G(M)$  is the convex hull of the Minkowski average of all permutations of  $M$ .

Define the **grading order**  $\preceq^*$  on  $\mathcal{M}$  by:

$$M \preceq^* M' \Leftrightarrow G(M) \subseteq G(M'), \quad \forall M, M' \in \mathcal{M}. \quad (12)$$

The grading order ranks menu  $M'$  as *more free* than  $M$  if the grader can generate more grade distributions in the grading procedure for  $M'$  than in that for  $M$ .

We can understand the grading order by comparing it to an alternative ranking: **Freedom as monotonicity (FAM)** holds that: *Menu  $M'$  allows more freedom than menu  $M$  exactly if  $M \subseteq M'$* ; that is, exactly if whatever  $M$  allows one to choose,  $M'$  also allows one choose. FAM analyzes greater freedom as greater ability in a basic sense. But FAM is very weak; for example, disjoint choice sets can never be compared. Imposing neutrality is one way to increase the scope for comparisons. One way to incorporate neutrality would be to weaken FAM to the condition that:

$$\begin{aligned} & \text{Menu } M' \text{ allows more freedom than menu } M \text{ exactly if} \\ & \text{there exists a permutation } \pi \text{ such that } M \subseteq [M']^\pi. \end{aligned} \tag{13}$$

(13) however, fails to capture very basic intuitions about freedom. To illustrate this, let  $Z = \{1, 2\}$ , let  $(0, 1)$  and  $(1, 0)$  represent, respectively, the degenerate lotteries that put probability one on 1 and 2, and  $(\frac{1}{2}, \frac{1}{2})$  be the lottery that selects 1 and 2 with equal probability. Let  $\preceq$  mean “allows more freedom than”. Then (13) fails to imply that  $\{(\frac{1}{2}, \frac{1}{2})\} \preceq \{(1, 0), (0, 1)\}$ . That is, (13) fails to imply that a choice between alternatives 1 and 2 allows more freedom than a flip of the coin. Noting that the relation in (13) satisfies both the neutrality and monotonicity axioms, the problem stems from the failure of (13) to satisfy the independence axiom. The grading order, by contrast, does validate our basic intuition:  $\{(\frac{1}{2}, \frac{1}{2})\} \prec^* \{(1, 0), (0, 1)\}$ .

To summarize, consider two distinct decision problems: (1) **The simple choice problem**: The agent’s problem of choosing a lottery from a menu, and (2) **The grade choice problem**: The grader’s problem of choosing a grade distribution in the grading procedure. The grading order applies monotonicity to the choice set in the grade choice problem rather than in the simple choice problem.

We can also interpret the grading order as a more abstract version of the standard representation of the axioms for preference for flexibility via

expected indirect utility. In the preference for flexibility representation, Nature randomly determines the agent’s preferences, and a menu is evaluated by the agent’s ex ante expected utility given Nature’s randomization and the optimal interim choice by the agent. In the grading procedure, grading schemes replace preferences, nature’s distribution is uniform – guaranteeing neutrality – and expected utility induced by *optimal* choice is replaced by the *range of control* that the grader can achieve as she varies her strategy.

#### 4.2.2 The Consensus Theorem

I now come to the main result:

##### **Theorem 1 (*Consensus Theorem*)**

1. *The grading order is an NFP.*
2. *It is consensus among all NFW’s that  $M$  is preferred to  $M'$  if and only if  $M$  is preferred to  $M'$  by the grading order. Equivalently, the grading order is the coarsest NFP.<sup>16</sup>*

I elaborate on Part 2. Section 3.2 showed that there can be a great deal of disagreement among NFW’s about which menus offer more freedom. So neutrality does not pin down the ranking of menus (when the other freedom axioms have been assumed). The grading order captures exactly what is common to all freedom orders satisfying neutrality. By Proposition 4, this is what is consensus among all agents with preference for flexibility behind a perfect veil of ignorance that erases all information about the differences between specific outcomes.

A typical representation theorem shows that a certain functional form captures what is *consistent* with a set of axioms; in contrast, the Consensus Theorem shows that the grading order captures what is *entailed* by the freedom axioms. In other words,  $M \preceq^* M'$  exactly if it is a *logical consequence* of the axioms that  $M$  offers more freedom than  $M'$ . To make this point vivid,

---

<sup>16</sup>A preorder  $\preceq_0$  is coarser than a preorder  $\preceq_1$  if for all  $M, M' \in \mathcal{M}$ ,  $M' \preceq_0 M \Rightarrow M' \preceq_1 M$ .

imagine that all that we have to work with is the axioms, which capture certain intuitions about freedom that we all accept. We do not have any more detailed ranking of menus consistent with the axioms. We are asked: “based on these principles, that we all accept, can you argue that menu  $M'$  offers more freedom than menu  $M$ ?” If  $M \preceq^* M'$ , then the answer is yes, and otherwise the answer is no. It may well be that there is a specific agent with an NFP  $\preceq$  such that  $M \preceq M'$ , but if it is not the case that  $M \preceq^* M'$ , then the preference for  $M'$  over  $M$  represents something idiosyncratic about this agent rather than something that follows from our basic tenants about freedom. Insofar as we think that arguments about freedom should be *principle-based* rather than idiosyncratic, it is an attractive feature of the grading order that it captures precisely what follows from our principles.<sup>17</sup>

In connection with the above, it is interesting that the grading order strictly prefers larger to smaller deterministic menus:  $\forall M_1, M_2 \in \mathcal{M}_d$ , if  $|M_1| < |M_2|$ , then  $M_2 \prec^* M_1$ . Proposition 2 tells us that there are some NFP's that are indifferent between  $M_1$  and  $M_2$ . If the grading order captures consensus among all NFP's, then how can it take sides? The answer is that all NFP's agree that  $M_2$  is at least as good as  $M_1$ ; so this consensus is accepted by the grading order. However, only some NFP's hold that  $M_1$  is at least as good as  $M_2$ . This nonunanimous view is not adopted by the grading order. Since the grading order weakly prefers  $M_2$  to  $M_1$  but does not weakly prefer  $M_1$  to  $M_2$ , the grading order strictly prefers  $M_2$  to  $M_1$ . While the grading order  $\preceq^*$  is (weakly) coarser than all other NFP's  $\preceq$ , the *strict part*  $\prec^*$  of the grading order is not the coarser than the strict part  $\prec$  of all other NFP's, as we have just seen. The grading order is indeed able to say *more* about strict comparisons than some other NFP's. This is because when there is consensus on one side of a (weak) comparison, but a lack of consensus on the other, the grading order sides only with the consensus. When, however, the grading order is indifferent among two menus, then all NFP's are indifferent. So the *weak part*  $\sim^*$  of the grading order is (weakly) coarser than the weak part  $\sim$  of any NFP. The next proposition summarizes some properties of the

---

<sup>17</sup>It follows that if one rejects some comparison made by the grading order, one must reject one of the axioms characterizing freedom.

grading order. By Theorem 1, this has consequences for all NFP's.

**Proposition 5 (*Properties of the Grading Order*)**

1. *Nothing can compensate for complete absence of freedom:*  
 $\{\beta\} \sim \{\beta'\} \prec^* M, \quad \forall \beta, \beta' \in \Delta(Z), \forall M \in \mathcal{M} \setminus \{\{\beta''\} : \beta'' \in \Delta(Z)\}.$
2. *The grading order coincides with the cardinality order on deterministic menus:*  $|M_1| \leq |M_2| \Leftrightarrow M_1 \preceq^* M_2, \forall M_1, M_2 \in \mathcal{M}_d.$
3. *More generally:*  $\forall M_1, M_2 \in \mathcal{M}, ([\exists \pi \in \Pi, M_1 \subseteq \text{co}(M_2^\pi)] \Rightarrow M_1 \preceq^* M_2),$   
*where we may replace  $\preceq^*$  by  $\prec^*$  if we replace  $\subseteq$  by  $\subsetneq$ .*
4.  $M^p \preceq^* M^q \Leftrightarrow \left( \sum_{i=1}^{n-j} \binom{n-i}{j} (q_i - p_i) \leq 0, \text{ for } j = 1, \dots, n-1 \right), \forall p, q \in \mathcal{D}$

Property 1 says that the menus that are judged worst by the grading order are precisely the singleton menus, or in other words, the menus that provide minimum freedom. If an agent has access to only one lottery, it does not matter what that lottery is, the agent is unfree. Any expansion freedom – any menu containing two or more lotteries – is preferred to no freedom. It is not the case, however, that a menu with three lotteries is always preferred to a menu with two lotteries; a menu with two lotteries may be strictly preferred to a menu with three lotteries. Property 4 characterizes the CDML's that can be compared by the grading order. It shows that  $n - 1$  inequalities must be satisfied for CDML's to be compared. So the grading order is not a complete order on menus induced by CDML's.

## 5 Applications

### 5.1 Voting

Voting is a natural area in which to apply neutral freedom because in assessing freedom in a voting context, we typically want to be neutral about the alternatives or candidates. The grading order turns out to be closely related to the well-known notion of Banzhaf power. Banzhaf power is a measure of power in (deterministic) binary voting institutions when voter preferences

are uniformly distributed. A voter's Banzhaf power is her probability of being pivotal. Banzhaf power can be generalized in two ways: by considering a wider class of *stochastic* voting institutions and by allowing more general distributions over voter preferences. Here we do so.

Consider a binary election between candidates 0 and 1. A (strict) preference can then be represented as an element of the set  $\{0, 1\}$ , where 0 represents a preference for candidate 0, and 1 represents a preference for candidate 1. If there are  $k$  voters, then  $\{0, 1\}^k$  represents the set of all possible profiles of voter preferences.  $\{0, 1\}^k$  can also be used to represent the set of profiles of votes for candidates 0 and 1. Define  $\alpha : \{0, 1\}^k \rightarrow [0, 1]$  to be a **stochastic voting mechanism**. The interpretation is that on vote profile  $z \in \{0, 1\}^k$ , the voting mechanism is such that candidate 1 wins with probability  $\alpha(z)$ , and 0 wins with probability  $1 - \alpha(z)$ . There is no requirement that voters be treated symmetrically. Thus, the formalism can capture institutions such as the electoral college in the United States, which effectively gives voters from different states different powers to influence the outcome. The fact that the voting mechanism is stochastic allows it to capture either (i) the possibility that a voting institution would be designed to incorporate a random element, or (ii) the possibility that due to imperfect measurement of votes, the map from votes submitted to the actual outcome is a random function (especially when the election is close).

Assume that the voting mechanism  $\alpha$  is **monotone** in the sense that  $\forall z, z' \in \{0, 1\}^k, z \leq z' \Rightarrow \alpha(z) \leq \alpha(z')$ .<sup>18</sup> That is, if some voters switch their votes from candidate 0 to candidate 1 (but none switch from 1 to 0), then the probability that candidate 1 wins increases. Monotonicity implies that voting truthfully (i.e., voting for the candidate that one prefers) is a dominant strategy in  $\alpha$ .

Let  $\mu$  be a probability distribution over voter preferences in  $\{0, 1\}^k$ . The interpretation is that if  $z$  is a possible preference profile,  $\mu(z)$  is the probability that the preference profile prevailing in society is  $z$ . Voter preferences are independently distributed so that  $\mu(z) = \prod_{i=1}^k \mu_i(z_i)$ , where  $\mu_i(z_i)$  is the probability that voter  $i$  prefers candidate  $z_i$ . Let  $z_{-i} \in \{0, 1\}^{k-1}$  be the profile

---

<sup>18</sup> $z \leq z'$  means that for all  $i$ ,  $z_i \leq z'_i$ .

that results by eliminating from  $z$  the component  $z_i$ . Let  $(0, z_{-i})$  and  $(1, z_{-i})$  be respectively the vote profile (or preference profile) that results when 0 and 1 are substituted for  $z_i$  in  $z$ . Let  $\mu_{-i}(z_{-i}) = \mu(0, z_{-i}) + \mu(1, z_{-i}) = \prod_{j:j \neq i} \mu_j(z_j)$  give the marginal distribution of  $z_{-i}$ .

Define the *Generalized Banzhaf power (GB power)* of voter  $i$ ,  $B_i(\alpha, \mu)$  to be the probability that voter  $i$  is pivotal in voting mechanism  $\alpha$ . Formally:<sup>19</sup>

$$B_i(\alpha, \mu) = \sum_{z_{-i} \in \{0,1\}^{k-1}} [\alpha(1, z_{-i}) - \alpha(0, z_{-i})] \mu_{-i}(z_{-i}) \quad (14)$$

To map the above into our framework, assume that  $Z = \{0, 1\}$ . The outcomes 0 and 1 correspond, respectively, to candidate 0 and candidate 1 winning. Consider voter  $i$ . Suppose all voters know the probability distribution  $\mu$ , but at the time that each voter votes, voter  $i$  learns only his own preference and not the preference of any other voter. Since preferences are independent, the voter will not be able to infer anything about other voters' preferences that he did not already know prior to learning his own preference. Suppose that all voters other than  $i$  use their dominant strategy, voting for whichever candidate they prefer. Then if voter  $i$  votes for candidate 1, then from the standpoint of voter  $i$ 's (interim) beliefs, candidate 1 will win with probability:

$$\sum_{z_{-i} \in \{0,1\}^{k-1}} \alpha(1, z_{-i}) \mu_{-i}(z_{-i}), \quad (15)$$

and candidate 0 will win with the remaining probability. Let  $\beta_i^1(\alpha, \mu)$  be this lottery in which candidate 1 wins with probability (15). Likewise, let  $\beta_i^0(\alpha, \mu)$  be the lottery over candidates voter  $i$  will face if voter  $i$  votes for candidate 0, in which candidate 1 wins with probability  $\sum_{z_{-i} \in \{0,1\}^{k-1}} \alpha(0, z_{-i}) \mu_{-i}(z_{-i})$ . We call the menu

$$M_i(\alpha, \mu) = \{\beta_i^0(\alpha, \mu), \beta_i^1(\alpha, \mu)\}$$

---

<sup>19</sup>In the case that the voting mechanism is deterministic (i.e.,  $\alpha(z) \in \{0, 1\}, \forall z$ ), it is obvious that (14) represents the probability of being pivotal. In the Appendix, I explain why we can also regard (14) as the probability of being pivotal in stochastic mechanisms.

*i*'s **equilibrium menu**.<sup>20</sup> This terminology is justified because it is an equilibrium of the voting game for each voter to follow their dominant strategy and vote for their favorite candidate. In this equilibrium, the equilibrium menu represents the two lotteries that voter *i* can induce by voting for one candidate or the other. Because of independence of voter preferences, these two lotteries do not depend on which candidate voter *i* ends up preferring.

**Proposition 6** *For all monotone stochastic voting mechanisms,  $\alpha, \alpha'$ , and all distributions of voter preferences,  $\mu, \mu'$ , and all voters  $i$ :*

$$B_i(\alpha, \mu) \leq B_i(\alpha', \mu') \Leftrightarrow M_i(\alpha, \mu) \preceq^* M_i(\alpha', \mu'). \quad (16)$$

In words, voter *i* has greater generalized Banzhaf power in  $(\alpha', \mu')$  than in  $(\alpha, \mu)$  if and only if voter *i*'s equilibrium menu in  $(\alpha', \mu')$  is greater than voter *i*'s equilibrium menu in  $(\alpha, \mu)$  according to the grading order. The significance of this result is that the grading order tracks voters' (generalized) Banzhaf power in binary voting institutions. This theorem validates the grading order insofar as the grading order coincides with the most established measure of voting power for binary voting.

This suggests a way of generalizing Banzhaf power to multiple candidate elections. Let the set of candidates be  $Z = \{1, \dots, n\}$  and consider a class of voting mechanisms that share a common message space  $J$ . A stochastic voting mechanism  $\alpha$  now maps a vector  $j = (j_1, \dots, j_k) \in J^k$  into a probability measure  $\beta \in \Delta(Z)$  over candidates. There is a profile  $(v_1, \dots, v_k)$  of voter utility functions over candidates. The probability measure  $\mu$  over these profiles is such that voter utility functions are independent of one another. A strategy  $\sigma_i$  maps *i*'s utility function into a probability distribution over votes in  $J$ . This situation induces a Bayesian game. Let  $\sigma = (\sigma_i : i = 1, \dots, k)$  be a strategy profile that is a Bayesian Nash equilibrium of this game. Then let  $\beta_i^j(\alpha, \mu, \sigma)$  be the lottery over candidates that results when voter *i* selects vote *j* in mechanism  $\alpha$  given that voter utility functions are distributed according to  $\mu$  and voters use strategy profile  $\sigma$ .<sup>21</sup> Then *i*'s equilibrium menu

<sup>20</sup>In Sher (2015), I present a general framework for evaluating freedom in games.

<sup>21</sup>Notice that  $\beta_i^j(\alpha, \mu, \sigma)$  only depends on  $\mu_{-i}$  and  $\sigma_{-i}$ .

is  $M_i(\alpha, \mu, \sigma) = \{\beta_i^j(\alpha, \mu, \sigma) : j \in J\}$ . We can say that  $(\alpha, \mu, \sigma)$  gives the voter more power than  $(\alpha', \mu', \sigma')$  if and only if  $M_i(\alpha', \mu', \sigma') \preceq^* M_i(\alpha, \mu, \sigma)$ . Notice that unlike in the case of binary voting mechanism, the ranking may no longer be complete. This means that some voting mechanisms are un-ordered in terms of the freedom or power that they provide specific voters. Notice also that the ranking depends on the equilibrium: A voter may have more influence in one equilibrium than in another because in the former she may have more of an impact on the outcome.

Suppose that  $J = Z$ , so that a message is a vote for a particular candidate. Suppose further that the voting mechanism is such that a candidate who receives no votes or only one vote is elected with probability zero. Suppose that we restrict attention to equilibria in which all voters vote for only candidates  $z^0$  and  $z^1$ . There are likely to be such equilibria because if no one votes for other candidates, there is no point in voting for other candidates. In such equilibria, the above ranking of voting mechanisms reduces to generalized Banzhaf power of the binary mechanisms that would result if voting for candidates other than  $z^0$  and  $z^1$  is forbidden. This is a natural consequence. It says that majority voting when there are many candidates but only two candidates are serious contenders gives all voters the same effective freedom as majority voting when only these two candidates are on the ballot. Moreover, the measurement of freedom depends on the identity of the two serious candidates only through differences in the distribution of preferences over the two candidates induced by  $\mu$  and the possible asymmetric treatment of candidates encoded in  $\alpha$ . When there are multiple serious contenders in equilibrium, then the measure may depart from Generalized Banzhaf power. It would be interesting to compare the grading order to other power indices and generalizations of Banzhaf power to multi-candidate elections.

## 5.2 Value of Information

An agent must make a decision on the basis of a signal that he receives about the underlying state of the world, which is relevant to the decision. Let  $S = \{1, \dots, m\}$  be a set of states,  $T = \{1, \dots, n\}$  a set of signals, and  $P = (p_{st})$

be an  $n \times m$  matrix with nonnegative entries such that  $\sum_{s \in S} p_{st} = 1$  for all  $t \in T$ .  $p_{st}$  is the probability of signal  $t$  given state  $s$ .  $P$  is an **information structure**. One information structure  $P$  is **more informative** than another information structure  $Q$  (in the sense of Blackwell) if there exists a right stochastic  $n \times n$  matrix  $R$  (i.e., a matrix  $R = (r_{ij})$  with nonnegative entries and  $\sum_{j=1}^n r_{ij} = 1$  for all  $i$ ) such that  $Q = PR$ . In this case,  $Q$  is also said to be a **garbling** of  $P$ . Upon observing a signal – but not the state – the agent can make a decision, which amounts to choosing an element of the (finite) set  $D$  of decisions. A **strategy** is then a function from  $T$  to  $D$  specifying which action the agent will take as a function of the signal that the agent observes.  $\Sigma$  is the set of strategies.

In this context, the set of outcomes is  $Z = S \times D$ , so that an outcome is the taking of a specific decision in a specific state. Fix a prior  $\eta$  over the states, giving probability  $\eta_s$  to state  $s$ . For each strategy  $\sigma \in \Sigma$ , let  $\gamma(P, \eta, \sigma)$  be the lottery over state-decision pairs that would result from following strategy  $\sigma$ .  $\gamma(P, \eta, \sigma)$  is the lottery that puts probability  $\eta_s \sum_{t: \sigma(t)=d} p_{st}$  on  $(s, d)$ . Let  $M(P, \eta) = \{\gamma(P, \eta, \sigma) : \sigma \in \Sigma\}$  be the menu of lotteries that results as the agent varies his strategy.<sup>22</sup>

**Proposition 7** *If  $P$  is more informative than  $Q$ , then every agent with preference for flexibility prefers  $M(P, \eta)$  to  $M(Q, \eta)$ . Consequently  $M(Q, \eta) \preceq^* M(P, \eta)$ .*

The result says that a more informative information structure leads to a decision problem with more freedom of choice (according to the grading order). Freedom as control in this case means that the agent has more control over the distribution of state-action pairs. If the signal that the agent observes is completely noisy – that is, if the signal the agent observes is statistically independent of the state – then the agent’s menu effectively only allows him to choose individual actions independent of the state, giving him the minimal freedom possible in this problem. If the signal is perfectly informative,

<sup>22</sup>Since the preordered flexibility axioms imply that a menu is indifferent to its convex hull, nothing substantive would change if we substituted the set of randomized strategies for the set of pure strategies  $\Sigma$  used in the above construction.

then the agent can perfectly control the relation between actions and states, although he cannot control the distribution of states.

Proposition 7 validates the common intuition that additional information increases one’s freedom. It provides, moreover, an alternative rationale for providing agents with more information. The standard rationale in economic theory is in terms of Blackwell’s theorem – the notion that more information leads an agent to attain a higher expected utility. The rationale provided here is that providing an agent with more information enhances his freedom.

## 6 Neutrality without Equal Substitutability

This section relaxes the neutrality axiom to allow for the possibility that not all alternatives are equally good substitutes for one another. I explore two approaches.

### 6.1 Independent Sets

Let  $\preceq$  be a freedom weak order that is not necessarily a neutral freedom weak order. That is,  $\preceq$  satisfies the DLR axioms, Axioms 2-5, but not necessarily the neutrality axiom, Axiom 6. Let  $\nu$  be a cardinal representation of  $\preceq$  (see Section 2.4). For any  $Y \subseteq Z$ , I abuse notation by writing  $\nu(Y)$  instead of  $\nu(\{\delta_z : z \in Y\})$ , and then further abuse notation by writing  $\nu(z)$  instead of  $\nu(\{z\})$  for any  $z \in Z$ ; similarly, I often write  $z$  instead of the singleton  $\{z\}$  where this causes no confusion. For  $x, y$ , and  $z$  in  $Z$  and  $S \subseteq Z$  with  $\{y, z\} \subseteq S$  and  $x \notin S$ , say that  $x$  is **not a better substitute for  $y$  than for  $z$  in  $S$**  if:

$$\nu([S \setminus y] \cup x) - \nu(S \setminus y) \leq \nu([S \setminus z] \cup x) - \nu(S \setminus z). \quad (17)$$

The reason that I subtract  $\nu(S \setminus y)$  (resp.,  $\nu(S \setminus z)$ ) rather than  $\nu(S)$  in the left (resp., right) side of (17) is to control for the marginal value of  $y$  (resp.,  $z$ ) to  $S \setminus y$  (resp.,  $y$ ). Removing  $y$  from  $S$  reduces the value of  $S$  by some quantity. We are not interested in the magnitude of this reduction, but rather in how much  $x$  contributes once  $y$  has been removed: Does  $x$

contribute more once  $y$  has been removed than once  $z$  has been removed? If not, then  $x$  is not a better substitute for  $y$  than for  $z$ .

**Definition 1**  $I \subseteq Z$  is an **independent set** if  $\forall x, y, z \in I$ , and all  $S \subseteq I$  with  $\{y, z\} \subseteq S$  and  $x \notin S$ ,  $x$  is not a better substitute for  $y$  than for  $z$  in  $S$ . We sometimes write that  $I$  is a  $\nu$ -independent set or a  $\preceq$ -independent set to make the dependence of the notion of an independent set on  $\nu$ , or the (not necessarily neutral) freedom weak order  $\preceq$  that  $\nu$  cardinally represents, explicit.<sup>23</sup>

**Proposition 8** Let  $\nu$  cardinally represent a freedom weak order.  $I$  is a  $\nu$ -independent set if and only if there exist values  $\eta(1), \eta(2), \dots, \eta(|I|)$  (with  $\eta(1) = 0$ ) such that

$$\nu(S) = \sum_{w \in S} \nu(w) - \eta(|S|), \quad \forall \text{ nonempty } S \subseteq I. \quad (18)$$

Since  $\nu$  cardinally represents a weak freedom order, when (18) holds, we must also have – among other conditions<sup>24</sup> – the inequalities:  $0 = \eta(1) \leq \eta(2) \leq \dots \leq \eta(|I|)$ .

An independent set is not necessarily one in which the values of the elements enter additively; there can be an effect for the size of the set, but this effect does not favor any alternative over any other, so that no alternative is a superior substitute for any other alternative. This sort of interaction among alternatives is similar to that in Section 3, which studied neutral freedom orders. However, in the neutral case, all subsets of  $Z$  are independent, and, moreover,  $\nu(x) = \nu(y)$  for all  $x, y \in Z$ .

Rather than starting with  $\preceq$ , we can start with a set  $\mathcal{I}$  of independent sets and then construct freedom weak orders  $\preceq$  that generate  $\mathcal{I}$  as the set of independent sets. The requirement on this family is that it is closed under inclusion: If  $I \in \mathcal{I}$ , and  $J \subseteq I$ , then  $J \in \mathcal{I}$ .  $\mathcal{I}$  is the set of independent sets generated by  $\preceq$  if  $\mathcal{I} = \mathcal{I}_{\preceq} := \{I \subseteq Z : I \text{ is a } \preceq\text{-independent set}\}$ . For  $I \subseteq Z$ ,

<sup>23</sup>All cardinal representations  $\nu$  of a given freedom weak order  $\preceq$  generate the same set of  $\nu$ -independent sets.

<sup>24</sup>These other conditions are akin to those in the definition of the set  $V$  in (4).

let  $\mathcal{M}_I := \{M \in \mathcal{M} : \forall \beta \in M, \sum_{z \in I} \beta_z = 1\}$ . So,  $\mathcal{M}_I$  is the set of menus that contain only lotteries with support in  $I$ . Let  $\Pi_I = \{\pi \in \Pi : \pi(I) = I\}$ .  $\Pi_I$  is the set of permutations that map  $I$  into  $I$ . With these definitions, we can weaken the neutrality axiom:

**Axiom 7 (*Qualified Neutrality*)**  $M \sim M^\pi, \forall I \in \mathcal{I}_\preceq, \forall M \in \mathcal{M}_I, \forall \pi \in \Pi_I$ .

Intuitively, we first judge that certain sets of alternatives should be treated as independent. Recall the definition of independence in terms of lack of asymmetry of substitutability (Definition 1). Once we have judged which sets are independent – a value judgment – we apply neutrality. Call a relation  $\preceq$  that satisfies Axiom 1 (resp., Axiom 2), Axioms 3-5 and Axiom 7 a **qualified neutral freedom preorder** – **QNFP** (resp., **qualified neutral freedom weak order** – **QNFW**).

**Proposition 9** *For any QNFP  $\preceq$ ,  $\{\beta\} \sim \{\beta'\} \preceq M$ ,  $\forall \beta, \beta' \in \Delta(Z), \forall M \in \mathcal{M}$ . Consequently, if  $\nu$  represents  $\preceq$ , then  $\nu(z) = \nu(z')$ ,  $\forall z, z' \in Z$ .*

Qualified neutrality implies that the stand alone freedom value of each alternative is the same, and, moreover, that menus with a single alternative offer minimal freedom. So qualified neutrality accommodates the motivating example of Section 3, but qualified neutrality allows that some pairs of alternatives are treated as better substitutes than others. So some of the undesirably strong consequences of full neutrality are eliminated. An observation that helps to explain Proposition 9 is that all two element sets of alternatives are independent. This is because if  $I = \{y, z\}$ , it is vacuously true that for all  $x \in I \setminus \{y, z\}$ ,  $x$  is not a better substitute for  $y$  than for  $z$  in  $\{y, z\}$ . Note separately that if we assume that all sets are independent – that is,  $\mathcal{I}_\preceq = 2^Z$  – then we recover our previous theory; that is, in this case Axiom 7 coincides with Axiom 6.

## 6.2 Limiting the Acceptable Permutations

Next, let us consider another related approach. Suppose the three alternatives are careers as a violinist, a cellist, and a lawyer. We might hold the

view that we should treat the options of cellist and violinist as interchangeable in all contexts, and also hold the view that cellist is a better substitute for violinist than for lawyer. In particular, we may want to assume that

$$\{\textit{cellist}, \textit{lawyer}\} \sim \{\textit{violinist}, \textit{lawyer}\}. \quad (19)$$

However since  $\{\textit{cellist}, \textit{violinist}, \textit{lawyer}\}$  is not an independent set, Axiom 7 would not imply (19). The alternative approach is that we directly assume which alternatives are substitutable for which. To this end, let  $\Pi_0 \subseteq \Pi$ . Intuitively,  $\Pi_0$  is the set of acceptable permutations. So, in the example above,  $\Pi_0$  would include the permutation that interchanges cellist and violinist, but not the permutation that interchanges cellist and lawyer. This interpretation of  $\Pi_0$  is captured by the following axiom.

**Axiom 8 ( $\Pi_0$ -*neutrality*)**  $M \sim M^\pi, \quad \forall M \in \mathcal{M}, \forall \pi \in \Pi_0.$

Axiom 8 depends on the choice of  $\Pi_0$ . Intuitively, we first judge which interchanges are acceptable, and then impose neutrality with respect to these permutations.  $\preceq$  is a  **$\Pi_0$ -neutral freedom preorder -  $\Pi_0$ NFP** (resp.,  **$\Pi_0$ -neutral freedom preorder -  $\Pi_0$ NFW**) if it satisfies Axiom 1 (resp., Axiom 2), Axioms 3-5, and Axiom 8. For  $I \subseteq Z$ , define  $\Pi|_I$  as the the set of permutations of  $I$ .  $\Pi|_I$  differs from  $\Pi_I$  as defined above. Similarly define  $\Pi_0|_I = \{\pi \in \Pi|_I : \exists \pi_0 \in \Pi_0, \forall z \in I, \pi(z) = \pi_0(z)\}$  for  $\Pi_0 \subseteq \Pi$ .

**Definition 2**  $I \subseteq Z$  is  **$\Pi_0$ -independent** if  $\Pi_0|_I = \Pi|_I$ .  $\mathcal{I}_{\Pi_0}$  is the set of  $\Pi_0$ -independent sets.

$I$  is  $\Pi_0$ -independent if all interchanges within  $I$  are permitted according to  $\Pi_0$ . The following proposition relates  $\Pi_0$ -independent sets to  $\preceq$ -independent sets.

**Proposition 10** For any  $\Pi_0$ NFP  $\preceq$ ,  $\mathcal{I}_{\Pi_0} \subseteq \mathcal{I}_{\preceq}$ .

Axiom 8 does not imply – when conjoined with the other axioms for a freedom order – that singleton menus are indifferent, or even that each singleton menu offers no more freedom than each non-singleton menu. However this

implication does hold under a certain condition. Say that  $\Pi_0$  is **connected** if the directed graph with vertices  $Z$  and edge set  $E = \{(z, z') \in Z \times Z : \exists \pi \in \Pi_0, \pi(z) = z'\}$  is weakly connected.<sup>25</sup>

**Proposition 11** *Let  $\Pi_0$  be connected and let  $\preceq$  be a  $\Pi_0$ NFP. Then:*

$$\{\beta\} \sim \{\beta'\} \preceq M, \quad \forall \beta, \beta' \in \Delta(Z), \forall M \in \mathcal{M}. \quad (20)$$

In the cellist-violinist-lawyer example,  $\Pi_0$  is not connected, but in the example to be presented in Section 6.3, it is.

### 6.3 An Illustration

I now illustrate the ideas discussed in the preceding subsections in a more detailed structural example. Many of the ideas in this section are based on ideas in Nehring and Puppe (2008), particularly the discussion of the hypercube model.<sup>26</sup> Nehring and Puppe discuss symmetric treatment of alternatives and properties but they do not discuss the specific neutrality axioms that are central to this paper – Axioms 6, 7, and 8; these axioms are new here. Nehring and Puppe do not relate their model to the the stochastic control inherent in, e.g., voting mechanisms as I do below.<sup>27</sup>

Let an outcome be determined by  $n$  basic binary properties in the set  $N := \{1, \dots, n\}$ . Redefine the outcome set to be  $Z = \{0, 1\}^n$ . If, e.g.,  $n = 3$ ,  $(1, 0, 1)$  is the outcome that has the first and third properties but lacks the second. For  $z = (z_1, \dots, z_n), z' = (z'_1, \dots, z'_n) \in Z$ , the **Hamming distance** between  $z$  and  $z'$  is:  $d(z, z') = |\{i \in N : z_i \neq z'_i\}|$ . So, the Hamming distance of  $z$  from  $z'$  is the number of components – representing basic properties – on which  $z$  and  $z'$  differ. Define

$$\Pi_0 := \{\pi \in \Pi : \forall y, z \in Z, d(y, z) = d(\pi(y), \pi(z))\}. \quad (21)$$

---

<sup>25</sup>Graph  $(Z, E)$  is **weakly connected** if for any pair of vertices  $z, z'$ , there exists a sequence of vertices  $z_0, z_1, \dots, z_k$  in  $Z$  with  $z_0 = z, z_k = z'$  and  $(z_i, z_{i-1}) \in E$  or  $(z_{i-1}, z_i) \in E$  for  $i = 1, \dots, k$ .

<sup>26</sup>Related ideas are in Nehring and Puppe (2002).

<sup>27</sup>Formally, Nehring and Puppe do discuss lotteries over menus but not menus of lotteries as I do here. They also relate their approach to the preference for flexibility approach.

$\Pi_0$  is the set of permutations that preserve the Hamming distance between every pair of strings. A permutation  $\pi \in \Pi_0$  can be represented by a pair  $(\rho, J)$ , where  $\rho$  is a permutation of  $N$  and  $J \subseteq N$ , such that if  $z' = \pi(z)$ ,  $z'_i = z_{\rho(i)}$  if  $i \in J$  and  $z'_i = 1 - z_{\rho(i)}$  if  $i \notin J$  (see Section B.18 of the Appendix). That is, if we want to permute alternatives so as to preserve distance between every pair of alternatives, we can interchange components and we can interchange values 0 and 1 for specific components. If we apply the  $\Pi_0$ -neutrality axiom (Axiom 8) with  $\Pi_0$  given by (21), then we are effectively being neutral about the  $n$  binary properties rather than about the  $2^n$  alternatives. That is, we treat properties symmetrically and for any given property, we treat the possession of that property and the lack of possession of that property symmetrically.

**Observation 2**  $\Pi_0$  defined by (21) is connected. So by Proposition 11, any  $\Pi_0$ NFP  $\preceq$  satisfies (20).

We now explore the set of  $\Pi_0$ -independent sets when  $\Pi_0$  is defined by (21). Define a **partial string** to be an element  $s$  of  $\{0, 1\}^S$  for some  $S \subseteq N$ . Thus a partial string specifies whether an alternative has the properties in  $S$ . If  $S = \emptyset$ , we define  $\{0, 1\}^S := \{\perp\}$ , so that there is a degenerate empty partial string  $\perp$ .  $P$  is the set of partial strings. For  $z \in Z$  and  $S \subseteq N$ , define  $z_S = (z_i : i \in S)$ .

Now fix  $S \subseteq N$  and  $s \in \{0, 1\}^S$ , and consider a partition  $\mathcal{C}$  of the remaining properties  $N \setminus S$ , such that for all  $C, C' \in \mathcal{C}$ ,  $|C| = |C'|$ . That is,  $\mathcal{C}$  partitions  $N \setminus S$  into groups, each of which has the same cardinality. Then for each  $C \in \mathcal{C}$ , let  $x^{C,S,s}$  be the element of  $Z$  such that (i)  $x_S^{C,S,s} = s$ , (ii)  $x_C^{C,S,s}$  is a vector of 1's, and (iii)  $x_{N \setminus (C \cup S)}^{C,S,s}$  is a vector of 0's. Then define  $I(\mathcal{C}, S, s) = \{x^{C,S,s} : C \in \mathcal{C}\}$ . For example, if  $n = 3$ ,  $S = \emptyset$ ,  $s = \perp$ , and  $\mathcal{C} = \{\{1\}, \{2\}, \{3\}\}$ , then  $I(\mathcal{C}, S, s) = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ . If  $n = 9$ ,  $S = \{7, 8, 9\}$ ,  $s = (1, 0, 1) \in \{0, 1\}^{\{7,8,9\}}$ , and  $\mathcal{C} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$ , then  $I(\mathcal{C}, S, s) = \{(1, 1, 0, 0, 0, 0, 1, 0, 1), (0, 0, 1, 1, 0, 0, 1, 0, 1), (0, 0, 0, 0, 1, 1, 1, 0, 1)\}$ .

**Proposition 12** Suppose that  $\Pi_0$  is as in (21), and  $I(\mathcal{C}, S, s)$  is as described above. Then  $I(\mathcal{C}, S, s)$  is a  $\Pi_0$ -independent set.

It follows that for any  $\Pi_0$ NFP, if we restrict attention to menus of lotteries not over all elements of  $Z = \{0, 1\}^n$ , but only over a subset of  $Z$  of the form  $I(\mathcal{C}, S, s)$ , then the entire theory of Sections 2-4 applies. For example,  $\nu(M)$  depends only on the cardinality of  $M$  for deterministic menus containing only elements in  $I(\mathcal{C}, S, s)$ . However, if we consider all menus of lotteries on  $Z$ , then  $\preceq$  may no longer satisfy all the properties of Sections 2-4, and in particular, for deterministic menus  $M$ ,  $\nu(M)$  may not be a function just of the cardinality of  $M$ . So the structure studied in Sections 2-4 – the structure of NFP’s – is embedded into structure we now study.

We now investigate in more detail the structure that  $\nu(M)$  may have under our generalized theory. A partial string  $s \in \{0, 1\}^S$  has length  $\ell$  if  $|S| = \ell$ . For each deterministic menu  $M \in \mathcal{M}_d$  and length  $\ell$ , define:

$$P_\ell(M) = \{s \in P : \exists S \subseteq \{1, \dots, n\}, |S| = \ell, s \in \{0, 1\}^S, \exists z \in Z, \delta_z \in M, z_S = s\}.$$

We call  $P_\ell(M)$  **the set of partial strings of length  $\ell$  in  $M$** .  $|P_\ell(M)|$  is the number of partial strings in  $P_\ell(M)$ . It is important to realize that partial strings are distinct when their domains are distinct. So the partial string  $s$  of length 2 that assigns 0 to both properties 1 and 2 (so that  $s \in \{0, 1\}^{\{1,2\}}$ ) is distinct from the partial string  $s'$  of length 2 that assigns 0 to both properties 1 and 3 (so that  $s' \in \{0, 1\}^{\{1,3\}}$ ).

**Proposition 13** *Let  $\Pi_0$  be given by (21). Choose  $\lambda = (\lambda_\ell : \ell = 1, \dots, n) \in \mathbb{R}_+^n$ . Then there exists  $\Pi_0$ NFW  $\preceq$  with cardinal representation satisfying:*

$$\nu(M) = \sum_{\ell=1}^n \lambda_\ell |P_\ell(M)|, \quad \forall M \in \mathcal{M}_d. \quad (22)$$

Nehring and Puppe (2008) discuss value functions of the form (22).<sup>28</sup> In the same way that Section 3.2 showed that the axioms for a NFW did not pin down preferences (or the corresponding value function  $\nu$ ), Proposition 13 shows that the axioms for a  $\Pi_0$ NFW (with  $\Pi_0$  given by (21)) do not pin down preferences. Nehring and Puppe (2008) come to a similar conclu-

---

<sup>28</sup>Some of the terminology differs.

sion.<sup>29</sup> Proposition 13 shows that unlike the model of NFW's, the model we study here allows for different degrees of substitutability among outcomes. For example, suppose that in  $Z = \{0, 1\}^n$ ,  $n = 3$  and consider the deterministic menu  $M = \{(1, 0, 0), (0, 1, 0)\}$ . If the value function satisfies (22) and  $\lambda_2 > 0$ , then  $\nu(M \cup (0, 0, 1)) - \nu(M) > \nu(M \cup (1, 0, 1)) - \nu(M)$  because  $(0, 0, 1)$  contributes more new partial strings to  $S$  than does  $(1, 0, 1)$ . So not all alternatives are equally good substitutes. At the same time, by Observation 2, the stand-alone value of each alternative is the same:  $\nu((1, 0, 0)) = \nu((0, 1, 0)) = \nu((0, 0, 1)) = \nu((1, 0, 1))$ . The notion of distance captures the degree of substitutability for two element sets.<sup>30</sup>

**Proposition 14** *Assume that  $\preceq$  has a representation satisfying (22) for some  $\lambda \in \mathbb{R}_+^n$  with  $\lambda_1 > 0$ . Then:  $\forall x, y, z, w \in Z$ ,  $d(x, y) < d(z, w) \Rightarrow \{x, y\} \prec \{z, w\}$ .*

That is to say, a choice between alternatives that are farther apart offers more freedom.

The current model also has consequences for evaluating the control inherent in menus of lotteries. Let us return to the context of elections studied in Section 5.1. As in that section we assume that we are neutral about all candidates. However, we now care about how different the candidates or their platforms are from one another. Consider two possible elections from the standpoint voter  $i$ . Formally, these elections are to be thought of as stochastic voting mechanisms along the lines outlined in Section 5.1. However, I now present a more informal description. I present an election  $j$  from the standpoint of a specific voter  $i$ .

**Election j** (a) The candidates are  $x^j, y^j \in Z = \{0, 1\}^n$ . (b) The probability that voter  $i$  is pivotal is  $\rho^j$ . (c) Conditional on  $i$  not being pivotal, the lottery over  $x^j$  and  $y^j$  induced by the voting behavior of voters other than  $i$  is  $\beta^j = (\beta_{x^j}^j, \beta_{y^j}^j)$ , where  $\beta_{x^j}^j$  is the probability that  $x^j$  wins and  $\beta_{y^j}^j$  is the probability that  $y^j$  wins.

---

<sup>29</sup>They write, "Our overall conclusion from the above analysis is that symmetry alone does not suffice to determine the weights of options in the hypercube model."

<sup>30</sup>Nehring and Puppe (2002) show that, in general, in the hypercube model, the value of sets larger than two is not determined by the binary distances between the elements.

The menu of options for voter  $i$  induced by Election  $j$  is:

$$M_j = \{ \rho^j \delta_{x^j} + (1 - \rho^j) \beta^j, \rho^j \delta_{y^j} + (1 - \rho^j) \beta^j \}.$$

The next result, which is parallel to Proposition 6, compares two Elections  $j$ , where  $j = 1, 2$ , allowing, now that not only the probability of being pivotal, but also the degree of difference between the candidates, matters.

**Proposition 15** *Suppose that  $\preceq$  is a  $\Pi_0$ NFW with representation satisfying (22). Suppose, moreover, that  $\rho^2 > 0$ . If  $\rho^1 \leq \rho^2$  and  $d(x^1, y^1) \leq d(x^2, y^2)$ , then  $M_1 \preceq M_2$ . If, in addition, either  $\rho^1 < \rho^2$  or  $d(x^1, y^1) < d(x^2, y^2)$ , then  $M_1 \prec M_2$ .*

Observe that it is a consequence of Proposition 15 that the ranking does not depend on the lotteries  $\beta^1$  and  $\beta^2$ ; that is, the ranking does not depend on how others would vote conditional on  $i$  not being pivotal; the voting behavior of others only influences a voter's freedom by affecting the probability that the voter will be pivotal. This is an interesting consequence that follows from neutrality (Axiom 8, coupled with the fact that  $\Pi_0$  is connected). It reflects the fact that  $\preceq$  measures the freedom inherent in the election, rather than the utility that the voter receives from it.

## References

- Arrow, K. J. (1995), 'A note on freedom and flexibility', *Choice, welfare and development: a festschrift in honour of Amartya K. Sen* pp. 7–15.
- Barbera, S., Bossert, W. and Pattanaik, P. K. (2004), 'Ranking sets of objects', *Handbook of Utility Theory, Vol. 2. Extensions* pp. 893–977.
- Carter, I. (1999), *A measure of freedom*, Oxford University Press.
- Dekel, E., Lipman, B. L. and Rustichini, A. (2001), 'Representing preferences with a unique subjective state space', *Econometrica* **69**(4), 891–934.
- Dowding, K. and van Hees, M. (2009), Freedom of choice, in P. Anand, P. K. Pattanaik and C. Puppe, eds, 'The Handbook of Rational and

- Social Choice: an overview of new foundations and applications’, Oxford University Press, pp. 374–392.
- Dubra, J., Maccheroni, F. and Ok, E. A. (2004), ‘Expected utility theory without the completeness axiom’, *Journal of Economic Theory* **115**(1), 118–133.
- Dworkin, R. (1985), *A matter of principle*, Oxford University Press.
- Galaabaatar, T. (2010), ‘Menu choice without completeness’. Unpublished manuscript.
- Kochov, A. (2007), ‘Subjective states without the completeness axiom’, *University of Rochester*.
- Kreps, D. M. (1979), ‘A representation theorem for “preference for flexibility”’, *Econometrica* **47**, 565–577.
- Nehring, K. (1999), ‘Preference for flexibility in a Savage framework’, *Econometrica* **67**(1), 101–119.
- Nehring, K. and Puppe, C. (2002), ‘A theory of diversity’, *Econometrica* **70**(3), 1155–1198.
- Nehring, K. and Puppe, C. (2008), ‘Diversity and the metric of opportunity’. Working paper.
- Pattanaik, P. K. and Xu, Y. (1990), On ranking opportunity sets in terms of freedom of choice, Technical report, Université catholique de Louvain, Institut de Recherches Economiques et Sociales (IRES).
- Rawls, J. (2005), *Political liberalism*, Columbia University Press.
- Sen, A. (2001), *Development as freedom*, Oxford University Press.
- Sher, I. (2015), ‘Evaluating allocations of freedom’. Unpublished Manuscript.
- Sugden, R. (1998), ‘The metric of opportunity’, *Economics and Philosophy* **14**(2), 307–337.