

Additive-Belief-Based Preferences*

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Abstract

We introduce a new class of preferences — which we call additive-belief-based (ABB) utility — that captures a general yet tractable approach to belief-based utility, and that encompasses many popular models in the behavioral literature. We show that the general class of ABB preferences and two prominent special cases, which allow utility to depend on the level of each period’s beliefs but not on changes in beliefs across periods, are fully characterized by suitable relaxations of the standard Independence Axiom. We also identify the intersection of ABB preferences with the class of recursive preferences and characterize attitudes towards the timing of resolution of uncertainty for ABB preferences.

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1 Introduction

It is both intuitive and well documented that beliefs about future consumption or life events directly affect well-being. For example, an individual may enjoy looking forward to an upcoming vacation and particularly so if the risk of severe weather conditions became very unlikely; on the other hand, the same individual may worry about a future medical procedure he determined to undertake. Loewenstein (1987), using survey techniques, provides early evidence on the effects of anticipatory motives in economics. There is also widespread evidence from other fields discussing how anticipation of pain produces psychological-stress reactions: a notable framework is Lazarus (1966), while Berns et. al. (2006) provide evidence from fMRI studies.

As a result, decision-making models in which individuals derive utility not only from material outcomes but also from their current and future beliefs have become increasingly prominent (see, for

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example, Caplin and Leahy, 2001; Kőszegi and Rabin, 2009; and Ely, Frankel, and Kamenica, 2015). Models of this kind have proved useful in explaining intuitive behavioral phenomena that are hard to reconcile with the standard “consequentialist” model, and that are related to intrinsic attitudes towards information and selective choices of information sources. For example, belief-based models can rationalize asset pricing “puzzles” (Caplin and Leahy, 2001; Pagel, 2016), portfolio allocation decisions (Pagel, 2016), and patterns of over-consumption in the face of income shocks (Kőszegi and Rabin, 2009). In addition, they have implications for the optimal revelation of positive or negative news; the design of medical tests is explicitly studied in Caplin and Eliaz (2003), Caplin and Leahy (2004), Kőszegi (2003), and Schweizer and Szech (2016). In most of these models, overall utility is additively separable between material payoffs and beliefs.

In this paper we suggest a unified framework for these models and provide their testable implications. In particular, we introduce and analyze a new class of utility functions — which we call *additive-belief-based (ABB)* utility — that captures a general additively separable approach to belief-based utility. Our main results show that both the general class and two of its special subsets are fully characterized by simple relaxations of the familiar Independence axiom of expected utility, applied to our setting. Thus, while existing papers have focused on emphasizing some of the behavioral implications of a particular belief-based utility function, we demonstrate what types of behavior, as a general class, these models can potentially accommodate, or must rule out. Furthermore, we demonstrate how functional forms restrictions translate to non-standard behavior, such as intrinsic attitudes towards information, helping lay bare some of the driving mechanisms in this literature.

Existing models usually take one of two forms. The first, as in Caplin and Leahy (2001), allows individuals’ utility to depend on the (absolute) level of beliefs; i.e., on how likely it is that certain states/payoffs occur. In line with previous literature, we refer to these set of models as *anticipatory utility* models. In the second, as in Kőszegi and Rabin (2009), utility depends not on the level of beliefs, but on changes in beliefs in any given period. We refer to this class as *changing beliefs* models. Our ABB representation encompasses both frameworks. Moreover, we point out a useful partition of the class of anticipatory utility models into (i) *prior-anticipatory utility* models, where utility depends on beliefs at the beginning of the time period, before information has been received; and (ii) *posterior-anticipatory utility* models, where utility depends on beliefs at the end of the time period, after information has been received. One of our primary contributions is to show what behaviors can serve distinguish these different psychological motivations — for example, a preference for one shot resolution of uncertainty cannot be exhibited by individuals who have only anticipatory motives, but rather requires a concern for changing beliefs.

While individuals in these models gain utility from their beliefs, they cannot directly choose

them.¹ Rather, individuals hold prior beliefs, receive information, and form interim beliefs by applying Bayes' rule. Therefore, individuals can control their beliefs only by choosing particular information structures. And since individuals gain utility from their beliefs, they may exhibit non-degenerate preferences over information structures even if they cannot react to new information by altering their behavior, that is, even if they do not have actions to take in the interim stage. Indeed, this feature is what distinguishes these models from the standard model, in which individuals would be indifferent between all possible information structures when no actions are available.² To tightly link these models to observable behavior, we look at preferences over the combination of information structures and prior beliefs. These can naturally be elicited in experimental and field settings. Formally, taking advantage of the theoretical mapping between information structures and compound lotteries, we take as our domain of preferences the set of two-stage compound lotteries, that is, lotteries whose prizes are different lotteries over final outcomes.³

In order to introduce the class of ABB functions, let P be a typical two-stage compound lottery. In period 0 it induces prior beliefs $\phi(P)$; the individual knows that the overall probability to receive x_j in period 2 is $\phi(P)(x_j)$. In period 1, P generates a signal i with probability $P(p_i)$. Signal i generates posterior beliefs over outcomes; the individual now knows that in period 2 he will receive x_j with probability $p_i(x_j)$. In period 2, all uncertainty resolves and the individual receives x_j and has degenerate beliefs centering on this outcome (denoted δ_{x_j}). The total utility of this scenario, denoted $V_{ABB}(P)$, is given by:

¹A well known example in which individuals optimally choose their beliefs is Brunnermeier and Parker (2005). Bénabou and Tirole (2016) and Bénabou (2015) survey models where individuals can distort their beliefs.

²Extensive experimental work has confirmed that individuals exhibit preferences over information structures even in the absence of being able to condition actions on information (e.g., Chew and Ho, 1994; Ahlbrecht and Weber, 1997; Arai, 1997; Lovallo and Kahneman, 2000; Eliaz and Schotter, 2010; Von Gaudecker et al., 2011; Brown and Kim, 2014; Kocher, Krawczyk and Van Winden, 2014; Ganguly and Tassoff, 2016; Falk and Zimmerman, 2016; Nielsen, 2017; Masatlioglu, Raymond and Orhun, 2017). Such experiments are often motivated by belief-based preferences and individuals' emotional reactions to their beliefs.

³In particular, we assume that the individual cannot take intermediate actions that may affect his final payoffs. While many of the models we refer to do allow for such actions, we will omit them from our analysis for several reasons. First, for expository purposes, we focus on the simplest model that still incorporates all variables of interest into the utility function. Second, as we discuss in Section 4, individuals in our model have *intrinsic* preferences over information. It will be cleanest to characterize such preferences within a framework that rules out any *instrumental* value of information. Third, allowing for intermediate actions requires us to take a stand about the solution concept that governs how individuals determine which action to take (e.g., backward induction or personal equilibrium), rather than solely focus on the parameters that identify their preferences. The choice of solution concept also requires carefully modeling the timing of actions relative to when beliefs, and belief-based utility, is realized.

$$\begin{aligned}
V_{ABB}(P) &= \underbrace{\sum_j \phi(P)(x_j)u(x_j)}_{\text{expected utility from material payoffs}} \\
&+ \underbrace{\sum_i P(p_i)\nu_1(\phi(P), p_i)}_{\text{expected utility from beliefs in periods 0 and 1}} \\
&+ \underbrace{\sum_i P(p_i) \sum_j p_i(x_j)\nu_2(p_i, \delta_{x_j})}_{\text{expected utility from beliefs in periods 1 and 2}}
\end{aligned}$$

The first term represents the expected consumption utility of the two-stage lottery — the expected utility that the individual receives from material outcomes (in period 2). The second term represents period 1’s belief-based utility — the individual’s expected utility from having interim beliefs p_i in period 1, conditional on having prior beliefs $\phi(P)$. The last term represents period 2’s belief-based utility — the individual’s expected utility from x_j being realized in period 2, conditional on having interim beliefs p_i .

As a concrete example, suppose there are two outcomes, H (high) and L (low), so that beliefs are summarized by the probability of H . Suppose that ex-ante the two outcomes are equally likely. Then the expected Bernoulli utility over material outcomes is $\frac{1}{2}u(H) + \frac{1}{2}u(L)$. In period 1, the individual receives a binary signal: half the time its good, and beliefs move to $\frac{3}{4}$ with corresponding utility $\nu_1(\frac{1}{2}, \frac{3}{4})$; half the time its bad, and beliefs fall to $\frac{1}{4}$ with corresponding utility $\nu_1(\frac{1}{2}, \frac{1}{4})$. Expected belief-based utility in period 1 is thus $\frac{1}{2}\nu_1(\frac{1}{2}, \frac{3}{4}) + \frac{1}{2}\nu_1(\frac{1}{2}, \frac{1}{4})$. In period 2, the individual learns for sure whether he got H or L . After the good signal utility is either $\nu_2(\frac{3}{4}, 1)$ or $\nu_2(\frac{3}{4}, 0)$; and after the bad signal utility is either $\nu_2(\frac{1}{4}, 1)$ or $\nu_2(\frac{1}{4}, 0)$. Expected belief based utility in period 2 is thus $\frac{1}{2}(\frac{3}{4}\nu_2(\frac{3}{4}, 1) + \frac{1}{4}\nu_2(\frac{3}{4}, 0)) + \frac{1}{2}(\frac{1}{4}\nu_2(\frac{1}{4}, 1) + \frac{3}{4}\nu_2(\frac{1}{4}, 0))$. Overall ABB utility is the sum of these three components.

In Section 2, we first show how the prominent sub-classes of ABB functionals are related to each other, demonstrating that models that allow for changing beliefs nest those of prior anticipatory beliefs, which in turn nest those of posterior anticipatory beliefs. We then provide necessary and sufficient conditions for continuous preferences to be represented with an ABB functional. We show that ABB utility is characterized by two simple properties.

First, Prior Conditional Two Stage Independence (PTI) requires standard independence (in mixing the compound lotteries) to hold only if *all* compound lotteries involved in the mixing induce the same prior distribution over final outcomes. That is, if $\phi(P) = \phi(Q) = \phi(R)$, then P is preferred to Q if and only if the mixture of P and R is preferred to the (same-proportion) mixture of

Q and R . Second, Cross Sectional Two Stage Independence (CTI) requires consistency/uniformity across priors in the scale used to measure preferences.

We next turn to showing how, in addition to CTI, strengthening PTI allows us to characterize models of prior-anticipatory beliefs. The key behavior that distinguishes utility from changes in beliefs and utility from the level of beliefs is how broadly Independence (again over compound lotteries) holds. If individuals only care about the levels of their beliefs, then Independence should hold whenever the two lotteries involved in the initial comparison, but not necessarily the one they are both mixed with, induce the same prior distribution over outcomes. That is, the Strong Prior-Conditional Two Stage Independence (SPTI) axiom drops from PTI the requirement that $\phi(R)$ agrees with $\phi(P) = \phi(Q)$. We last show that imposing Independence on all mixtures in the first stage (a property that subsumes all requirements above) characterizes posterior-anticipatory beliefs. Thus, our results demonstrate how a simple set of familiar and easily tested conditions in terms of observed behavior allows distinguishing between different types of belief-dependent utility.

ABB preferences are not the only class of preferences that have been developed to explain informational preferences, even in the absence of the ability to condition actions on that information. A different vein of the literature, primarily developed by Kreps and Porteus (1978) and extended by Segal (1990), focuses on recursive preferences over compound lotteries (and information). We show that in the context of two-stage compound lotteries, the intersection of the two models is precisely the class of preferences that admit a posterior-anticipatory beliefs representation.

Lastly, we investigate what types of restrictions on the functional forms are equivalent to well-known types of intrinsic (i.e., non-instrumental) informational preferences, such as preferences for early resolution of uncertainty (Kreps and Porteus, 1978) or preferences for one-shot resolution of uncertainty (Dillenberger, 2010). In doing so, we provide characterizations that generalize some earlier results, for example those of Kőszegi and Rabin (2009), which were made in the context of specific functional forms.⁴ Our results allow us to determine how different classes of models (recursive and ABB) can, or cannot, accommodate different intrinsic attitudes towards information.

2 The Model

2.1 Preliminaries

Consider a set of prizes X , which is assumed to be a closed subset of some metric space. A simple lottery p on X is a probability distribution over X with a finite support. Let $\Delta(X)$ (or simply Δ)

⁴The specification of Kőszegi and Rabin (2009) functional form is as follows: given lottery p , let $c_p(i)$ be the payoff at percentile i of the distribution induced by p . Then $\nu_1 = \kappa_1 \int \mu(u(c_{\phi(P)}(i)) - u(c_{p_i}(i)))d_i$, $\nu_2 = \kappa_2 \int \mu(u(c_{p_i}(i)) - u(c_{\delta_x}(i)))d_i$, where μ is a gain-loss utility function that is continuous, strictly increasing, twice differentiable for $x \neq 0$ with $\mu''(x) \leq 0$ for $x > 0$ and $\mu''(x) \geq 0$ for $x < 0$, and satisfying $\mu(0) = 0$, $\mu(y) + \mu(-y) < \mu(x) + \mu(-x)$ whenever $y < x \geq 0$, and $\frac{\lim_{x \rightarrow 0} \mu''(|x|)}{\lim_{x \rightarrow 0} \mu''(-|x|)} = \lambda > 1$.

be the set of all simple lotteries on X . For any lotteries $p, q \in \Delta$ and $\alpha \in (0, 1)$, we let $\alpha p + (1 - \alpha)q$ be the lottery that yields prize x with probability $\alpha p(x) + (1 - \alpha)q(x)$. Denote by δ_x the degenerate lottery that yields x with probability 1 and let $\bar{X} = \{\delta_x : x \in X\}$; we will often abuse notation and refer to δ_x simply as x . Similarly, denote by $\Delta(\Delta(X))$ (or simply Δ^2) the set of simple lotteries over Δ , that is, compound lotteries. For $P, Q \in \Delta^2$ and $\alpha \in (0, 1)$, denote by $R = \alpha P + (1 - \alpha)Q$ the lottery that yields simple (one-stage) lottery p with probability $\alpha P(p) + (1 - \alpha)Q(p)$. Denote by D_p the degenerate, in the first stage, compound lottery that yields p with certainty. Define a *reduction operator* $\phi : \Delta^2 \rightarrow \Delta$ that maps compound lotteries to reduced one-stage lotteries by $\phi(Q) = \sum_{p \in \Delta} Q(p)p$.⁵ We refer to $\phi(Q)$ as prior beliefs (or simply a prior). When there is no risk of confusion, we sometimes refer to ϕ as the prior itself, without specifying the compound lottery that induced it. Our primitive is a binary relation \succsim over Δ^2 .

2.2 Functional Forms

We first formally define additive-belief-based utility.

Definition 1. *An additive-belief-based (ABB) representation is a tuple (u, ν_1, ν_2) consisting of continuous functions $u : X \rightarrow \mathbb{R}$, $\nu_1 : \Delta \times \Delta \rightarrow \mathbb{R}$, and $\nu_2 : \Delta \times \bar{X} \rightarrow \mathbb{R}$, such that $V_{ABB} : \Delta^2 \rightarrow \mathbb{R}$ defined as*

$$V_{ABB}(P) = \sum_j \phi(P)(x_j)u(x_j) + \sum_i P(p_i)\nu_1(\phi(P), p_i) + \sum_i P(p_i) \sum_j p_i(x_j)\nu_2(p_i, \delta_{x_j})$$

represents \succsim .

The general ABB functional form allows utility to depend on changes in beliefs in period 1 and period 2. If utility depends on changes in beliefs, then ν_1 and ν_2 are functions of both their arguments. Alternatively, many models in the literature assume that individuals do not care about changes, but rather about the levels of their beliefs. Individuals may care about their beliefs in any given period in one of two ways. The first case supposes that utility depends on beliefs at the beginning of any period, that is, ν_1 is solely a function of $\phi(P)$ and ν_2 is solely a function of p_i . We call this functional form prior-anticipatory utility and define it as follows.

Definition 2. *A prior-anticipatory representation is an ABB representation with the restrictions that $\nu_1(\phi(P), p_i) = \hat{\nu}_1(\phi(P))$ and $\nu_2(p_i, \delta_{x_j}) = \hat{\nu}_2(p_i)$.*

In the second case, utility is derived from beliefs at the end of any period (that period's posterior beliefs, after receiving information), that is, ν_1 is solely a function of p_i and ν_2 is solely a function of δ_{x_j} . We call this posterior-anticipatory utility and the functional form is given by:

⁵Compound lotteries are isomorphic to the set of prior beliefs over outcomes plus a potential information structure. We can associate an information structure with the set of posterior beliefs it induces — the set of second-stage lotteries.

Definition 3. A posterior-anticipatory representation is an ABB representation with the restrictions that $\nu_1(\phi(P), p_i) = \bar{\nu}_1(p_i)$ and $\nu_2(p_i, \delta_{x_j}) = \bar{\nu}_2(\delta_{x_j})$.

Clearly, both anticipatory representations above are subsets of V_{ABB} . More surprisingly, prior-anticipatory representation nests posterior-anticipatory representation.

Lemma 1. If \succsim has a posterior-anticipatory representation, then it has a prior-anticipatory representation.

2.3 Characterization

We now characterize the functionals we have described using the relation \succsim . As will become apparent, our approach to restrict preferences is to impose Independence-type conditions on particular subsets of Δ^2 . The first two axioms are standard.

Weak Order (WO) The relation \succsim is complete and transitive.

Continuity (C) The relation \succsim is continuous.

Our key axiom is Prior Conditional Two-Stage Independence (PTI). PTI requires the Independence axiom to hold within the set of compound lotteries which share the same reduced form probabilities over outcomes (that is, the same prior beliefs). Observe that the set $\mathcal{P}(p) := \{Q \in \Delta^2 | \phi(Q) = p\}$ is convex for any $p \in \Delta$. Thus, PTI says that Independence holds along “slices” of the compound lottery space, where all elements of the slice have the same reduced form probabilities.

Prior Conditional Two-Stage Independence (PTI): For any $P, P', Q \in \Delta^2$ and $\alpha \in [0, 1]$, if $\phi(P) = \phi(P') = \phi(Q)$, then $P \succsim P'$ if and only if $\alpha P + (1 - \alpha)Q \succsim \alpha P' + (1 - \alpha)Q$.

Recall that we identify preferences over compound lotteries with preferences with preferences over the combination of information structures and prior beliefs. PTI then requires that within a set of information structures that correspond to the same prior beliefs, the individual is an expected utility maximizer over their posterior beliefs; “non-standard” behavior (i.e. a reversal in the relative ranking of posteriors) may arise only when comparing across underlying prior beliefs.

In addition to PTI, we need to link the evaluations made across different prior beliefs. This is the content of the following axiom, which states that they are performed using the same “measurement rod”.⁶ CTI ensures that relative preferences of one set of posterior beliefs corresponding to prior ϕ , compared to a second set of posterior beliefs corresponding to prior ϕ' , are not altered by mixing, so long as the mixing preserves the prior associated with each posterior belief.

⁶We could more compactly state PTI and CTI as a single axiom, by not requiring $\phi(Q) \neq \phi(S)$ in the statement of CTI. Note that under this modification, taking $Q = S$ implies PTI. We have decided to state PTI and CTI as two separate requirements since they are conceptually different and play different roles in deriving the result representation. Indeed, we will see below that fixing CTI, both anticipatory representations are obtained by strengthening PTI.

Cross Sectional Two-Stage Independence (CTI): For any $P, Q, R, S \in \Delta^2$ and $\alpha \in [0, 1]$, if $\phi(P) = \phi(Q) \neq \phi(R) = \phi(S)$, $P \succsim R$, and $Q \succsim S$, then $\alpha P + (1 - \alpha)Q \succsim \alpha R + (1 - \alpha)S$.

Our first main result shows that PTI and CTI, along with the standard two axioms above, are all we need to characterize preferences that admit an ABB representation.

Proposition 1. *The relation \succsim satisfies WO, C, PTI, and CTI, if and only if it has an ABB representation.*

All proofs are in Appendix 6.1. To prove Proposition 1, we define a prior-conditional representation $V_{PC} = \sum_i P(p_i) \nu(\phi(P), p_i)$, which is an expected utility functional (over the second-stage lotteries p_i 's) for a fixed prior $\phi(P)$. We then show that the relation \succsim has a prior-conditional representation if and only if it has an ABB representation. By PTI, an immediate application of the Mixture Space Theorem yields that fixing ϕ we have a prior-conditional representation. But while fixing ϕ the rankings within $\mathcal{P}(\phi)$ will not be affected by any monotone transformation of the prior-conditional utility, the rankings across different slices might. Axiom CTI rules this out: it guarantees that all such transformations are ϕ -independent, and can be taken without loss of generality to be the identity. We will revisit the observation that the general ABB representation can be compactly written as a simple expected utility functional in Section 2.4, when discussing its limited uniqueness properties.

PTI is not very restrictive, as it requires mixing not to reverse rankings only when all lotteries involved in the mixing have the same reduced form probabilities. A natural way to strengthen it is to suppose that only the compound lotteries involved in the *original* preference comparison need to have the same reduced form probabilities — the common compound lottery that they are mixed with need not. This means that the pair of lotteries which are compared after the mixing will have the same reduced form probabilities as each other, but need not have the same reduced form probabilities as the original pair. The next axiom formalizes this intuition.

Strong Prior Conditional Two-Stage Independence (SPTI): For any $P, P', Q \in \Delta^2$ and $\alpha \in [0, 1]$, if $\phi(P) = \phi(P')$, then $P \succsim P'$ if and only if $\alpha P + (1 - \alpha)Q \succsim \alpha P' + (1 - \alpha)Q$.

SPTI rules out complementarity between the prior distribution and the corresponding information systems. That is, irrespectively of the underlying prior beliefs, the individual consistently chooses among information systems based on the expected utility criterion over posterior beliefs; the relative value of posterior beliefs under prior ϕ does not change if we move to prior ϕ' . Thus, violations of expected utility may occur only when the decision maker compares two compound lotteries that do not refine the same prior beliefs.

SPTI clearly implies PTI, but it is logically independent of CTI. Replacing PTI with SPTI yields our second characterization result:

Proposition 2. *The relation \succsim satisfies WO, C, CTI, and SPTI, if and only if it has a prior-anticipatory representation.*

Similar to the previous proposition, the main step in proving Proposition 2 is showing that \succsim has a prior-anticipatory representation if and only if it has a (prior-separable) representation of the form $V_{ps} = \nu_{ps1}(\phi(P)) + \sum_i P(p_i)\nu_{ps2}(p_i)$.

In order to characterize posterior-anticipatory representations, we further strengthen when Independence applies. The next axiom implies both CTI and SPTI, as it requires Independence to hold when mixing *any* two compound lotteries in the first stage.

Two-Stage Independence (TI): *For any $P, P', Q \in \Delta^2$ and $\alpha \in [0, 1]$, $P \succsim P'$ if and only if $\alpha P + (1 - \alpha)Q \succsim \alpha P' + (1 - \alpha)Q$.*

TI implies that prior beliefs do not matter when considering preferences over information structures. Our next result shows that it is equivalent to a posterior anticipatory representation.

Proposition 3. *The relation \succsim satisfies WO, C, and TI, if and only if it has a posterior-anticipatory representation.*

2.4 Special Cases and Uniqueness

The functional forms we have previously derived above are quite general. In many cases, we may want to suppose further restrictions on the set of functionals we consider.

One typical assumption within the literature is that, in either stage, the individual receives the same utility (often normalized to 0) from beliefs that do not change. We describe these preferences as belief stationarity invariant (BSI).

Definition 4. *An ABB representation is belief-stationarity invariant (BSI) if $\nu_1(p_i, p_i) = \nu_1(q_i, q_i) = \nu_2(\delta_x, \delta_x) = \nu_2(\delta_y, \delta_y) = 0$ for all x, y, p_i, q_i .*

A second type of assumption is that the utility derived from beliefs does not depend on which period those beliefs are realized. We call this belief time invariance (BTI).⁷

Definition 5. *An ABB representation is belief time invariant (BTI) if $\nu_1 = \nu_2$ over their relevant shared domain.*

In order to relate BSI and BTI to behavior, we discuss a certain restriction on preferences over compound lotteries. The next axiom is due to Segal (1990).

⁷BTI rules out situations where individuals may get a stronger or weaker “kick” from beliefs if they occur sooner (for example, via discounting). We can weaken BTI to allow for such considerations, and say that BPS representation is pseudo-belief time invariant (PBTI) if for some scalar $\kappa > 0$, $\nu_1 = \kappa\nu_2$ over their relevant shared domain. However, in the end of the proof of Proposition 4 we show that PBTI does not restrict preferences alone and, furthermore, that BSI and PBTI in conjunction have no observable implications as well, as long as κ can be chosen arbitrarily (i.e., is not fixed in a given value).

Time Neutrality (TN): For any $p \in \Delta$, if $P = D_p$ and $Q = \sum_i p(x_i)\delta_{x_i}$, then $P \sim Q$.

Time Neutrality supposes that the individual is indifferent between a compound lottery that fully resolves in period 1 and one that resolves only in period 2 (so that the information structure reveals no information in period 1), provided that they both induce the same probability distribution over final outcomes.

Although BSI and BTI do not restrict preferences alone, in conjunction they do.

Proposition 4. *Suppose \succsim has an ABB representation. The following statements are true:*

1. *The relation \succsim has a representation which is belief stationarity invariant.*
2. *The relation \succsim has a representation which is belief time invariant.*
3. *The relation \succsim has a representation which is both belief stationarity invariant and belief time invariant, if and only if it satisfies Time Neutrality.*

The fact that we can obtain either a BSI or a BTI representation without loss of generality raises the question to what extent are ABB preferences uniquely identified. The uniqueness property can be broken up into two parts: First, an immediate application of the mixture space theorem implies that if V and V' are both ABB representations of the same preference relation, then they differ by a positive affine transformation.

Proposition 5. *Suppose \succsim has an ABB representation V . The ABB representation V' also represents \succsim if and only if there exist scalars $\alpha > 0$ and β such that $V' = \alpha V + \beta$.*

Second, there are individual terms that can be subtracted from one component and absorbed in another, leaving the numerical value intact. In Appendix 6.2 we show that the uniqueness results of the sub-components u , ν_1 , and ν_2 are more subtle because any outcome that generates material utility must also appear in the support of the beliefs entering ν_1 and ν_2 . Thus, one should expect that, without any further restrictions, there is some freedom to assign utility that is generated by any x appearing in the support of the lottery to either material utility or belief-based utility. In particular, this suggests that attitudes towards risk cannot be uniquely identified — attitudes towards final outcomes can be adjusted across all three functions that compose the representation.

If, instead, we focus on the standard normalization applied in the literature (i.e. the one imposed by BSI), then u is unique up to an affine transformation, while ν_1 and ν_2 are unique up to common scaling. Since any ABB preferences have a BSI representation, this uniqueness result is entirely general.⁸

⁸We only consider transformations that generate different actual values for all sub-functions involved in the transformation, and do not consider transformations that add or subtract elements to a specific sub-functioinal that leave its own actual value unchanged. For example, $\sum_x p(x)\gamma_\nu(p, \delta_x) = \sum_x p(x)(\gamma_\nu(p, \delta_x) + \epsilon_p(x))$ whenever $\sum_x p(x)\epsilon_p(x) = 0$.

Proposition 6. *Suppose \succsim has an ABB representation (u, ν_1, ν_2) that satisfies BSI. The ABB representation (u', ν'_1, ν'_2) also represents \succsim and satisfies BSI if and only if there exist scalars $\alpha > 0$ and β_u such that $u'(x) = \alpha u + \beta_u$; $\nu'_1(\rho, p) = \alpha \nu_1(\rho, p)$; and $\nu'_2(p, \delta_x) = \alpha \nu_2(p, \delta_x)$.*

3 ABB and Recursive Preferences

ABB preferences are not the only preferences used to model decisions over compound risk; an alternative specification is of preferences that are recursive. Recursive preferences have also played an extensive role in a variety of models attempting to capture, among other things, choices over compound lotteries and information (see Kreps and Porteus, 1978; Segal, 1990; Grant, Kajii, and Polak, 1998; Dillenberger, 2010; Dillenberger and Segal (2017); and Sarver, 2018).

Segal (1990) was the first to formally discuss recursive preferences on the domain of compound lotteries. In the definition below, $\mathbb{C}\mathbb{E}_W(p)$ denotes the certainty equivalent of $p \in \Delta$ corresponding to the real function W on Δ , that is, $W(p) = W(\delta_{\mathbb{C}\mathbb{E}_W(p)})$.⁹

Definition 6. *Suppose preferences over Δ^2 can be represented by the functional V . We say that preferences have a recursive representation (V_1, V_2) , where $V_i : \Delta \rightarrow \mathbb{R}$, if and only if for all $P = \sum_i P(p_i)D_{p_i}$, we have $V(P) = V_1(\sum_i P(p_i)\delta_{\mathbb{C}\mathbb{E}_{V_2}(p_i)})$.*

Segal (1990) provided a behavioral equivalent for these functional forms using a substitution axiom he called Compound Independence, which we refer to as Recursivity.

Recursivity (R): *For any $p, q \in \Delta$, $Q \in \Delta^2$, and $\alpha \in [0, 1]$, $D_p \succsim D_q$ if and only if $\alpha D_p + (1 - \alpha)Q \succsim \alpha D_q + (1 - \alpha)Q$.*

Similarly to our previous main assumptions, Recursivity applies Independence to a particular “slice” of compound lotteries: the original pair of lotteries being compared must be degenerate in the first stage. This slice is, however, orthogonal to that considered by CTI and PTI (or SPTI). Segal (1990) shows that the relation \succsim satisfies WO, C, and R, if and only if it admits a recursive representation.

One immediate question is to what extent these two classes of utility, ABB and recursive, are related. A next result shows that their intersection is exactly those preferences which admit a posterior-anticipatory representation.

Proposition 7. *The following are equivalent:*

- *The relation \succsim satisfies WO, C, PTI, CTI, and R*
- *The relation \succsim has a posterior-anticipatory representation*

⁹For the certainty equivalent to be well-defined, we need to impose some order on the set X . It will be the case whenever we take the set of prizes to be an interval $X \subset \mathbb{R}$ and both functions V_i in Definition 6 are monotone with respect to first-order stochastic dominance.

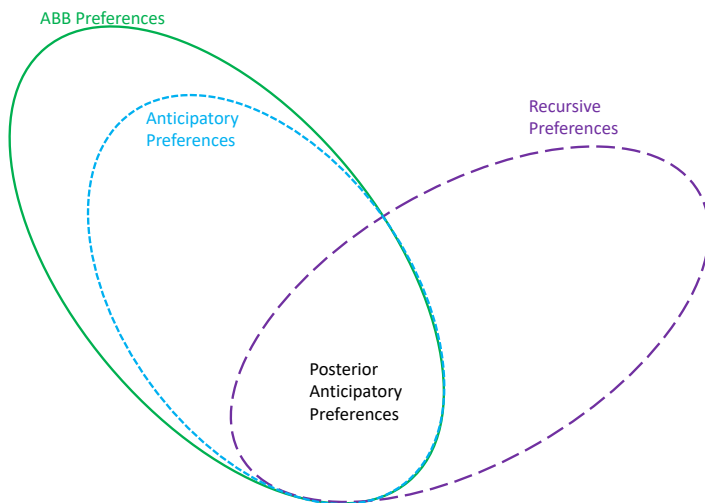


Figure 1: Relationships between models

- The relation \succsim has a recursive representation where V_1 is expected utility

Figure 1 depicts the relationships discussed here and the results of the last section.

In general, ABB models can be directly tested (and falsified) in the natural domain of information preferences given a fixed prior — this is precisely the subdomain where axiom PTI bites. In contrast, recursive preferences, *as a general class*, have no observable restrictions when the prior is fixed; testing the assumption of Recursivity (Axiom R) requires observing preferences as the prior changes. Proposition 7 then implies that this latter property applies as well to the subclass of preferences that have posterior-anticipatory representation. The proposition further implies that any posterior-anticipatory representation captures individuals who have “emotions over emotions”. This is because recursive models transform any two-stage compound lottery into a simple lottery over the second-stage certainty equivalents. Those certainty equivalents capture all second-period utility from beliefs changing in the second stage (e.g., disappointment/elation). Since first-period preferences take the certainty equivalents as the possible outcomes, those anticipated second period emotions are part and parcel of the “material” payoffs of the first-stage preferences. In all ABB models which do not have posterior-anticipatory representation, future changes in beliefs are independent of the effects of previous changes in beliefs. In other words, first period’s beliefs-based utility, as captured by ν_1 , is all about changes in material outcomes and does not depend on second period’s belief-based utility, ν_2 .

In addition to axiom R, Segal (1990) also introduced several other restrictions on preferences over compound lotteries (such as the Reduction of Compound Lotteries axiom and the requirement that Independence holds among both the set of full early resolving lotteries and fully late resolving

lotteries). To complete our analysis, in Appendix 6.3 we establish their relationship with CTI, PTI, SPTI, and TI, and further interpret these connections via the functional forms.

4 ABB and the Timing of Uncertainty Resolution

Individuals with ABB utility will have intrinsic preferences over information, that is, they may prefer one information structure to another even in the absence of the ability to condition actions on either of them. Many papers looking at specific examples of ABB functional forms derive results regarding preferences over information, while focusing on two concepts: preferences for early versus late resolution of uncertainty and preferences for one-shot versus gradual resolution of uncertainty. In an analogous vein, characterizations of these informational attitudes have been a major focus of the decision-theoretic literature. However, there do not exist equivalent characterizations for ABB preferences. Our results will allow us to compare how different classes of models (recursive and ABB) can accommodate (or not) different non-instrumental attitudes towards information.

Many authors, such as Kreps and Porteus (1978) and Grant, Kajii, and Polak (1998), conjecture that individuals not only prefer uncertainty to be fully resolved earlier (in period 1 rather than in period 2) but also that they always prefer Blackwell-more-informative signals in period 1, that is, earlier resolution of uncertainty. Drawing on Grant, Kajii, and Polak (1998), we can define a preference for early resolution of uncertainty.

Definition 7. *The relation \succsim displays a preference for early resolution of uncertainty if*

$$\beta\alpha D_q + (1 - \beta)\alpha D_p + (1 - \alpha)Q \succsim \alpha D_r + (1 - \alpha)Q$$

for any $Q \in \Delta^2$ and $p, q, r \in \Delta$ such that $r = \beta p + (1 - \beta)q$.

That is, preference for early resolution of uncertainty implies affinity towards spiting any branch that leads to some posterior beliefs r into several branches, whenever the split consists of a mean-preserving spread of r . Preference for late resolution of uncertainty is analogously defined, by requiring the reverse ranking for the lotteries above.

We now characterize preferences that exhibit preferences for either earlier or later resolved lotteries. Similarly to known results about recursive preferences, attitude towards the resolution of uncertainty is characterized in our model by the curvature of the appropriate components. In the next result we refer to preferences which have a prior-anticipatory representation, but do not have a posterior-anticipatory representation, as having a prior*-anticipatory representation.

Proposition 8. *The following statements are true:*

1. *Suppose \succsim has ABB representation. Then \succsim exhibits a preference for early (resp., late) resolution of uncertainty if and only if $\nu_1(\rho, \cdot) + \sum_x \nu_2(\cdot, x)$ is convex (resp., concave).*

2. Suppose \succsim has a prior*-anticipatory representation. Then \succsim exhibits a preference for early (resp., late) resolution of uncertainty if and only if ν_2 is convex (resp., concave).
3. Suppose \succsim has a posterior-anticipatory representation. Then \succsim exhibits a preference for early (resp., late) resolution of uncertainty if and only if $\bar{\nu}_1$ is convex (resp., concave).

A distinct notion of preferences for resolution of uncertainty is discussed by Dillenberger (2010). He supposes that individuals satisfy Time Neutrality (axiom TN described earlier) and that they prefer either compound lotteries in which *all* uncertainty is resolved in period 1 or in period 2 to any other compound lotteries which induce the same prior beliefs. He defines this as a preference for one-shot resolution of uncertainty.

Definition 8. *The relation \succsim exhibits a preference for one-shot resolution of uncertainty (PORU) if for all $P, Q, R \in \Delta^2$ such that $\phi(P) = \phi(Q) = \phi(R) = p$, if $P = D_p$ and $Q = \sum_i p(x_i)\delta_{x_i}$, then $P \sim Q \succsim R$.*

Proposition 9. *The following statements are true:*

1. Suppose \succsim has an ABB representation. Then \succsim exhibits a preference for one-shot resolution of uncertainty if and only if

$$\sum_i P(p_i) \sum_x p_i(x) \nu_1(\phi(P), \delta_x) \geq \sum_i P(p_i) \sum_x p_i(x) \nu_1(\phi(P), p_i) + \sum_i P(p_i) \sum_x p_i(x) \nu_1(p_i, \delta_x)$$

2. If \succsim has a prior-anticipatory representation, then it can never exhibit strict preference for one-shot resolution of uncertainty.¹⁰

For item (1), suppose \succsim has an ABB representation. Observe first that PORU implies TN. Therefore, by Proposition 4, if \succsim exhibits PORU then it must have an ABB representation that satisfies both BSI and BTI, and in particular $\nu_1 = \nu_2$. And since the expected utility from material payoffs is the same in all lotteries compared, the result follows.¹¹ The intuition behind item (2) derives from Corollary 1 of Appendix 6.3, where we show that if \succsim has a prior-anticipatory representation, then TN implies that ν_2 is an expected utility functional, and thus does not generate any anomalous preferences towards information. The rest of the utility functional depends only

¹⁰That is, there are no triple as in Definition 8 for which $P \sim Q \succ R$.

¹¹Note that the inequality in item (1) has the form of (the opposite of) the triangle inequality, which should hold ‘on average’. Indeed, one example of a function that satisfies it is $\nu_1(\rho, p) = -d(\rho, p)$, where d is some standard distance measure on the unit simplex; the inequality then holds term by term, and thus also in expectation. In this case, the utility loss from beliefs moving equals the total expected distance traveled by beliefs. Indeed, for such function the inequality holds term by term (this is the negative of the triangle inequality) and thus also in expectation. One interpretation for such functional form would be that the agent is simply averse to any changes in beliefs, presumably due to some hidden costs of adjustments (think of an agent who pre-committed to some belief-contingent action, or someone who placed bets on his intermediate beliefs) that are ‘large enough’, overwhelming any effect of good news.

Changing Beliefs	Prior*-Anticipatory Utility	Posterior-Anticipatory Utility
Caplin and Eliaz (2003)	<i>See text</i>	Kreps and Porteus (1979)
Mullainathan and Shleifer (2005)		Epstein and Zin (1989)
Kőszegi and Rabin (2009)		Caplin and Leahy (2001)
Kőszegi (2010)		Kőszegi (2003)
Ely, Frankel and Kamenica (2015)		Caplin and Leahy (2004)
Pagel (2016)		Kőszegi (2006)
		Eliaz and Spiegel (2006)
	Eliaz and Schotter (2010)	
	Szech and Schweitzer (2016)	

Table 1: Some models nested by ABB

on ϕ , the reduced form probability of the lottery, independently of the pattern of resolution of uncertainty. The result suggests preferences for one-shot resolution of uncertainty as a sufficient condition to rule out anticipatory preferences.

5 Discussion

We now discuss how our general framework applies to some specific functional forms used in the literature. In order to provide a sense of the breadth of models that our approach encompasses, Table 1 provides a list of papers which use functional forms nested by ABB. Many of these models also allow individuals to take intermediate actions, so their domain and representation may appear different than that presented in this paper.¹²

Some models in the literature fit into the framework of changing beliefs models — they have an ABB representation, but do not have any anticipatory representation. These include models that are explicitly meant to capture utility derived from changing beliefs, such as Kőszegi and Rabin (2009) and Pagel (2016). Ely, Frankel, and Kamenica (2015) also model individuals who care about changes in their beliefs. Their model of surprise has the same structure as our ABB representation, but is formally not within the class of models we study because it is discontinuous. Their model of suspense is similar in spirit, but formally different from ABB models not only due to its lack of continuity, but also because of a non-linear transformation that is applied to the expected utility from changes in beliefs. In other models, utility may not be derived from changes in beliefs per se. Rather, utility is derived from levels of beliefs, but the function that determines the levels depends on the prior beliefs. These include the models of Mullainathan and Shleifer (2005), where individuals seek signals that confirm their priors, as well as Caplin and Eliaz (2003), which takes on

¹²As in the previous section, we denote models that have prior-anticipatory but not posterior-anticipatory representation as having prior*-anticipatory utility.

the form of a prior-dependent Kreps-Porteus representation. Within our domain, we also capture the model of Kőszegi (2010), who explicitly models expectations (i.e. beliefs) that interact with material payoffs (although his domain also allows for actions and material payoffs in period 1).

Other models in the literature adhere to the anticipatory-utility framework. In particular, our posterior-anticipatory utility delivers the Caplin and Leahy (2001) representation when applied to our domain, and thus all the models that are based on their framework are captured by our model. These include Kőszegi (2003), Caplin and Leahy (2004), Kőszegi (2006), Eliaz and Spiegel (2006), Eliaz and Schotter (2010), and Szech and Schweitzer (2016).

We know of no existing models that explicitly capture pure prior-anticipatory motivations, that is, models that admit a prior-anticipatory, but not posterior-anticipatory, representation. To better understand the gap between the two anticipatory representations, in Proposition 13 of Appendix 6.3 we provide an alternative characterization of posterior-anticipatory representation, which amounts to adding to the axioms underlying the prior-anticipatory representation an additional Independence requirement on the subset of early resolving lotteries (compound lotteries in which all uncertainty is resolved in the first stage). Descriptively, we believe prior-anticipatory motives are important, as they can accommodate a behavior that violates expected utility over early resolving lotteries, in accordance with frequently observed experimental results, such as the Allais paradox. To see that this sub-class is nonempty, let f and g be two arbitrary non-expected utility functions.¹³ In Claim 3 of Appendix 6.1 we establish that the function $V(P) = f(\phi(P)) + \sum_i P(p_i)g(p_i)$ is a prior*-anticipatory utility.

Our model is related to several axiomatizations which are also distinct from the literature on recursive preferences. As previously discussed, our functional form nests that of Caplin and Leahy (2001). Caplin and Leahy provide an axiomatization of their functional form, but take as their domain the set of “psychological lotteries”, which include lotteries not just over material outcomes, but also over psychological states (i.e., beliefs). Thus, their domain includes objects (psychological states) which are explicitly not observable, and not directly choosable. Our approach, which confines attention to preferences over compound lotteries, ensures that all our restrictions are stated solely in terms of preferences over observable objects.

Recently Gul, Natenzon and Pesendorfer (2016) have introduced a model that shares some key features of our model. Although the domain and objects of choice are quite different than ours, there are many key similarities in that both papers consider utility functions where individuals gain utility from beliefs and from material payoffs in a way that is additively separable. However, while we suppose individuals calculate the expectation of a belief-based utility using objective probabilities, Gul, Nautenzon and Pesendorfer (2016) allow for non-additive measures.

The ABB representation is also reminiscent of the functional forms used in the rational inatten-

¹³Prominent examples of non-expected utility models include Rank-Dependent Utility (Quiggin, 1982) and the Betweenness class of preferences (Chew, 1983; Dekel, 1986).

tion literature (e.g. Sims, 2003). For example, if we assume $\nu_2 = 0$, then we can interpret ν_1 as the cost of beliefs shifting from the prior to the posterior. There are two caveats to this analogy. First, while in our current framework individuals do not take intermediate actions, in rational inattention models individuals only pay the cost of learning in order to match actions to states.¹⁴ The second difference is more subtle. The rational inattention literature typically supposes that individuals choose information in a way that is unobserved by the analyst — in an extreme case, information is obtained solely by self-reflection. Our domain, on the other hand, specifically supposes we observe individuals choosing between known, objectively given information structures. Therefore, one can interpret the cost of information in our setting as the amount that must be paid conditional on choosing a particular information structure. This is distinct from a rational inattention story, where even in the face of objective information, an individual could avoid paying costs by simply not processing that information.

6 Appendix

6.1 Proofs

Proof of Lemma 1: By Definition 2, a prior-anticipatory representation is given by $\sum_i P(p_i) \sum_j p_i(x_j)u(x_j) + \nu_1(\phi(P)) + \sum_i P(p_i)\nu_2(p_i)$. Note that preferences admit such a representation if and only if they can be represented by the functional $\hat{\nu}_1(\phi(P)) + \sum_i P(p_i)\nu_2(p_i)$, for some arbitrary non-expected utility functional $\hat{\nu}_1$. Similarly, by Definition 3, a posterior-anticipatory representation is given by $\sum_i P(p_i) \sum_j p_i(x_j)[u(x_j) + \nu_2(x_j)] + \sum_i P(p_i)\nu_1(p_i)$, which is equivalent to a representation of the form $\sum_i P(p_i) \sum_j p_i(x_j)\hat{u}(x_j) + \sum_i P(p_i)\nu_1(p_i)$. Clearly the second representation is a subset of the first. \square

Proof of Proposition 1. We first define a prior-conditional representation as $V_{PC} = \sum_i P(p_i)\nu_{PC}(\phi(P), p_i)$.

Claim 1. *The relation \succsim has a prior-conditional representation if and only if it has an ABB representation.*

Proof of Claim 1. Consider the first two terms in the ABB representation. Observe that the term $\sum_i P(p_i) \sum_j p_i(x_j)u(x_j) + \sum_i P(p_i)\nu_1(\phi(P), p_i)$ can be rewritten as $\sum_i P(p_i)\hat{\nu}_1(\phi(P), p_i)$. Similarly, any $\sum_i P(p_i)\hat{\nu}_1(\phi(P), p_i)$ can be rewritten as $\sum_i P(p_i) \sum_j p_i(x_j)u(x_j) + \sum_i P(p_i)\nu_1(\phi(P), p_i)$.

Consider now the third term in the ABB representation. Note that any $\sum_i P(p_i) \sum_j p_i(x_j)\nu_2(p_i, \delta_{x_j})$ can be rewritten as $\sum_i P(p_i)\hat{\nu}_2(p_i)$, since p_i embeds all the x_j s in it's support. Moreover, given any $\sum_i P(p_i)\hat{\nu}_2(p_i)$, we can rewrite it as $\sum_i P(p_i) \sum_j p_i(x_j)\nu_2(p_i, \delta_{x_j})$.

¹⁴This is, however, not a particularly large difference since, as have previously mentioned, we could extend our domain to include such actions.

Thus preferences have an ABB representation if and only if they can be represented by

$$\sum_i P(p_i) \hat{v}_1(\phi(P), p_i) + \sum_i P(p_i) \hat{v}_2(p_i)$$

Simplifying further, observe that any $\sum_i P(p_i) \hat{v}_1(\phi(P), p_i) + \sum_i P(p_i) \hat{v}_2(p_i)$ can be rewritten as $\sum_i P(p_i) \tilde{v}(\phi(P), p_i)$; and any $\sum_i P(p_i) \tilde{v}(\phi(P), p_i)$ can be rewritten as $\sum_i P(p_i) \hat{v}_1(\phi(P), p_i) + \sum_i P(p_i) \hat{v}_2(p_i)$. We have just proved that \succsim has a representation of the form V_{ABB} if and only if it has a representation of the form $\sum_i P(p_i) \tilde{v}(\phi(P), p_i)$.

We now use our new representation for Claim 2.

Claim 2. *The relation \succsim has a prior-conditional representation if and only if it satisfies WO, C, PTI, and CTI.*

Proof of Claim 2. Observe that the prior-conditional representation holds if and only if for any fixed ϕ preferences are expected utility, which is known to be equivalent to WO, C, and PTI. Each of these conditional expected utility functionals are ordinally unique up to a monotone transformation f_ϕ , which a priori may depend on the reduced form probabilities. We now show that this is impossible, so that f is independent of ϕ and thus can be taken without loss of generality to be affine.

We partition the set of compound lotteries into two types of orthogonal equivalence classes. The first is the set \mathcal{I} of equivalence classes induced by \succsim ; denote an arbitrary class by I (with $P, Q \in I$ if and only if $P \sim Q$), and for any $P \in \Delta$ let $I(P) = \{Q : P \sim Q\}$. The second set includes the equivalence classes induced by ϕ , with a generic element $\mathcal{P}(\phi) = \{P \in \Delta : \phi(P) = \phi\}$.¹⁵ Note that if there are disjoint $I, I' \in \mathcal{I}$ that have nonempty intersection with $\mathcal{P}(\phi)$, then all equivalence classes of \succsim that contain an element of $\mathcal{P}(\phi)$ must form a convex set. To see this, let $I(\phi) = \{I(P) : P \in \mathcal{P}(\phi)\}$ and consider any two elements $I, I' \in I(\phi)$ with corresponding $P \in I \cap \mathcal{P}(\phi)$ and $P' \in I' \cap \mathcal{P}(\phi)$. Since by PTI standard Independence holds on $\mathcal{P}(\phi)$, preferences must be continuous over mixtures of P and P' and so for any $\alpha \in [0, 1]$, $\alpha P + (1 - \alpha)P'$ must be ranked in terms of \succsim between P and P' (and conversely, for any Q ranked in terms of \succsim between P and P' , we can find an $\alpha \in [0, 1]$ such that $\alpha P + (1 - \alpha)P' \sim Q$.)

We next show that for any $I \in \mathcal{I}$, f_ϕ is independent of ϕ on I . We do it by dividing the set \mathcal{I} into two classes and show that f is independent of ϕ within each class. The classes have non-intersecting interiors, but may overlap at their boundary points.

1. This class includes all $I \in \mathcal{I}$ such that $I = \bigcup_{\phi \in \Phi} \mathcal{P}(\phi)$ for some set of priors Φ . This means that if a given $\mathcal{P}(\phi)$ has an element in one such I , then all elements of $\mathcal{P}(\phi)$ must be in I . We

¹⁵If $\mathcal{P}(\phi) \subseteq I$ for some I , then, for this ϕ , the decision maker is simply indifferent to the resolution of uncertainty.

can then normalize the utility of all elements of $\{\mathcal{P}(\phi) : \phi \in \Phi\}$ to some constant k . Then f is independent of any $\phi \in \Phi$ on I (since all f_ϕ s take in k as an argument and must output the same number).

2. This class includes all $I \in \mathcal{I}$ such that there exists a ϕ where both $I \cap \mathcal{P}(\phi) \neq \emptyset$ and $(\mathcal{I} \setminus I) \ni J \cap \mathcal{P}(\phi) \neq \emptyset$ hold. Thus, if a given $\mathcal{P}(\phi)$ has an element in one such I , then another element of $\mathcal{P}(\phi)$ must be in a different I' (which by construction is also part of this class). Denote the set of these I s as \mathbb{I} . Denote the closure of any set Z as $\text{cl}(Z)$.

Pick a ϕ such that $I(\phi) \subseteq \text{cl}(\mathbb{I})$ and either (i) there exists a worst \succsim -equivalence class that a member of $\mathcal{P}(\phi)$ is in, denoted $I_{\min}(\phi)$, and this is the lowest element of $\text{cl}(\mathbb{I})$: $P \in I_{\min}(\phi) \implies Q \succsim P, \forall Q \in I \in \mathbb{I}$; or (ii) there exists a sequence of $P_n \in \mathcal{P}(\phi)$ such that, for each n , $P_n \in I_n \in \mathbb{I}$ and the sequence I_n converges to the lowest element of $\text{cl}(\mathbb{I})$. In either case we denote by $\underline{I}(\phi)$ an element of the greatest lower bound of the set of indifference classes each containing some $P \in \mathcal{P}(\phi)$. Thus, we pick out a ϕ such that $I(\phi)$ is not singleton, and there exists $I \in \text{cl}(I(\phi))$ that is included in the worst indifference class of \mathbb{I} .

Set $I^1 = \text{cl}(I(\phi))$. Inductively, set $I^n = I(\phi)$ for a ϕ such that $I_{\min}(\phi)$ is the lowest element of $\text{cl}(\mathbb{I} \setminus \bigcup_{i=1 \dots n-1} I^i)$. We refer to each I^n as a sub-class. For any n , we denote by ϕ^n the ϕ that generates I^n . Note that this construction spans the entire \mathbb{I} ; that each I^i is a convex set and is closed (by construction); that given any two I^k and I^j , their intersection can only consist of a single equivalence class I ; and that there must be a countable number of these sub-classes.

Start with I^1 . There are two cases to consider. The first case is where $I(\phi^1) = I^1$. In this case, the maximal and minimal \succsim -equivalence classes of I^1 contain members of $\mathcal{P}(\phi^1)$. For any I^i , let \bar{I}^i as be a lottery in the \succsim -maximal $I \in I^i$. ~~Similarly, define~~ and \underline{I}^i as a lottery in the \succsim -minimal $I \in I^i$. Note that by the uniqueness result of the vNM functional on each slice, we can normalize $\sum_j \bar{I}^1(p_j) \tilde{\nu}(\phi^1, p_j) = 1$ and $\sum_j \underline{I}^1(p_j) \tilde{\nu}(\phi^1, p_j) = 0$. Using the fact that within a slice Independence is satisfied, for any $P \in \mathcal{P}(\phi^1)$ we can assign a number $\lambda(P) \in [0, 1]$ such that $\sum_j P(p_j) \tilde{\nu}(\phi^1, p_j) = \lambda(P)$.

In the second case, $I(\phi^1) \subset I^1$. In this case, the maximal and minimal \succsim -equivalence classes of I^1 are reached in the limit by a sequence $\{I_k\}$ each contains a member of $\mathcal{P}(\phi^1)$. Suppose it is the minimal \succsim -equivalence class (the other case is analogous). Then, as described above, we denote by \underline{I}^1 an element of the greatest lower bound of $\{I_k\}$ and repeat the normalization process described in the previous case, but where $\lambda(P)$ is identified as the limit of our mixing operation using \bar{I}^i and a sequence of elements in $I_k \cap \mathcal{P}(\phi^1)$.

We have now completed our normalization for I^1 . We extend the normalization process inductively for any I^i , where $i > 1$, considering three different cases. The proof below

supposes that $I(\phi^i) = I^i$. When $I(\phi^i) \subset I^i$ the mixture operation involving $\overline{I^i}$ and $\underline{I^i}$ can be interpreted as mixing with a sequence of lotteries and taking the limit.

Remark 1. *In what follows, we abuse notation and denote by $I^i \cap \mathcal{P}(\phi)$ a subset of $\mathcal{P}(\phi)$, where each of its elements belongs to some $I \in I^i$. We also write $P \in I^i$ rather than “ $P \in I \in I^i$ ”.*

- (a) The first case is where $I^{i-1} \cap I^i = \emptyset$. In this case, there is a “gap” in the indifference classes between I^{i-1} and I^i . Suppose that $\sum_j \overline{I^{i-1}}(p_j) \tilde{\nu}(\phi^{i-1}, p_j) = \kappa$. We fix a $\delta > 0$ and set $\sum_j \underline{I^i}(p_j) \tilde{\nu}(\phi^i, p_j) = \kappa + \delta$ and $\sum_j \overline{I^i}(p_j) \tilde{\nu}(\phi^i, p_j) = \kappa + 1 + \delta$. For any $P \in \mathcal{P}(\phi^i)$ assign a number $\lambda(P) \in [\kappa + \delta, \kappa + \delta + 1]$ such that $\sum_j P(p_j) \tilde{\nu}(\phi^i, p_j) = \lambda(P)$.

Now consider any $\phi' \neq \phi^i$ such that $I^i \cap \mathcal{P}(\phi') \neq \emptyset$. Let $\underline{R^i(\phi')}$ be a \succeq -minimal element in $I^i \cap \mathcal{P}(\phi')$. Similarly, let $\overline{R^i(\phi')}$ be a \succeq -maximal element in $I^i \cap \mathcal{P}(\phi')$. Find $P \in I^i \cap \mathcal{P}(\phi^i)$ such that $P \sim \underline{R^i(\phi')}$ and set $\sum_j \underline{R^i(\phi')}(p_j) \tilde{\nu}(\phi', p_j) = \lambda(P)$. Similarly, find $P' \in I^i \cap \mathcal{P}(\phi^i)$ such that $P' \sim \overline{R^i(\phi')}$ and set $\sum_j \overline{R^i(\phi')}(p_j) \tilde{\nu}(\phi', p_j) = \lambda(P')$.

- (b) The second case is where $I^{i-1} \cap I^i \neq \emptyset$ (recall that by construction these two sets can only overlap at a single indifference class) and there is no $\tilde{\phi}$ such that both $\mathcal{P}(\tilde{\phi}) \cap I^{i-1}$ and $\mathcal{P}(\tilde{\phi}) \cap I^i$ consist of more than a singleton. In this case, there is no “gap” in the indifference classes between I^{i-1} and I^i but there is no prior ϕ such that $I(\phi)$ has non-trivial intersection with both I^{i-1} and I^i . Suppose that $\sum_j \overline{I^{i-1}}(p_j) \tilde{\nu}(\phi^{i-1}, p_j) = \kappa$. We set $\sum_j \underline{I^i}(p_j) \tilde{\nu}(\phi^i, p_j) = \kappa$ and $\sum_j \overline{I^i}(p_j) \tilde{\nu}(\phi^i, p_j) = \kappa + 1$. For any $P \in \mathcal{P}(\phi^i)$ assign a number $\lambda(P) \in [\kappa, \kappa + 1]$ such that $\sum_j P(p_j) \tilde{\nu}(\phi^i, p_j) = \lambda(P)$.

As in the previous case, now consider any $\phi' \neq \phi^i$ such that $I^i \cap \mathcal{P}(\phi') \neq \emptyset$. Let $\underline{R^i(\phi')}$ be a \succeq -minimal element in $I^i \cap \mathcal{P}(\phi')$. Similarly, let $\overline{R^i(\phi')}$ be a \succeq -maximal element in $I^i \cap \mathcal{P}(\phi')$. Find $P \in I^i \cap \mathcal{P}(\phi^i)$ such that $P \sim \underline{R^i(\phi')}$ and set $\sum_j \underline{R^i(\phi')}(p_j) \tilde{\nu}(\phi', p_j) = \lambda(P)$. Similarly, find $P' \in I^i \cap \mathcal{P}(\phi^i)$ such that $P' \sim \overline{R^i(\phi')}$ and set $\sum_j \overline{R^i(\phi')}(p_j) \tilde{\nu}(\phi', p_j) = \lambda(P')$.

- (c) The third case is where $I^{i-1} \cap I^i \neq \emptyset$ (recall that by construction these two sets can only overlap at a single indifference class) and there exists a $\tilde{\phi}$ such that both $\mathcal{P}(\tilde{\phi}) \cap I^{i-1}$ and $\mathcal{P}(\tilde{\phi}) \cap I^i$ consist of more than a singleton. In this case, there is no “gap” in the indifference classes between I^{i-1} and I^i and there is a prior, denoted $\tilde{\phi}$ below, such that $I(\tilde{\phi})$ has non-trivial intersection with both I^{i-1} and I^i .

Suppose that $\sum_j \overline{I^{i-1}}(p_j) \tilde{\nu}(\phi^{i-1}, p_j) = \kappa$. This implies that there are $P, P' \in \mathcal{P}(\tilde{\phi})$ such that $\sum_j P(p_j) \tilde{\nu}(\tilde{\phi}, p_j) = \kappa$ and $\sum_j P'(p_j) \tilde{\nu}(\tilde{\phi}, p_j) = \kappa'$ for some $\kappa' < \kappa$. Observe that given its cardinal uniqueness, this fully pins down $\tilde{\nu}(\tilde{\phi}, \cdot)$. Moreover, observe that there exists a P'' such that $\sum_j P''(p_j) \tilde{\nu}(\tilde{\phi}, p_j) = \kappa'' > \kappa$.

By construction there exists some $Q \in \mathcal{P}(\phi^i)$ such that $Q \sim P''$. We set $\sum_j \underline{I^i}(p_j) \tilde{\nu}(\phi^i, p_j) = \kappa$. Moreover, we can find a $\lambda(Q)$ such that $\lambda(Q) \underline{I^i} + (1 - \lambda(Q)) \overline{I^i} \sim Q$. Let ι solves

$\lambda(Q)\kappa + (1 - \lambda(Q))\iota = \kappa''$ and set $\sum_j \overline{I^i}(p_j)\tilde{\nu}(\phi^i, p_j) = \iota$.

As in the previous cases, now consider any $\phi' \neq \phi$ such that $I^i \cap \mathcal{P}(\phi') \neq \emptyset$. Let $\underline{R^i}(\phi')$ be a \succsim -minimal element in $I^i \cap \mathcal{P}(\phi')$. Similarly, let $\overline{R^i}(\phi')$ be a \succsim -maximal element in $I^i \cap \mathcal{P}(\phi')$. Find $P \in I^i \cap \mathcal{P}(\phi^i)$ such that $P \sim \underline{R^i}(\phi')$ and set $\sum_j \underline{R^i}(\phi')(p_j)\tilde{\nu}(\phi', p_j) = \lambda(P)$. Similarly, find $P' \in I^i \cap \mathcal{P}(\phi^i)$ such that $P' \sim \overline{R^i}(\phi')$ and set $\sum_j \overline{R^i}(\phi')(p_j)\tilde{\nu}(\phi', p_j) = \lambda(P')$.

Note that, by construction, we have $f_\phi(\lambda(P)) = f_{\phi'}(\lambda(P))$ and $f_\phi(\lambda(P')) = f_{\phi'}(\lambda(P'))$. By CTI, for any $\alpha \in [0, 1]$, $\alpha P + (1 - \alpha)P' \sim \alpha \underline{R^i}(\phi') + (1 - \alpha)\overline{R^i}(\phi')$ and thus $f_\phi(\alpha\lambda(P) + (1 - \alpha)\lambda(P')) = f_{\phi'}(\alpha\lambda(P) + (1 - \alpha)\lambda(P'))$. This implies that within I^i we can take, without loss of generality, $f_{\phi'} = f_\phi$. By repeating this same process over all ϕ' 's that have an element in I^i we can show that for all relevant priors ϕ' we can set $f_{\phi'} = f_\phi$ on I^i . In other words, on each I^i we can take f to be independent of the prior.¹⁶

We now turn to piecing together the entire utility function. First, recall that we have a countable set I^1, I^2, \dots (ordered in terms of increasing preferences). Take the set $\mathbb{G}^1 = \mathcal{I} \setminus \mathbb{I}$. These are the “leftover” equivalence classes that compose the first class we considered. Note that we can also group these together into convex sets. We do so inductively. Pick an I such that I is the worst equivalence class in $\text{cl}(\mathbb{G})$. Then find $\hat{I} \in \text{cl}(\mathbb{G})$ with the property that for any $P \in \hat{I}$ there is no $I' \in \mathbb{I}$ with $Q \prec P$ for some $Q \in I'$. Then denote G^1 as the set of equivalence classes from I to \hat{I} (inclusive); $G^1 = \{I'' \mid P \succsim Q \succsim R \text{ for all } P \in \hat{I}, Q \in I'', \text{ and } R \in I\}$. Now set $\mathbb{G}^2 = \mathbb{G}^1 \setminus G^1$, and find G^2 using the same process, and continue in this fashion to get a collection $\{G^l\}$.

We have already shown that f is independent of ϕ on any G^l or I^i . On any one of these ranges denote the relevant f as f_j where $j \in \{G^l, I^i\}$. Suppose there is an indifference class I such that I is in two elements of $\{G^l, I^i\}$, call them H and H' . Suppose without loss of generality that for some $P \in I$ with $\sum_j P(p_j)\tilde{\nu}(\phi, p_j) = a$, we have $f_H(a) \geq f_{H'}(a)$. Then, since both f_H and $f_{H'}$ are strictly increasing, we can simply set $f_H^*(\cdot) = f_{H'}(\cdot) - [f_{H'}(a) - f_H(a)]$, generating continuity at a . Denote the adjusted collection of transformation functions by $\{h_H\}$. We can now simply take a single function f defined by $f = h_H$ on H and performing a monotone transformation to recover our utility function $V(P) = f^{-1}(f(\sum_j P(p_j)\tilde{\nu}(\phi(P), p_j)))$.

This proves the equivalence in the proposition. \square

Proof of Proposition 2. First we define a prior-separable representation as $V_{ps} = \nu_{ps1}(\phi(P)) + \sum_i P(p_i)\nu_{ps2}(p_i)$

¹⁶Observe that if we mix two lotteries in I^i that induce different priors, then their mixed lottery, which induce some prior ϕ'' either lives in I^i and the same calibration exercise can be performed for ϕ'' , or outside of I^i , in which case it should be relegated to the step where we deal with the relevant $I^j \neq I^i$.

Claim 3. *The relation \succsim has a prior-separable representation if and only if it has a prior-anticipatory representation.*

Proof of Claim 3. Recall from the proof of Lemma 1 that \succsim has a prior-anticipatory representation if and only if it has a representation $\hat{\nu}_1(\phi(P)) + \sum_i P(p_i)\nu_2(p_i)$. Note that this is simply the sum of a utility function defined over the reduced lottery and a recursive utility that is expected utility in the first stage.

We now use our new representation for Claim 4.

Claim 4. *The relation \succsim has a prior-separable representation if and only if it satisfies WO, C, CTI, and SPTI.*

Proof of Claim 4. It is easy to check that the axioms are necessary for the representation. For sufficiency, observe that SPTI implies PTI, which, in turns, implies that there exists a representation of the form $\sum_i P(p_i)\tilde{\nu}(\phi(P), p_i)$. Moreover, by SPTI, if $\sum_i P(p_i)\tilde{\nu}(\phi(P), p_i) = \sum_i Q(p_i)\tilde{\nu}(\phi(P), p_i)$, then $\sum_i(\alpha P + (1 - \alpha)R)(p_i)\tilde{\nu}(\phi((\alpha P + (1 - \alpha)R)), p_i) = \sum_i(\alpha Q + (1 - \alpha)R)(p_i)\tilde{\nu}(\phi((\alpha P + (1 - \alpha)R)), p_i)$ for any R , which is true if and only if $\tilde{\nu}$ is additively separable in it's first argument: $\sum_i P(p_i)\tilde{\nu}(\phi(P), p_i) = \hat{\nu}_1(\phi(P)) + \sum_i P(p_i)\nu_2(p_i)$. To see this, observe that with n sub-lotteries, the utility function V_{ps} can be thought of as a function of $n + 1$ arguments — the n sub-lotteries and the prior beliefs. Since the representation is additively separable, conditional on the prior, preferences must satisfy separability (i.e, preferential independence in Debreu, 1960) across the sub-lotteries (and all subsets of the sub-lotteries). Further observe that SCTI implies that all subsets of the sub-lotteries and the prior also satisfy separability (preferential independence). Thus, by Debreu (1960) (see also Wakker, 1993) the representation must be additively separable in all components.

This proves the equivalence in the proposition. \square

Proof of Proposition 3. First we define a prior-separable expected utility representation as $V_{pseu} = \sum_x \phi(P)(x)\nu_{pseu1}(x) + \sum_i P(p_i)\nu_{pseu2}(p_i)$.

Claim 5. *The relation \succsim has a prior-separable expected utility representation if and only if it has a posterior-anticipatory representation.*

Proof of Claim 5. From Lemma 1, \succsim has a posterior-anticipatory representation if and only if it has a representation $\sum_i P(p_i) \sum_j p_i(x_j)\hat{u}(x_j) + \sum_i P(p_i)\nu_1(p_i)$. This is simply the sum of an expected utility functional defined over the reduced lottery and a recursive utility that is expected utility in the first stage.

We now use the new representation for Claim 6.

Claim 6. *The relation \succsim has a prior-separable expected utility representation if and only if it satisfies WO, C and TI.*

Observe that by the mixture space theorem, \succsim satisfies WO, C, and TI if and only if it can be represented by the functional $\sum_i P(p_i)\dot{\nu}(p_i)$. Moreover, if preferences can be represented by $\sum_i P(p_i)\dot{\nu}(p_i)$ then clearly they have a prior-separable expected utility representation (where $\nu_{pseu1}(x) = 0$). Similarly, any prior anticipatory representation can be written as $\sum_i P(p_i)\dot{\nu}(p_i)$ where $\dot{\nu}(p_i) = \nu_{pseu2}(p_i) + \sum_x p_i(x)\nu_{pseu1}(x)$.

This proves the equivalence in the proposition. \square

Proof of Proposition 4. We prove each of the statements in order.

- We first show that if \succsim has an ABB representation then it has a BSI representation in a series of two claims.

Claim 7. *There exists an equivalent representation $(u, \hat{\nu}_1, \hat{\nu}_2)$ which satisfies the condition $\hat{\nu}_1(\rho, \rho) = 0$ for all ρ .*

Proof of Claim 7. Denote as $N(p)$ the number of elements in the support of p and sum up below only amongst those elements with positive probability. Define: $\hat{\nu}_1(\rho, p) = \nu_1(\rho, p) - \nu_1(p, p)$ and $\hat{\nu}_2(p, \delta_x) = \nu_2(p, \delta_x) + \frac{\nu_1(p, p)}{N(p)p(x)}$. By construction, $\hat{\nu}_1(\rho, \rho) = 0$. Moreover, preferences did not change as the new representation gives utility:

$$\sum_x \rho(x)u(x) + \sum_p P(p)\hat{\nu}_1(\rho, p) + \sum_p \sum_x P(p)p(x)\hat{\nu}_2(p, \delta_x)$$

or

$$\sum_x \rho(x)u(x) + \sum_p P(p)[\nu_1(\rho, p) - \nu_1(p, p)] + \sum_p \sum_x P(p)p(x)[\nu_2(p, \delta_x) + \frac{\nu_1(p, p)}{N(p)p(x)}]$$

or

$$\sum_x \rho(x)u(x) + \sum_p P(p)\nu_1(\rho, p) - \sum_p P(p)\nu_1(p, p) + \sum_p \sum_x P(p)p(x)\nu_2(p, \delta_x) + \sum_p P(p)\nu_1(p, p) \sum_x \frac{1}{N(p)}$$

or

$$\sum_x \rho(x)u(x) + \sum_p P(p)\nu_1(\rho, p) - \sum_p P(p)\nu_1(p, p) + \sum_p P(p)\nu_1(p, p) + \sum_p \sum_x P(p)p(x)\nu_2(p, \delta_x)$$

which is the original utility function. \square

Claim 8. *There exists an equivalent representation $(\tilde{u}, \hat{\nu}_1, \tilde{\nu}_2)$, which satisfies the condition $\tilde{\nu}_2(\delta_x, \delta_x) = 0$ for all $x \in X$.*

Proof of Claim 8. Define $\tilde{\nu}_2(p, \delta_x) = \hat{\nu}_2(p, \delta_x) - \hat{\nu}_2(\delta_x, \delta_x)$ and $\tilde{u}(x) = u(x) + \hat{\nu}_2(\delta_x, \delta_x)$. Note that $\tilde{\nu}_2(\delta_x, \delta_x) = 0$ for all x . Observe that this does not change preferences since utility under this representation is:

$$\sum_x \tilde{u}(x)\rho(x) + \sum_p P(p)\hat{\nu}_1(\rho, p) + \sum_p \sum_x P(p)p(x)\tilde{\nu}_2(p, \delta_x)$$

or

$$\sum_x \rho(x)[u(x) + \hat{\nu}_2(\delta_x, \delta_x)] + \sum_p P(p)\hat{\nu}_1(\rho, p) + \sum_p P(p)p(x)[\hat{\nu}_2(p, \delta_x) - \hat{\nu}_2(\delta_x, \delta_x)]$$

or

$$\sum_x \rho(x)u(x) + \sum_p P(p)\hat{\nu}_1(\rho, p) + \sum_p \sum_x P(p)p(x)\hat{\nu}_2(p, \delta_x) + \sum_x \rho(x)\hat{\nu}_2(\delta_x, \delta_x) - \sum_p \sum_x P(p)p(x)\hat{\nu}_2(\delta_x, \delta_x)$$

which is simply the original utility function. \square

Thus, we have a utility representation $(\tilde{u}, \hat{\nu}_1, \tilde{\nu}_2)$ which satisfies BSI.

- We next show that \succsim always has a representation which is belief-time invariant. Take the representation $(\tilde{u}, \hat{\nu}_1, \tilde{\nu}_2)$ defined in the previous part. Define

$$\tilde{\nu}'_2(p, \delta_x) = \tilde{\nu}_2(p, \delta_x) + [\hat{\nu}_1(p, \delta_x) - \tilde{\nu}_2(p, \delta_x)]$$

and

$$\hat{\nu}'_1(\rho, p) = \hat{\nu}_1(\rho, p) - \sum_x p(x)[\hat{\nu}_1(p, \delta_x) - \tilde{\nu}_2(p, \delta_x)]$$

Observe $(\tilde{u}, \hat{\nu}'_1, \tilde{\nu}'_2)$ represents the same preferences. Utility under the second representation is:

$$\sum_x \tilde{u}(x)\rho(x) + \sum_p P(p)\hat{\nu}'_1(\rho, p) + \sum_p \sum_x P(p)p(x)\tilde{\nu}'_2(p, \delta_x)$$

or

$$\begin{aligned} & \sum_x \tilde{u}(x)\rho(x) + \sum_p P(p)[\hat{\nu}_1(\rho, p) - \sum_x p(x)[\hat{\nu}_1(p, \delta_x) - \tilde{\nu}_2(p, \delta_x)]] \\ + & \sum_p \sum_x P(p)p(x)[\tilde{\nu}_2(p, \delta_x) + [\hat{\nu}_1(p, \delta_x) - \tilde{\nu}_2(p, \delta_x)]] \end{aligned}$$

or

$$\begin{aligned} & \sum_x \tilde{u}(x)\rho(x) + \sum_p P(p)\hat{\nu}_1(\rho, p) - \sum_p \sum_x P(p)p(x)[\hat{\nu}_1(p, \delta_x) - \tilde{\nu}_2(p, \delta_x)] \\ + & \sum_p \sum_x P(p)p(x)\tilde{\nu}_2(p, \delta_x) + \sum_p \sum_x P(p)p(x)[\hat{\nu}_1(p, \delta_x) - \tilde{\nu}_2(p, \delta_x)] \end{aligned}$$

or

$$\sum_x \tilde{u}(x)\rho(x) + \sum_p P(p)\hat{\nu}_1(\rho, p) + \sum_p \sum_x P(p)p(x)\tilde{\nu}_2(p, \delta_x)$$

which are the original preferences.

Moreover, observe that by construction

$$\hat{\nu}'_1(p, \delta_x) = \hat{\nu}_1(p, \delta_x) - [\hat{\nu}_1(\delta_x, \delta_x) - \tilde{\nu}_2(\delta_x, \delta_x)] = \hat{\nu}_1(p, \delta_x) - [0 - 0]$$

Also

$$\tilde{\nu}'_2(p, \delta_x) = \tilde{\nu}_2(p, \delta_x) + [\hat{\nu}_1(p, \delta_x) - \tilde{\nu}_2(p, \delta_x)] = \hat{\nu}_1(p, \delta_x)$$

Thus we satisfy BTI. However, we no longer satisfy BSI. This is because

$$\hat{\nu}'_1(\rho, \rho) = \hat{\nu}_1(\rho, \rho) - \sum_x \rho(x)[\hat{\nu}_1(\rho, \delta_x) - \tilde{\nu}_2(\rho, \delta_x)]$$

no longer necessarily equals 0.

- We now show that \succsim has a representation which is both belief-stationary invariant and belief-time invariant if and only if it satisfies TN.

For the only if part, observe that for $P = D_p$

$$\begin{aligned} V_{ABB}(P) &= E_p(u) + \nu_1(p, p) + \sum_j p(x_j)\nu_2(p, \delta_{x_j}) \\ &= E_p(u) + \sum_j p(x_j)\nu_2(p, \delta_{x_j}) \end{aligned}$$

where the second equality is by BSI.

For $Q = \sum_i p(x_j)\delta_{x_j}$ we have

$$\begin{aligned} V_{ABB}(Q) &= E_p(u) + \sum_j p(x_j)\nu_1(p, \delta_{x_j}) + \sum_j p(x_j)\nu_2(\delta_{x_j}, \delta_{x_j}) \\ &= E_p(u) + \sum_j p(x_j)\nu_1(p, \delta_{x_j}) \end{aligned}$$

where the second equality is again by BSI.

By BTI, $\sum_j p(x_j)\nu_2(p, \delta_{x_j}) = \sum_j p(x_j)\nu_1(p, \delta_{x_j})$, which implies $V_{ABB}(P) = V_{ABB}(Q)$, that is, TN is satisfied.

To prove the other direction, we can simply assume preferences satisfy BSI. Observe that time neutrality implies that

$$\sum_x \tilde{u}(x)\rho(x) + \hat{\nu}_1(\rho, \rho) + \sum_x \rho(x)\tilde{\nu}_2(\rho, \delta_x) = \sum_x \tilde{u}(x)\rho(x) + \sum_x \rho(x)\hat{\nu}_1(\rho, \delta_x) + \sum_x \rho(x)\tilde{\nu}_2(\delta_x, \delta_x)$$

or, taking the fact that BSI holds

$$\sum_x \rho(x)\tilde{\nu}_2(\rho, \delta_x) = \sum_x \rho(x)\hat{\nu}_1(\rho, \delta_x)$$

Observe that $\hat{\nu}_1(\rho, \delta_x)$ only appears as a term as part of the sum $\sum_x \rho(x)\hat{\nu}_1(\rho, \delta_x)$. Thus, we cannot separately identify the individual parts of $\sum_x \rho(x)\hat{\nu}_1(\rho, \delta_x)$. Since $\sum_x \rho(x)\tilde{\nu}_2(\rho, \delta_x) = \sum_x \rho(x)\hat{\nu}_1(\rho, \delta_x)$ we can simply suppose without loss of generality that $\rho(x)\tilde{\nu}_2(\rho, \delta_x) = \rho(x)\hat{\nu}_1(\rho, \delta_x)$ term by term.

- Lastly, as we mention in Footnote 2.4, we show that if \succsim has an ABB representation, then it has a representation which is both belief-stationary invariant and pseudo-belief-time invariant.¹⁷ First, normalize the representation using claims 7 and 8 so that it satisfies BSI. We then normalize the representation so that BTI holds as in the second part of the proof of this proposition. As we have mentioned there, $\hat{\nu}'_1(\rho, \rho)$ no longer necessarily equals 0. But, since we started with a BSI representation, we already had that $\hat{\nu}'_1(\delta_x, \delta_x) = \tilde{\nu}'_2(\delta_x, \delta_x) = 0$ so those values do not change.

In order to simplify notation, call the functionals after these two steps u, ν_1 , and ν_2 respectively. Thus, $\nu_1 = \nu_2$ over their shared domain, and $\nu_2(\delta_x, \delta_x) = 0 = \nu_1(\delta_x, \delta_x)$.

¹⁷Since \succsim always has a representation which is belief-time invariant. This immediately implies that there is also a PBTI representation, where $\kappa = 1$.

Now we will define a representation that satisfies both BSI and PBTI. We do this in a way that mirrors Claim 7. Denote as $N(p)$ the number of elements with positive probability in p and sum up below only amongst those elements. Define: $\hat{\nu}_1(\rho, p) = \nu_1(\rho, p) - \nu_1(p, p)$. Importantly, this redefinition implies $\hat{\nu}_1(\rho, \delta_x) = \nu_1(\rho, \delta_x) - \nu_1(\delta_x, \delta_x) = \nu_1(\rho, \delta_x)$.

We then turn to solve for $\hat{\nu}_2$. Denote $z(p) = \nu_1(p, p)$. For our representation to satisfy PBTI we need that $\hat{\nu}_2(p, \delta_x) = \kappa \hat{\nu}_1(p, \delta_x) = \kappa \nu_1(p, \delta_x) = \kappa \nu_2(p, \delta_x)$ for some κ . If p has $N(p)$ outcomes in its support, then these are $N(p)$ equations and $N(p) + 1$ unknowns. We also need it to be the case that $\sum p(x) \hat{\nu}_2(p, \delta_x) = z(p)$. Substituting in we get $\kappa \sum p(x) \nu_2(p, \delta_x) = z(p)$ or $\kappa = \frac{z(p)}{\sum p(x) \nu_2(p, \delta_x)}$. Observe that this uniquely pins down κ and so uniquely pins down $\hat{\nu}_2$ for each p . Thus, PBTI is satisfied. Moreover, observe that by construction $\hat{\nu}_2(\delta_x, \delta_x) = 0$ still and $\hat{\nu}_1(\rho, \rho) = 0$, and so BSI is satisfied as well. \square

Proof of Proposition 5. From Claim 1 we know that we can confine attention to a prior-conditional representation of \succsim . Observe that fixing $\phi(P)$, $\sum_i P(p_i) \nu_{PC}(\phi(P), p_i)$ is an expected utility functional, and so possesses the same uniqueness results; i.e. it is unique up to affine transformations of scalars $\alpha_P > 0$ and β_P . But, since $\sum_i P(p_i) \nu_{PC}(\phi(P), p_i) \geq \sum_i Q(q_i) \nu_{PC}(\phi(Q), q_i)$ if and only if $\beta_P + \alpha_P \sum_i P(p_i) \nu_{PC}(\phi(P), p_i) \geq \beta_Q + \alpha_Q \sum_i Q(q_i) \nu_{PC}(\phi(Q), q_i)$, it must be the case that $\beta_P = \beta_Q$ and $\alpha_P = \alpha_Q$. \square

Proof of Proposition 6. To see the result for a BSI representation, first take the uniqueness result for general ABB preferences (Proposition 10 in Appendix 6.2). Suppose that (u, ν_1, ν_2) is a BSI representation. We first show that any transformation where $\gamma_1(x) \neq 0$ for some x cannot generate a BSI representation. Suppose that there is some x_i such that $\gamma_1(x_i) \neq 0$. Consider the two-stage lottery $D_{\delta_{x_i}}$. Then $\nu'_1(\delta_{x_i}, \delta_{x_i}) = 0 - \gamma_1(x_i) \neq 0$ so this cannot be a BSI representation. Next we show that any transformation where $\gamma_u(x) \neq 0$ for some x cannot generate a BSI representation. Suppose that there is some x_i such that $\gamma_u(x_i) \neq 0$. Consider the two-stage lottery $D_{\delta_{x_i}}$. Then $\nu'_2(\delta_{x_i}, \delta_{x_i}) = 0 - \gamma_u(x_i) \neq 0$ so this cannot be a BSI representation. For similar reasons $\beta_1 = \beta_2 = 0$. Lastly, we show that any transformation where $\gamma_\nu(p, \delta_x) \neq 0$ for some p and x cannot generate a BSI representation. If there were p_i and x_j such that $\gamma_\nu(p_i, \delta_{x_j}) \neq 0$, then $\nu'_1(p, p) = 0 + p(x) \gamma_\nu(p_i, \delta_{x_j}) \neq 0$, violating a BSI representation. \square

Proof of Proposition 7. We first show that \succsim has a posterior-separable expected utility representation (i.e. it satisfies WO, C, and TI) if and only if it satisfies WO, C, PTI, CTI, and R.

Necessity is immediate. To show sufficiency, note that \succsim has a representation of the form $V_{PC} = \sum_i P(p_i) \nu_{PC}(\phi(P), p_i)$ if it satisfies WO, C, PTI and CTI. If R is satisfied, then it must be the case that ν_{PC} is independent of the first argument. Thus we have a representation of the form

$\sum_i P(p_i)\hat{\nu}_{PC}(p_i)$, which is equivalent to the following representation: $\sum_i P(p_i)\sum_j p_i(x_j)\hat{u}(x_j) + \sum_i P(p_i)\nu_1(p_i)$.

Recall that \succsim has a posterior-anticipatory representation if and only if it has a representation $\sum_i P(p_i)\hat{\nu}(p_i)$. By Segal (1990), this is a recursive representation where V_1 is expected utility. This representation clearly satisfies WO, C, and TI. \square

Proof of Proposition 8. For item (1), observe that we can ignore the first term of the ABB representation, as it is the same under any two compound lotteries with the same reduced form probabilities. Suppose then that $\nu_1(\rho, \cdot) + \sum_x \nu_2(\cdot, x)$ is convex. Then, by Grant, Kajii and Polak (1998) the individual must exhibit a preference for early resolution of uncertainty. Conversely, if the term above is not convex, then it must be concave in a local neighborhood of some p_i . We can replicate the argument in Grant, Kajii and Polak (1998). Take some compound lottery that delivers as one sub-lottery p_i , and take a linear bifurcation of p_i so that the new sub-lotteries are arbitrarily close to p_i . Then by Grant, Kajii and Polak (1998) the individual must be worse off (since locally the utility function is concave).

For item (2), take any P and Q as specified in the statement of the proposition. Observe that $\phi(P) = \phi(Q)$. Direct calculations then show that $P \succsim Q$ if and only if $\beta\hat{\nu}_2(p_1) + (1 - \beta)\hat{\nu}_2(p_2) \geq \hat{\nu}_2(\beta p_1 + (1 - \beta)p_2)$. And since the triple p_1, p_2 , and β were arbitrary, the inequality holds if and only if $\hat{\nu}_2$ is convex. Similarly, the inequality is reversed if and only if $\hat{\nu}_2$ is concave.

For item (3), simply replace in the entire paragraph above $\hat{\nu}_2$ with $\hat{\nu}_1$. \square

Proof of Proposition 9. For item (i), note that by Definition 7, PORU implies TN. By Proposition 4, the representation satisfies both BSI and BTI. We now use these functional form restrictions when calculating the values of the lotteries in question. We have $V(D_p) = V(\sum_i p(x_i)\delta_{x_i}) = \sum_j \phi(P)(x_j)u(x_j) + \sum_x \phi(P)(x)\nu_1(\phi(P), \delta_x)$, which is the left hand side of the inequality in (i) in addition to the expected utility from material payoffs. PORU implies that these two compound lotteries are better than any other P with the same reduced probabilities $\phi(P)$. Indeed, the value of any such P is $V(p) = \sum_j \phi(P)(x_j)u(x_j) + \sum_i P(p_i)\nu_1(\phi(P), p_i) + \sum_{p_i} P(p_i)\sum_x p_i(x)\nu_1(p_i, \delta_x)$, which is the right hand side of the inequality in addition to the same expected utility from material payoffs.

For item (ii), first recall that if \succsim has a prior-anticipatory representation, then it can also be represented by the functional $\hat{\nu}_1(\phi(P)) + \sum_i P(p_i)\nu_2(p_i)$. Corollary 1 below shows that in this case TN implies that ν_2 is an expected utility functional, and thus the second term does not generate any anomalous preferences towards information. Clearly the first term $\hat{\nu}_1(\phi(P))$ depends only on the prior beliefs, independently of the pattern of resolution of uncertainty. The individual is thus indifferent among all lotteries that induce the same prior beliefs, and in particular cannot display strict PORU. \square

6.2 Uniqueness of ABB representations

In this section we present more detailed uniqueness results.

Proposition 10. *Suppose \succsim has an ABB representation (u, ν_1, ν_2) . The ABB representation (u', ν'_1, ν'_2) also represents \succsim if and only if there exists scalars $\alpha > 0, \beta_u, \beta_1, \beta_2$, and continuous functions $\gamma_u : X \rightarrow \mathbb{R}$, $\gamma_1 : X \rightarrow \mathbb{R}$, and $\gamma_\nu : \Delta \times X \rightarrow \mathbb{R}$ such that*

- $u'(x) = \alpha u(x) + \beta_u + \gamma_u(x) + \gamma_1(x)$
- $\nu'_1(\rho, p) = \alpha \nu_1(\rho, p) + \beta_1 + \sum_x p(x) \gamma_\nu(p, \delta_x) - \sum_x \rho(x) \gamma_1(x)$
- $\nu'_2(p, \delta_x) = \alpha \nu_2(p, \delta_x) + \beta_2 - \gamma_\nu(p, \delta_x) - \gamma_u(x)$

Proof of Proposition 10. We first show that if

- $u'(x) = \alpha u + \beta_u + \gamma_u(x) + \gamma_1(x)$
- $\nu'_1(\rho, p) = \alpha \nu_1(\rho, p) + \beta_1 + \sum_x p(x) \gamma_\nu(p, \delta_x) - \sum_x \rho(x) \gamma_1(x)$
- $\nu'_2(p, \delta_x) = \alpha \nu_2(p, \delta_x) + \beta_2 - \gamma_\nu(p, \delta_x) - \gamma_u(x)$

then (u', ν'_1, ν'_2) represents the same preferences as (u, ν_1, ν_2) .

Consider the utility function generated by the former representation.

$$\sum_x u' \rho(x) + \sum_p P(p) \nu'_1(\rho, p) + \sum_p \sum_x P(p) p(x) \nu'_2(p, \delta_x)$$

or

$$\begin{aligned} & \sum_x \rho(x) [\alpha u + \beta_u + \gamma_u(x) + \gamma_1(x)] \\ & + \sum_p P(p) [\alpha \nu_1(\rho, p) + \beta_1 + \sum_x p(x) \gamma_\nu(p, \delta_x) - \sum_x \rho(x) \gamma_1(x)] \\ & + \sum_p \sum_x P(p) p(x) [\alpha \nu_2(p, \delta_x) + \beta_2 - \gamma_\nu(p, \delta_x) - \gamma_u(x)] \end{aligned}$$

or

$$\begin{aligned} & \alpha \sum_x \rho(x) u + \beta_u + \sum_x \rho(x) \gamma_u(x) + \sum_x \rho(x) \gamma_1(x) \\ & + \alpha \sum_p P(p) \nu_1(\rho, p) + \beta_1 + \sum_p \sum_x P(p) p(x) \gamma_\nu(p, \delta_x) - \sum_p P(p) \sum_x \rho(x) \gamma_1(x) \\ & + \alpha \sum_p \sum_x P(p) p(x) \nu_2(p, \delta_x) + \beta_2 - \sum_p \sum_x P(p) p(x) \gamma_\nu(p, \delta_x) - \sum_p \sum_x P(p) p(x) \gamma_u(x) \end{aligned}$$

Denoting $\beta = \beta_u + \beta_1 + \beta_2$ and recalling that $\sum_p \sum_x P(p)p(x) = \sum_x \rho(x)$ we get

$$\begin{aligned} & \alpha \left[\sum_x \rho(x)u + \sum_p P(p)\nu_1(\rho, p) + \sum_p \sum_x P(p)p(x)\nu_2(p, \delta_x) \right] + \beta \\ + & \sum_x \rho(x)\gamma_u(x) + \sum_p \sum_x P(p)p(x)\gamma_\nu(p, \delta_x) + \sum_x \rho(x)\gamma_1(x) \\ - & \sum_p \sum_x P(p)p(x)\gamma_\nu(p, \delta_x) - \sum_x \rho(x)\gamma_u(x) - \sum_x \rho(x)\gamma_1(x) \end{aligned}$$

or

$$\alpha \left[\sum_x \rho(x)u + \sum_p P(p)\nu_1(\rho, p) + \sum_p \sum_x P(p)p(x)\nu_2(p, \delta_x) \right] + \beta$$

which clearly are the same preferences as (u, ν_1, ν_2) .

To prove the other direction, suppose that (u, ν_1, ν_2) and (u', ν'_1, ν'_2) represent the same preferences.

Define $\hat{u}(x) = u(x) - u(x)$; $\hat{\nu}_2(p, \delta_x) = \nu_2(p, \delta_x) - \nu_2(p, \delta_x)$; and $\hat{\nu}_1(\rho, p) = \nu_1(\rho, p) + \sum_x \rho(x)u(x) + \sum_x p(x)\nu_2(p, \delta_x)$. These represent the same preferences as (u, ν_1, ν_2) but we can write $V(P) = \sum_p P(p)\hat{\nu}_1(\phi(P), p)$.

Now define $\hat{u}'(x) = u'(x) - u'(x)$; $\hat{\nu}'_2(p, \delta_x) = \nu'_2(p, \delta_x) - \nu'_2(p, \delta_x)$; and $\hat{\nu}'_1(\rho, p) = \nu'_1(\rho, p) + \sum_x \rho(x)u'(x) + \sum_x p(x)\nu'_2(p, \delta_x)$. These represent the same preferences as (u', ν'_1, ν'_2) but we can write $V'(P) = \sum_p P(p)\hat{\nu}'_1(\phi(P), p)$.

Since $V(P) = \sum_p P(p)\hat{\nu}_1(\phi(P), p)$ and $V'(P) = \sum_p P(p)\hat{\nu}'_1(\phi(P), p)$ we know that $\hat{\nu}'_1(\phi(P), p)$ must be an affine transformation of $\hat{\nu}_1(\phi(P), p)$; so that $\hat{\nu}'_1(\phi(P), p) = \alpha\hat{\nu}_1(\phi(P), p) + \beta$. Thus $V'(P) = \sum_p P(p)\alpha\hat{\nu}_1(\phi(P), p) + \beta$. Clearly, $\sum_p \alpha P(p)\hat{\nu}_1(\phi(P), p) + \beta$ has an ABB representation $(\alpha u + \beta_u, \alpha\nu_1 + \beta_1, \alpha\nu_2 + \beta_2)$, where $\beta_u + \beta_1 + \beta_2 = \beta$.

By construction $\alpha\hat{u} = \hat{u}' = 0$ and $\alpha\hat{\nu}_2 = \hat{\nu}'_2 = 0$. Thus we can say $u'(x) = u'(x) - \alpha u(x) + \alpha u(x)$; $\nu'_2(p, \delta_x) = \nu'_2(p, \delta_x) - \alpha\nu_2(p, \delta_x) + \alpha\nu_2(p, \delta_x)$; and $\nu'_1(\phi(P), p) = \alpha\nu_1(\phi(P), p) - \sum_x \rho(x)[u'(x) - \alpha u(x)] - \sum_x p(x)[\nu'_2(p, \delta_x) - \alpha\nu_2(p, \delta_x)] + \beta$. Moreover, it is easy to verify that we can arbitrarily divide β among the terms.

Define $\gamma_u(x) = 0$; $\gamma_1(x) = u'(x) - \alpha u(x)$; and $\gamma_\nu(p, \delta_x) = -[\nu'_2(p, \delta_x) - \alpha\nu_2(p, \delta_x)]$. Then $u'(x) = \alpha u(x) + \gamma_u(x) + \gamma_1(x)\beta_u$; $\nu'_2(p, \delta_x) = \alpha\nu_2(p, \delta_x) - \gamma_\nu(p, \delta_x) - \gamma_u(x) + \beta_1$ and $\nu'_1(\phi(P), p) = \alpha\nu_1(\phi(P), p) + \sum_x p(x)\gamma_\nu(p, \delta_x) - \sum_x \rho(x)\gamma_1(x)$. Thus we have constructed the transformation. \square

For completeness, we now show that if we suppose utility depends only on the levels of beliefs, stronger uniqueness results also obtain. In this case, both belief-based functionals are unique up to

expected utility preferences. Thus, the only part of the utility function not uniquely identified (up to standard transformations) are an individual's expected utility attitudes towards final outcomes.

Proposition 11. *Suppose \succsim has an anticipatory representation (u, ν_1, ν_2) . The ABB representation (u', ν'_1, ν'_2) also represents \succsim if and only if there are scalars $\alpha > 0, \beta_u, \beta_1, \beta_2$ and continuous functions $\gamma_u : X \rightarrow \mathbb{R}$ and $\gamma_\nu : X \rightarrow \mathbb{R}$ such that*

- $u'(x) = \alpha u + \beta_u + \gamma_u(x) + \gamma_\nu(x)$
- $\nu'_1(\rho) = \alpha \nu_1(\rho) + \beta_1 - \sum_x \rho(x) \gamma_\nu(x)$
- $\nu'_2(p) = \alpha \nu_2(p) + \beta_2 - \sum p(x) \gamma_u(x)$ ¹⁸

Proof of Proposition 11. The proof is analogous to the one of Proposition 10. We show necessity for prior anticipatory preferences as an example. Consider the utility function generated by the latter representation.

$$\sum_x u' \rho(x) + \sum_p P(p) \nu'_1(\rho) + \sum_p \sum_x P(p) p(x) \nu'_2(p)$$

or

$$\begin{aligned} & \sum_x \rho(x) [\alpha u(x) + \beta_u + \gamma_u(x) + \gamma_\nu(x)] + \sum_p P(p) [\alpha \nu_1(\rho) + \beta_1 - \sum_x \rho(x) \gamma_\nu(\delta_x)] \\ & + \sum_p \sum_x P(p) p(x) [\alpha \nu_2(p) + \beta_2 - \gamma_u(x)] \end{aligned}$$

or

$$\begin{aligned} & \alpha \sum_x \rho(x) u(x) + \beta_u + \sum_x \rho(x) \gamma_u(x) + \sum_x \rho(x) \gamma_\nu(x) + \alpha \nu_1(\rho) + \beta_1 - \sum_x \rho(x) \gamma_\nu(\delta_x) \\ & + \sum_p P(p) \alpha \nu_2(p) + \beta_2 - \sum_p \sum_x P(p) p(x) \gamma_u(x) \end{aligned}$$

Denoting $\beta = \beta_u + \beta_1 + \beta_2$ and recalling that $\sum_p \sum_x P(p) p(x) = \sum_x \rho(x)$ we get

$$\alpha \left[\sum_x \rho(x) u(x) + \nu_1(\rho) + \sum_p P(p) \alpha \nu_2(p) \right] + \beta$$

which clearly are the same preferences as (u, ν_1, ν_2) . \square

¹⁸Observe that in a posterior-anticipatory representation p is a degenerate lottery.

6.3 Other Properties

In addition to axiom R, Segal (1990) also introduced several other restrictions on preferences over compound lotteries. We first quickly review them. The strongest assumption is called Reduction of Compound Lotteries (ROCL), which supposes that individuals only care about the reduced form probabilities of any given compound lottery.

Reduction of Compound Lotteries (ROCL): *For all $P, Q \in \Delta^2$, if $\phi(P) = \phi(Q)$ then $P \sim Q$.*

To introduce the next assumption, we first define two special subsets of Δ^2 .

- $\Gamma = \{D_p | p \in \Delta\}$, the set of degenerate lotteries in Δ^2 . Γ is the set of late resolving lotteries.
- $\Lambda = \{Q \in \Delta^2 | Q(p) > 0 \Rightarrow p = \delta_x \text{ for some } x \in X\}$, the set of compound lotteries whose outcomes are degenerate in Δ . Λ is the set of early resolving lotteries.

We define the restriction of \succsim to the subsets Γ and Λ as \succsim_Γ and \succsim_Λ , respectively.¹⁹ Assumption (I) imposes Independence on these two induced relations.

Independence (I): *The relations \succsim_Γ and \succsim_Λ satisfy Independence.*

Of course, one could also suppose Independence on either subset of preferences, for example, only \succsim_Λ (the definition for preferences over late resolving lotteries is analogously defined).

Independence over Early Resolving Lotteries (I_Λ): *The relation \succsim_Λ satisfies Independence.*

The last assumption is **Time Neutrality (TN)**, discussed previously.

Segal (1990), among other things, relates his proposed axioms to one another. In particular, he shows that if \succsim satisfies WO and C, then (i) ROCL implies TN; (ii) ROCL and R imply I, and ROCL and I imply R; and (iii) R, I, and TN, imply ROCL. We can extend Segal's reasoning to include CTI, PTI, SPTI, and TI.²⁰

Proposition 12. *Suppose \succsim satisfies WO and C. The following statements are true.*²¹

1. (i) ROCL implies SPTI (ii) TI implies SPTI, R, and CTI; (iii) SPTI implies PTI.
2. (i) R, CTI, and PTI jointly imply TI (and so SPTI); (ii) CTI, SPTI and TN jointly imply ROCL
3. TN, R, CTI, and PTI jointly imply I (and so ROCL).

¹⁹Both Γ and Λ are isomorphic to Δ , and therefore \succsim_Γ and \succsim_Λ can be interpreted as the the individual's preferences over simple lotteries in the appropriate period.

²⁰We do not discuss the implications of SPTI and PTI individually, nor CTI individually, although our results can be extended. We do so because of the focus of our analysis is on CTI together with at least one of SPTI and PTI holding.

²¹Some items have already been established earlier; we add them here for completeness.

Proof of Proposition 12. We show each part in turn

1. Observe that ROCL implies that all lotteries with the same reduced form probabilities are indifferent, which immediately implies SPTI. It is clear that TI implies SPTI, R and CTI since they are just TI applied to particular subsets of mixtures. We discussed previously that SPTI implies PTI.
2. Axioms R, PTI and CTI have been already shown to imply a posterior-anticipatory representation, which implies TI (and so SPTI). CTI and SPTI implies that we have a representation of the form $\hat{\nu}_1(\phi(P)) + \sum_i P(p_i)\nu_2(p_i)$. Over early resolving lotteries this takes the structure $\hat{\nu}_1(\phi(P)) + \sum_i P(\delta_{x_i})\nu_2(\delta_{x_i})$, which is simply a non-expected utility functional over the reduced form probabilities. TN implies this must be true also for any lottery with structure $\hat{\nu}_1(\phi(P)) + \nu_2(\phi(P))$, and so $\nu_2(\phi(P)) = \sum_i P(\delta_{x_i})\nu_2(\delta_{x_i})$, and so ν_2 satisfies reduction, and so ROCL is satisfied.
3. Last, from item (2) R, CTI, and PTI imply SPTI, and we know that CTI, SPTI, and TN imply ROCL. \square

All relationships in Proposition 12 are interpreted via the lens of restrictions on preferences. In the context of our paper, it is perhaps more instructive to interpret them via the functional forms.

Corollary 1. *Suppose \succsim has a prior-anticipatory representation. Then (i) TN or ROCL implies that ν_2 is an expected utility functional; and (ii) R or I_Λ implies that \succsim has a posterior-anticipatory representation.*

If \succsim has a posterior-anticipatory representation and satisfies TN, then it is expected utility.

Proof of Corollary 1: A prior-anticipatory representation implies that SPTI is satisfied. From the previous proof we know that SPTI and TN jointly imply ROCL, and that ROCL alone implies TN. Given the representation $\hat{\nu}_1(\phi(P)) + \sum_i P(p_i)\nu_2(p_i)$, TN implies that $\nu_2(p) = \sum_i p(\delta_{x_i})\nu_2(\delta_{x_i})$, and so ν_2 is expected utility. As shown previously, if R is satisfied then a posterior-anticipatory representation is implied. Moreover, Proposition 13 below shows that if I_Λ is satisfied, then a posterior-anticipatory representation is implied.

The representation of posterior-anticipatory preferences has the form $\sum_i P(p_i) \sum_j p_i(x_j)\hat{u}(x_j) + \sum_i P(p_i)\nu_1(p_i)$. Observe that over Λ these preferences have the structure $\sum_i P(p_i) \sum_j p_i(x_j)\hat{u}(x_j) + \sum_i P(\delta_{x_i})\nu_1(\delta_{x_i})$, which is expected utility. Thus, if TN is satisfied, preferences over Γ must also satisfy Independence. Given R, I and TN imply standard expected utility. \square

These results allude to an alternative characterization of posterior-anticipatory preferences.

Proposition 13. *The following are equivalent:*

- The relation \succsim satisfies WO, C, CTI, SPTI, and I_Λ
- The relation \succsim has a posterior-anticipatory representation

Proof of Proposition 13: We use the fact that the relation \succsim has a prior-separable expected utility representation if and only if it has a posterior-anticipatory representation. Given this, the result follows from the following claim.

Claim 9. *The relation \succsim has a prior-separable expected utility representation if and only if it satisfies WO, C, CTI, SPTI, and I_Λ .*

Proof of Claim 9. It is easy to check that any V_{pseu} representation satisfies CTI, SPTI and I_Λ . For the other direction, notice that we have (given CTI and SPTI) a representation of the form $\hat{\nu}_1(\phi(P)) + \sum_i P(p_i)\nu_2(p_i)$. Moreover, I_Λ implies that Independence is satisfied over lotteries in Λ . Observe that within Λ the representation has the form $\hat{\nu}_1(\phi(P)) + \sum_i P(\delta_{x_i})\nu_2(\delta_{x_i})$. The second terms is simply an expected utility functional on Λ . Thus, the first term must be expected utility over the reduced form probabilities in order for Independence to be satisfied. \square

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