An Anscombe–Aumann Approach to Second-Order Expected Utility*

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Abstract

We present an axiomatization of the Second-Order Expected Utility model in the environment of Anscombe and Aumann (1963) where the domain of the preference relation is the set of lotteries over all acts whose prize are lotteries. We replace the axiom of reversal of order in compound lotteries by an extension of monotonicity in the prizes that is a strengthening of the Dominance axiom introduced by Seo (2009). This extends the contributions of Grant et al. (2009) by allowing for a general representation result without restricting the decision maker's attitude towards subjective uncertainty.

1 Introduction

In Anscombe and Aumann (1963), a decision maker ranks objective (or roulette) lotteries $p \in \mathcal{L}(Z)$ with prizes in some set Z.[1] His preference ordering $\succeq^{ca}$ on $\mathcal{L}(Z)$ satisfies the axioms of expected utility theory and can then be numerically represented by the functional

$$p \longmapsto \mathbb{E}_p(u) := \sum_{z \in Z} p(z)u(z)$$

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[1] Lotteries are assumed to be simple in the sense that their support is finite.

for some utility function $u : Z \to \mathbb{R}$. In order to accommodate subjective uncertainty, Anscombe and Aumann (1963) introduce a finite set S of states of nature. An act (or horse lottery) is then defined as a function $h : S \to \mathcal{L}(Z)$ from states of nature to objective lotteries on prizes in Z. The set $\mathcal{L}(Z)^S$ of acts is denoted by $\mathcal{H}$. Anscombe and Aumann (1963) assumed that the decision maker can not only rank objective lotteries (elements of $\mathcal{L}(Z)$), but can also rank lotteries on acts (elements of $\mathcal{L}(\mathcal{H})$). His preference ordering $\succeq$ on $\mathcal{L}(\mathcal{H})$ is assumed to extend the preference ordering $\succeq^{ca}$ on $\mathcal{L}(Z)$ by identifying constant acts with objective lotteries. The preference $\succeq$ is also assumed to satisfy the axioms of expected utility theory and can then be numerically represented by a functional

$$P \longmapsto \sum_{f \in \mathcal{H}} P(f)U(f)$$

for some utility function $U : \mathcal{H} \to \mathbb{R}$. In addition to the assumptions of expected utility theory, the two preference orderings $\succeq^{ca}$ and $\succeq$ are connected by two additional axioms: the *monotonicity* in the prizes and the *reversal of order in compound lotteries*. The first axiom requires that if two acts f and g are identical except for the lotteries $f(s)$ and $g(s)$ associated with one state s, then the ranking (according to $\succeq$) between the horse lotteries f and g is governed by the ranking (according to $\succeq^{ca}$) between the lotteries $f(s)$ and $g(s)$. The second axiom says that spinning a roulette wheel on acts before the realization of the state of nature is equivalent to spinning the same roulette wheel after subjective uncertainty is resolved. Formally, any lottery $P \in \mathcal{L}(\mathcal{H})$ on acts is equivalent to the degenerate lottery on the act g with the prize $g(s)$ defined as the compound lottery of P and $f(s)$:

$$g(s) := \sum_{f \in \mathcal{H}} P(f)f(s).$$

Under monotonicity and the reversal of order in compound lotteries, Anscombe and Aumann (1963) proved that there exists a unique probability measure μ over the set S of states of nature such that

$$U(f) = \sum_{s \in S} \mu(s)\mathbb{E}_{f(s)}(u).$$

This is the so-called standard *subjective expected utility* model of Anscombe and Aumann (1963). This elegant model has been challenged by the famous paradox of Ellsberg (1961) since it does not accommodate a decision maker's aversion to bets on events for which probabilities are not specified. To account for uncertainty aversion, numerous extensions of subjective expected utility have been proposed, including Schmeidler (1989)'s Choquet expected utility, Gilboa and Schmeidler (1989)'s maxmin expected utility, Segal (1987)'s application of anticipated utility, Tversky and Kahneman (1992)'s cumulative prospect theory (see also Wakker and Tversky (1993) for an axiomatization), Neilson (1993, 2010)'s Second-Order Expected utility (SOEU) model and a number of more recent models like the smooth ambiguity model of Klibanoff et al. (2005), the variational preferences of Maccheroni et al. (2006), the Second-Order Subjective Expected Utility (SOSEU) model of Seo (2009) and the second-order variational preferences of Nascimento and Riella (2013).

We follow this literature and propose a new axiomatization of the Second-Order Expected Utility (SOEU) model that was first axiomatized by Neilson (1993, 2010). Imposing the Savage axioms to the horse lotteries and the von Neumann–Morgenstern axioms to the roulette lotteries, Neilson (2010) obtained the following SOEU representation

$$U(f) = \sum_{s \in S} \mu(s) v\left(\mathbb{E}_{f(s)}(u)\right)$$

yielding a subjective probability measure over states and two utility functions, the function u governing risk attitudes and the function v governing ambiguity attitudes. This is a particular case of the smooth ambiguity model of Klibanoff et al. (2005) and the second-order subjective expected utility of Seo (2009). Neilson (2010) proved that it can still account for "Ellsbergian" choices. This model has been used by Nau (2006), Ergin and Gul (2009), Chew and Sagi (2008), Grant et al. (2009), Strzalecki (2011) and Al-Najjar and Castro (2014). It is also very popular in applied work: see for instance Snow (2010, 2011), Gollier (2011), Alary et al. (2013), Hoy et al. (2014) and Huang and Tzeng (2018).

To get an axiomatization of the SOEU model based on the axioms of Anscombe and Aumann (1963), we drop reversal of order in compound lotteries and propose a natural extension of monotonicity to lotteries over acts. In our representation result, the probability μ is unique and the functions u and $v \circ \mathbb{E}^u$ are cardinally unique.[2] We then show that the function v is concave if, and only if, we add the standard Uncertainty Aversion axiom of Schmeidler (1989). This complements the analysis in Seo (2009) who obtains an axiomatization of Second-Order Subjective Expected Utility (SOSEU) by considering a weaker axiom

[2] $\mathbb{E}^u$ is mapping $p \mapsto \mathbb{E}_p(u)$ from $\mathcal{L}(Z)$ to $\mathbb{R}$.

than our extended monotonicity.[3] It also extends the contributions of Grant et al. (2009) by allowing for a general representation result without restricting the decision maker's attitude towards subjective uncertainty.

2 The Setup

2.1 Notations

Fix a non-empty space X. By a *lottery* (or a *simple probability distribution*) on X, we mean a map $p \in \mathbb{R}_+^X$ such that its support $\{x \in X : p(x) > 0\}$ is finite and $\sum_{x \in X} p(x) = 1$. For any $A \subseteq X$, we slightly abuse notation and write $p(A) := \sum_{a \in A} p(a)$ that can be interpreted as the probability of getting an element of A. The set of all lotteries on X is denoted by $\mathcal{L}(X)$. For any $x \in X$, we let $\delta_x \in \mathcal{L}(X)$ be the point mass at x, defined by $\delta_x(y) = 0$ if $y \neq x$ and $\delta_x(x) = 1$. Lotteries of the form δ_x are called degenerate lotteries. Observe that every lottery $p \in \mathcal{L}(X)$ can be written as a convex combination of degenerate lotteries

$$p = \sum_{x \in X} p(x) \delta_x$$

where the above sum is finite since the support of p is finite. For any lottery $p \in \mathcal{L}(X)$ and any real map $u \in \mathbb{R}^X$, the *expected value of u with respect to p* is defined as the real number

$$\mathbb{E}_p(u) := \sum_{x \in X} p(x) u(x).$$

If u is interpreted as a utility function on X, then $\mathbb{E}_p(u)$ corresponds to the *expected utility* of the lottery p.

[3] In the SOSEU model, the decision maker evaluates a lottery P on acts according to

$$\sum_{f \in \mathcal{H}} P(f) \int_{\mathcal{L}(S)} v\left(\mathbb{E}_{\mu \circ f}(u)\right) d\nu(\mu)$$

where ν is a second-order belief (a belief on beliefs over states of nature) and $\mu \circ f$ is the lottery on prizes obtained by combining μ and f, i.e., $(\mu \circ f)(z) = \sum_{s \in S} \mu(s) f(s)$ for any $z \in Z$. Our representation result cannot be directly obtained from the representation result in Seo (2009). This is because Seo (2009) does not provide any characterization of the support of the second order belief ν.

2.2 Anscombe–Aumann Setting

Let S be a finite set of *states of nature* to represent situations involving subjective uncertainty. Let Z denote a set of *outcomes* or *prizes*. A *purely subjective act* is a function $f \in Z^S$. In order to provide a representation result of preferences on purely subjective acts, we follow Anscombe and Aumann (1963) by considering an enrichment of items to which preference must apply. We denote by $\mathcal{H}$ the set of functions from S into $\mathcal{L}(Z)$ that are called *acts* (or *horse race lotteries*).[4] The decision maker is assumed to be able to rank lotteries on acts. Therefore, the decision maker's preference $\succeq$ is a binary relation on $\mathcal{L}(\mathcal{H})$. Identifying an act $f \in \mathcal{H}$ with the degenerate lottery $\delta_f \in \mathcal{L}(\mathcal{H})$, we can view $\mathcal{H}$ as a subset of $\mathcal{L}(\mathcal{H})$. The ranking induced by $\succeq$ on degenerate lotteries δ_f for any $f \in \mathcal{H}$ is denoted by $\succeq^a$.[5] Similarly, identifying the lottery $p \in \mathcal{L}(Z)$ with the degenerate lottery $\delta_{p\mathbf{1}_S}$ on the constant act $p\mathbf{1}_S : s \mapsto p$, we can view $\mathcal{L}(Z)$ as a subset of $\mathcal{L}(\mathcal{H})$ and denote by $\succeq^{ca}$ the induced ranking.[6]

We should be careful when considering compound lotteries in $\mathcal{L}(\mathcal{L}(Z))$. In this paper, a lottery $P \in \mathcal{L}(\mathcal{H})$ such that the support of P only contains constant acts is called a compound lottery and is identified as an element of $\mathcal{L}(\mathcal{L}(Z))$. Using the terminology of Anscombe and Aumann (1963), the compound lottery P corresponds to a first spin of a roulette wheel before the horse race. The prizes of this first roulette are constant acts $f : s \mapsto p_f$ with known chance $P(f)$. Then, after the horse race, independently of the realized state s, a second roulette wheel is spun which prizes are elements of Z and chances are given by $p_f \in \mathcal{L}(Z)$.

Typical elements of $\mathcal{L}(\mathcal{H})$ are denoted by P, Q, and R. We use f, g, and h for elements in $\mathcal{H}$. Typical elements of $\mathcal{L}(\mathcal{L}(Z))$ are denoted by $\bar{P}$, $\bar{Q}$, and $\bar{R}$. Finally, p, q, and r are typical elements of $\mathcal{L}(Z)$. Using this notation, an element $P \in \mathcal{L}(\mathcal{H})$ can be written

$$P = \sum_{f \in \mathcal{H}} P(f)\delta_f$$

where each $f : S \mapsto \mathcal{L}(Z)$ can be decomposed as follows

$$\forall s \in S, \quad f(s) = \sum_{z \in Z} f(s)(z)\delta_z.$$

[4] By identifying a prize z with the corresponding degenerate lottery δ_z, the set Z^S of purely subjective acts can be embedded in $\mathcal{H}$.

[5] The upper-script "a" stands for act.

[6] The upper-script "ca" stands for constant act.

A typical lottery on acts $P \in \mathcal{L}(\mathcal{H})$ is depicted in Figure 1.

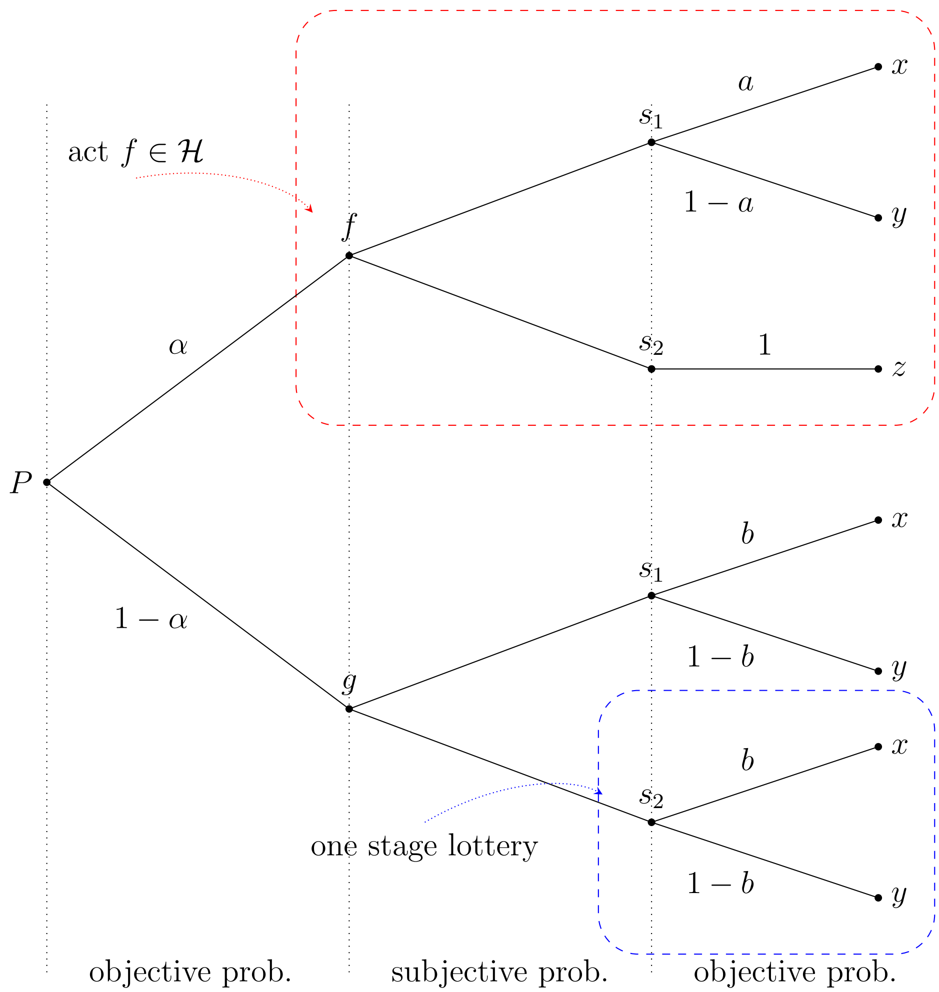


Figure 1: $P \in \mathcal{L}(\mathcal{H})$ is a lottery on acts. The first and the last nodes are governed by the objectives probabilities $(\alpha, 1-\alpha)$, $(a, 1-a)$ and $(b, 1-b)$. The act g is constant and can be viewed as a one-stage lottery in $\mathcal{L}(Z)$.

We refer to an element in $\mathcal{L}(Z)$ as a one-stage lottery and refer to an element in $\mathcal{L}(\mathcal{L}(Z))$ as a two-stage (or compound) lottery. A constant act (taking the same value for every $s \in S$) is viewed as a one-stage lottery. If $f, g \in \mathcal{H}$ and $E \subset S$ with $E \notin \{\emptyset, S\}$, then $f_E g$ denotes the act with $f_E g(s) = f(s)$ if $s \in E$ and $f_E g(s) = g(s)$ if $s \notin E$. When $E = \{s\}$ is a singleton, we use the simpler notation $f_s g$ instead of $f_{\{s\}} g$.

The spaces $\mathcal{L}(Z)$, $\mathcal{H}$ and $\mathcal{L}(\mathcal{H})$ are convex subsets of linear subspaces and are therefore endowed with their natural mixture. Formally, if p and q are one-stage lotteries in $\mathcal{L}(Z)$, then

$\alpha p + (1-\alpha)q$ is also a one-stage lottery defined by $[\alpha p + (1-\alpha)q](z) = \alpha p(z) + (1-\alpha)q(z)$ for each prize $z \in Z$. Similarly, if f and g are two acts in $\mathcal{H}$ and $\alpha \in [0,1]$, then $\alpha f + (1-\alpha)g$ is also an act defined by the componentwise mixture

$$[\alpha f + (1-\alpha)g](s) := \alpha f(s) + (1-\alpha)g(s), \quad \text{for all } s \in S$$

where $\alpha f(s) + (1-\alpha)g(s)$ is the mixture of the two one-stage lotteries $f(s)$ and $g(s)$. Finally, if P and Q in $\mathcal{L}(\mathcal{H})$ are two lotteries on acts, then $\alpha P + (1-\alpha)Q$ is also a lottery on acts defined by

$$\alpha P + (1-\alpha)Q := \sum_{f \in \mathcal{H}} [\alpha P(f) + (1-\alpha)Q(f)]\delta_f.$$

If p and q are one-stage lotteries, then $\alpha p + (1-\alpha)q$ is also a one-stage lottery.[7] It should not be confused with the two stage lottery $\alpha\delta_{p\mathbf{1}_S} + (1-\alpha)\delta_{q\mathbf{1}_S}$ (also denoted by $\alpha\delta_p + (1-\alpha)\delta_q$) that is a compound lottery where the first lottery occurs before the outcome of the subjective uncertainty and the second occurs after. See Figure 2 for an illustration.

2.3 Basic Axioms

The two following axioms are standard.

Order. The preference $\succeq$ is complete and transitive.

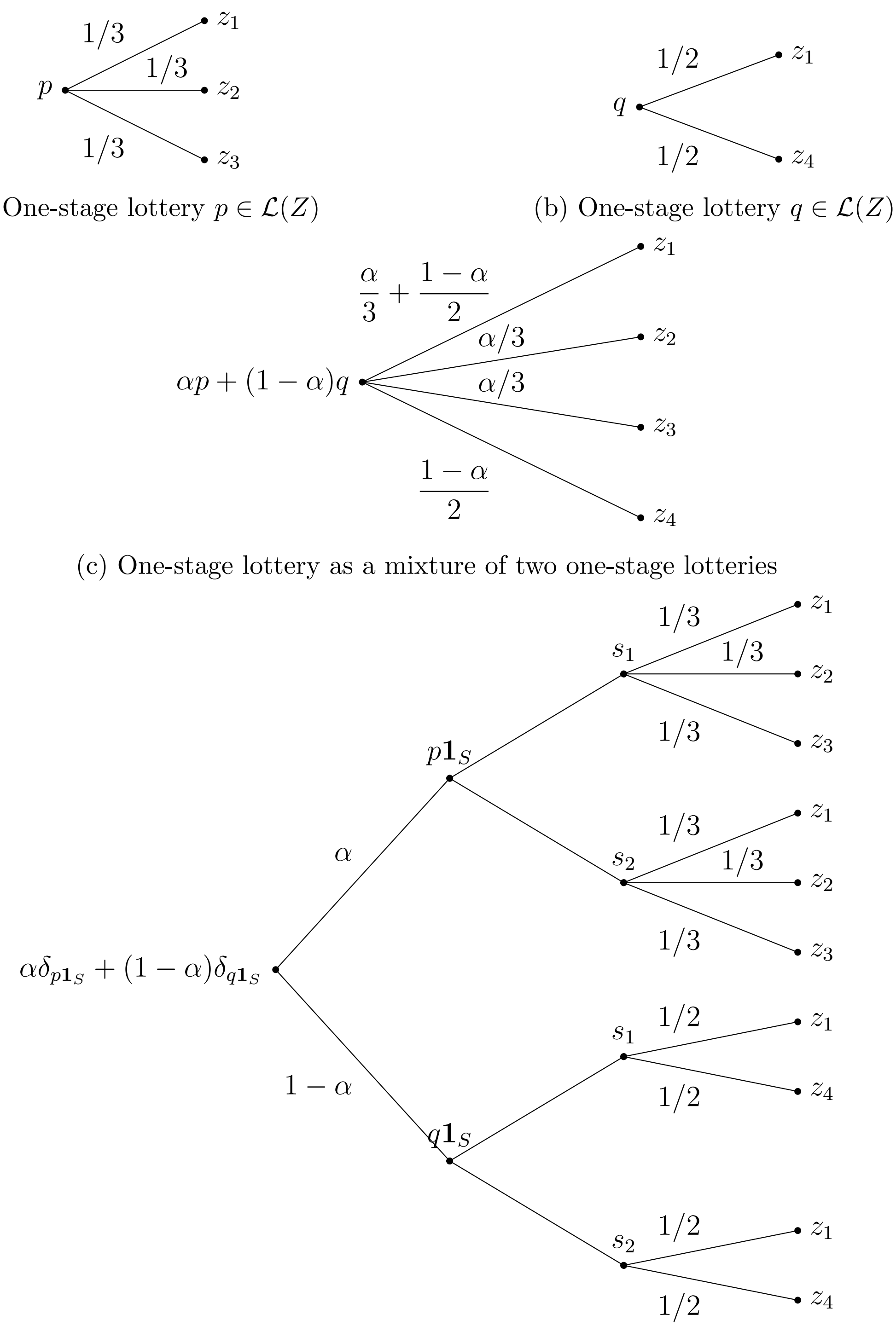
Mixture Continuity. If $P, Q, R \in \mathcal{L}(\mathcal{H})$, then the sets $\{\alpha \in [0,1] : \alpha P + (1-\alpha)Q \succeq R\}$ and $\{\alpha \in [0,1] : R \succeq \alpha P + (1-\alpha)Q\}$ are closed.

For any one-stage lottery $p \in \mathcal{L}(Z)$, we can consider the degenerate lottery $\delta_{p\mathbf{1}_S} \in \mathcal{L}(\mathcal{H})$ which assigns with probability one the constant act $p\mathbf{1}_S : s \mapsto p$. Denote by $\succeq^{ca}$ the induced preference on constant acts in $\mathcal{L}(Z)$ defined by

$$p \succeq^{ca} q \Longleftrightarrow \delta_{p\mathbf{1}_S} \succeq \delta_{q\mathbf{1}_S}.$$

Since one-stage lotteries involve *objective probabilities* in the sense that everyone agrees on the likelihood of getting each alternative in Z, we impose the standard independence axiom.

[7] We can interpret $\alpha p + (1-\alpha)q$ as the successive spin of two different roulette wheels, but both spins occurring after the realization of the subjective uncertainty.



(a) One-stage lottery $p \in \mathcal{L}(Z)$

(b) One-stage lottery $q \in \mathcal{L}(Z)$

(c) One-stage lottery as a mixture of two one-stage lotteries

(d) Two-stage lottery

Figure 2: $\alpha p + (1-\alpha)q$ is a mixture of one-stage lotteries but $\alpha\delta_p + (1-\alpha)\delta_q$ is a mixture of two-stage lotteries.

Second-Stage Independence. For any $\alpha \in (0, 1]$ and one-stage lotteries $p, q, r \in \mathcal{L}(Z)$,

$$\alpha p + (1-\alpha)r \succeq^{ca} \alpha q + (1-\alpha)r \Longleftrightarrow p \succeq^{ca} q.$$

The interpretation of this assumption becomes transparent when the mixture one-stage lottery $\alpha p + (1-\alpha)r$ is viewed as a compound lottery compounded from two roulette wheels which spins occur after the realization of the subjective uncertainty (represented by the states in S). In a Lottery A, the first spin determines whether, with probability α, you get the lottery p, or, with probability $1-\alpha$, you get r. Then, conditional on the result of the first spin, there is a second spin which chances to get a prize in Z are defined by p or r. In a Lottery B, the first spin determines whether, with probability α, you get the lottery q, or, with probability $1-\alpha$, you get r. Then, conditional on the result of the first spin, there is a second spin which chances to get a prize in Z are defined by q or r. The Second-Stage Independence axiom requires that the decision maker's prefers Lottery A to Lottery B if, and only if, he prefers p to q. This axiom is related to dynamic consistency as explained in Gilboa (2008).

In a second-stage lottery $p \in \mathcal{L}(Z)$, the chances $(p(z))_{z\in Z}$ are objectively defined. Similarly, the chances $(P(f))_{f\in\mathcal{H}}$ involved in a lottery on acts $P \in \mathcal{L}(\mathcal{H})$ are also objective. Therefore, replacing the set Z by $\mathcal{H}$ in the Second-Stage Independence axiom, we get the following natural axiom.

First-Stage Independence. For any $\alpha \in (0, 1]$ and lotteries $P, Q, R \in \mathcal{L}(\mathcal{H})$,

$$\alpha P + (1-\alpha)R \succeq \alpha Q + (1-\alpha)R \Longleftrightarrow P \succeq Q.$$

As before, the interpretation of the First-Stage Independence axiom becomes transparent when the mixture lottery $\alpha P + (1-\alpha)Q$ is viewed as a compound lottery compounded from two roulette wheels which spins occur before the realization of the subjective uncertainty (represented by the states in S). The prizes associated to the first spin are lotteries over acts, while the prize associated to the second spin are acts.

Remark 2.1. It is important to observe that first-stage independence is not related to the following property (Independence axiom on acts): for any $\alpha \in (0, 1]$ and acts $f, g, h \in \mathcal{H}$,

$$\alpha f + (1-\alpha)h \succeq \alpha g + (1-\alpha)h \Longleftrightarrow f \succeq g.$$

This is because the mixture of lotteries $\alpha\delta_f + (1-\alpha)\delta_g$ is different than the degenerate lottery $\delta_{\alpha f+(1-\alpha)g}$ on the mixture act $\alpha f + (1-\alpha)g$.

3 Extended Monotonicity and SOEU

For any act $f \in \mathcal{H}$, we can consider the degenerate lottery $\delta_f \in \mathcal{L}(\mathcal{H})$ which assigns with probability one the act f. We can define $\succeq^a$ the induced preference on $\mathcal{H}$ as follows

$$f \succeq^a g \Longleftrightarrow \delta_f \succeq \delta_g.$$

We start by recalling the standard Monotonicity axiom applied to acts.

Monotonicity. For any acts $f, g \in \mathcal{H}$,

$$[f(s) \succeq^{ca} g(s), \quad \forall s \in S] \Longrightarrow f \succeq^a g.$$

Observe that the Monotonicity axiom deals only with acts in $\mathcal{H}$. In order to extend it to lotteries over acts, we introduce the following notation.

Fix an act $f \in \mathcal{H}$ and a state $s \in S$. Recall that the property $f(s) \succeq^{ca} g(s)$ in the statement of the Monotonicity axiom is equivalent to $f(s)\mathbf{1}_S \succeq^a g(s)\mathbf{1}_S$. We then consider the following notation: $\Psi(f, s) := f(s)\mathbf{1}_S$ is the constant act $s' \mapsto f(s)$ taking the value $f(s)$ for every $s' \in S$. Observe that monotonicity can be stated as follows: for any acts $f, g \in \mathcal{H}$,

$$[\Psi(f, s) \succeq^a \Psi(g, s), \quad \forall s \in S] \Longrightarrow \delta_f \succeq \delta_g,$$

or, equivalently,

$$\left[\delta_{\Psi(f,s)} \succeq \delta_{\Psi(g,s)}, \quad \forall s \in S\right] \Longrightarrow \delta_f \succeq \delta_g. \tag{3.1}$$

Recall that a lottery $P \in \mathcal{L}(\mathcal{H})$ on acts can be canonically decomposed as

$$P = \sum_{f \in \mathcal{H}} P(f)\delta_f. \tag{3.2}$$

We then let $\Psi(P, s) \in \mathcal{L}(\mathcal{H})$ be the lottery on (constant) acts defined by

$$\Psi(P, s) := \sum_{f \in \mathcal{H}} P(f)\delta_{\Psi(f,s)}$$

where each act $f : s' \mapsto f(s')$ in the canonical decomposition (3.2) is replaced by the constant act $\Psi(f, s) : s' \mapsto f(s)$. See Figure 3 for an illustration.

Replacing the LHS of the Monotoncity axiom (3.1) by

$$\sum_{f\in\mathcal{H}} P(f)\delta_{\Psi(f,s)} \succeq \sum_{g\in\mathcal{H}} Q(g)\delta_{\Psi(g,s)}, \quad \forall s \in S$$

we get the following extension of the Monotonicity axiom to lotteries on acts.

Extended Monotonicity. For any $P, Q \in \mathcal{L}(\mathcal{H})$,

$$[\Psi(P, s) \succeq \Psi(Q, s), \quad \forall s \in S] \Longrightarrow P \succeq Q.$$

To interpret extended monotonicity, consider an agent who is not certain of the true state. If he believes that the state is s, then the two-stage lottery induced by P is $\Psi(P, s)$. Extended dominance means that if the decision maker prefers the two-stage lottery induced by P to the one induced by Q independently of the true state s, then he must prefer P to Q. Choosing degenerate lotteries $P = \delta_f$ and $Q = \delta_g$ on single acts $f, g \in \mathcal{H}$, we easily prove that extended monotonicity is stronger than monotonicity. It will follow from our characterization result that extended monotonicity is a much stronger axiom than monotonicity. Seo (2009) used the terminology "extended AA dominance" for our axiom of extended monotonicity.

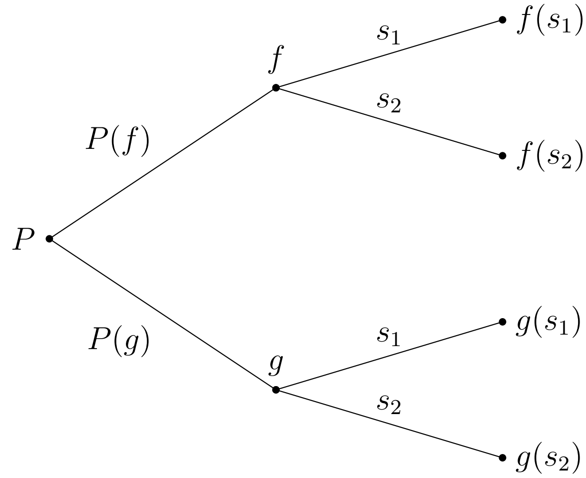
Remark 3.1. As pointed out by Seo (2009), extended monotonicity implies Krep's reversal-of-order-style axiom (see (Kreps 1988, p. 107)). It states that if two lotteries P and Q over "purely subjective" (or Savage) acts map naturally to the same lottery over outcomes for each state s, then they must be indifferent.

We now state the formal definition of the Second Order Expected Utility (SOEU) representation.

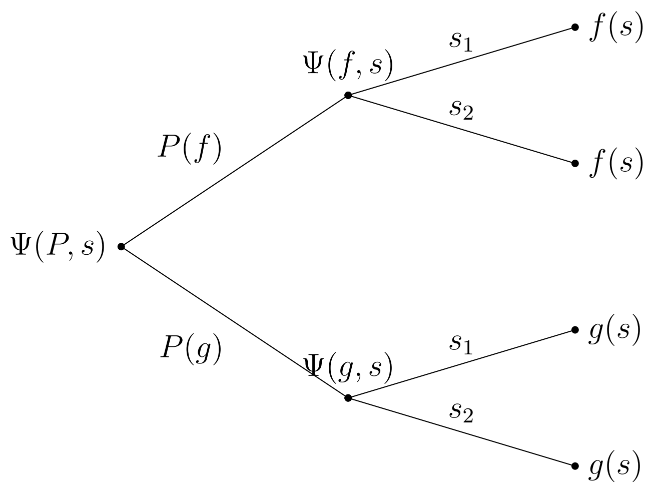
Definition 3.1. A SOEU representation of the preference $\succeq$ is a probability measure $\mu \in \mathcal{L}(S)$, a function $u : Z \to \mathbb{R}$, and a strictly increasing function $v : \mathrm{co}(u(Z)) \to \mathbb{R}$ such that I represents $\succeq$ on $\mathcal{L}(\mathcal{H})$, where

$$I(P) = \sum_{f\in\mathcal{H}} P(f) \sum_{s\in S} \mu(s)v(\mathbb{E}_{f(s)}(u)). \tag{3.3}$$

The main contribution of this paper is the following new representation result that obtains



(a) $P \in \mathcal{L}(\mathcal{H})$ is an objective lottery on acts



(b) $\Psi(P, s)$ is an objective lottery on constant acts

Figure 3: If the decision maker believes that s occurs with probability one, then P and $\Psi(P, s)$ are equivalent.

an axiomatization of SOEU without any restriction on the decision maker's attitude towards subjective uncertainty.[8]

Theorem 3.1. *Preference $\succeq$ on $\mathcal{L}(\mathcal{H})$ satisfies order, mixture continuity, first-stage independence, second-stage independence, and extended monotonicity if, and only if, it has an SOEU representation (μ, u, v).*

Proof. We start by proving necessity of the axioms. Assume that $\succeq$ is represented by the function I in (3.3). The proof that order, mixture continuity and first-stage independence are necessary is a straightforward exercise and is omitted. Observe that the preference $\succeq^{ca}$ on constant acts is represented by the function $p \mapsto \mathbb{E}_p(u)$ and therefore second-stage independence is satisfied. To prove extended monotonicity, fix $P \in \mathcal{L}(\mathcal{H})$ and recall that $\Psi(P, s)$ is the following lottery on constant acts

$$\Psi(P, s) := \sum_{f \in \mathcal{H}} P(f) \mathbf{1}_{f(s)\mathbf{1}_S}.$$

This implies that

$$I(\Psi(P, s)) = \sum_{f \in \mathcal{H}} P(f) v(\mathbb{E}_{f(s)}(u))$$

and we get that

$$I(f) = \sum_{s \in S} \mu(s) I(\Psi(P, s)).$$

Extended monotonicity then follows immediately.

We now provide the proof of sufficiency. When $P \sim Q$ for all $P, Q \in \mathcal{L}(\mathcal{H})$, then the representation is trivial. It is sufficient to choose any arbitrary constant function u.[9] Thus, assume that $\succeq$ satisfies non-degeneracy, i.e., $P \succ Q$ for some $P, Q \in \mathcal{L}(\mathcal{H})$. Since the preference $\succeq$ satisfies order, mixture continuity and first-stage independence, we can apply Theorem 5.15 in Kreps (1988) to deduce that there exists a function $U : \mathcal{H} \to \mathbb{R}$ such that $\succeq$ is represented by the function $I : \mathcal{L}(\mathcal{H}) \to \mathbb{R}$ defined by

$$I(P) = \sum_{f \in \mathcal{H}} P(f) U(f).$$

Moreover, the function U is unique up to a positive affine transformation. That is, if $I'(P) =$

[8] This corresponds to one of the open problems suggested by Grant et al. (2009).
[9] Observe that in this case, the probability μ is indeterminate.

$\sum_{f\in\mathcal{H}} P(f)U'(f)$ also represents $\succeq$ for some function $U' : \mathcal{H} \to \mathbb{R}$, then there exist two real numbers $a > 0$ and $b \in \mathbb{R}$ such that $U' = aU + b$. We normalize U such that

$$\sum_{f\in\mathcal{H}} U(f)R^0(f) = 0$$

for some lottery R^0 on constant acts. Recall that is R^0 can be identified with a compound lottery in $\mathcal{L}(\mathcal{L}(Z))$.

Let U^{ca} be the restriction of U to constant acts in $\mathcal{H}$. Since constant acts can be identified with lotteries on Z, we can assume that U^{ca} is a function defined on $\mathcal{L}(Z)$. The function U^{ca} is a representation of the preference $\succeq^{ca}$ on constant acts in $\mathcal{L}(Z)$ that satisfies order, mixture continuity and second-stage independence. Applying again Theorem 5.15 in Kreps (1988), we deduce the existence of a function $u : Z \to \mathbb{R}$ such that $\succeq^{ca}$ is represented by the function $I^{ca} : \mathcal{L}(Z) \to \mathbb{R}$ defined by

$$I^{ca}(p) = \sum_{z\in Z} p(z)u(z).$$

Moreover, the function u is unique up to a positive affine transformation. The functions U^{ca} and I^{ca} represent the same preference $\succeq^{ca}$. Observe moreover that $I^{ca}(\mathcal{L}(X)) = \text{co}(u(X))$ is an interval of the real line.

Lemma 3.1. *There exists a strictly increasing function $v : \text{co}(u(X)) \to \mathbb{R}$ such that $U^{ca} = v \circ I^{ca}$.*

Proof. We follow almost verbatim the arguments of Lemma B.9 in Seo (2009). Fix an arbitrary $t \in \text{co}(u(X))$. There exists a lottery $p \in \mathcal{L}(X)$ such that $t = I^{ca}(p)$. Observe that for any other lottery $q \in \mathcal{L}(X)$ satisfying $t = I^{ca}(q)$, we have $q \sim^{ca} p$. We then have $U^{ca}(p) = U^{ca}(q)$ and we can define $v(t) := U^{ca}(p)$. To prove that v is strictly increasing, let $t, t' \in \text{co}(u(X))$ such that $t' > t$. There exist $p, p' \in \mathcal{L}(X)$ such that $t = I^{ca}(p)$ and $t' = I^{ca}(p')$. In particular, we have $p' \succ^{ca} p$. This implies that $U^{ca}(p') > U^{ca}(p)$ and we get that $v(t') > v(t)$. □

We propose to show that extended monotonicity translates into the following no-arbitrage-type property. Identify any act $f \in \mathcal{H}$ as an asset with state-dependent payoff given by the random variable

$$(U^{ca}(\Psi(f, s)))_{s\in S}.$$

A portfolio is a simple function $\theta : \mathcal{H} \to \mathbb{R}$ with finite support, that is the set $\{f \in \mathcal{H} : \theta(f) \neq 0\}$ is finite. We let Θ be the linear space of portfolios. The payoff of portfolio θ in state s is then

$$\Pi(\theta, s) := \sum_{f \in \mathcal{H}} \theta(f) U^{ca}(\Psi(f, s)).$$

The price of each asset f is $U(f)$ and the cost of portfolio θ is then

$$c(\theta) := \sum_{f \in \mathcal{H}} \theta(f) U(f).$$

Extended monotonicity implies the following (weak) no-arbitrage property.

Lemma 3.2. *Any portfolio with non-negative payoffs cannot have a negative price. Formally, for every portfolio* $\theta \in \Theta$

$$[\Pi(\theta, s) \geq 0, \quad \forall s \in S] \Longrightarrow c(\theta) \geq 0.$$

Proof of Lemma 3.2. We can adapt the arguments in Seo (2009). We provide the details for the sake of completeness. Fix an arbitrary portfolio $\theta \in \Theta$ such that

$$\Pi(\theta, s) \geq 0, \quad \forall s \in S. \tag{3.4}$$

We can always write $\theta = \alpha P - \beta Q$ where $P, Q \in \mathcal{L}(\mathcal{H})$ and $\alpha, \beta \geq 0$. We provide the details of the proof when $\alpha \geq \beta$. The other case can be treated similarly. If $\alpha = 0$ then $\beta = 0$ and $\theta = 0$. The desired result is then obvious. Therefore, assume that $\alpha > 0$. Note that (3.4) implies

$$\sum_{f \in \mathcal{H}} P(f) U^{ca}(\Psi(f, s)) \geq \frac{\beta}{\alpha} \sum_{f \in \mathcal{H}} Q(f) U^{ca}(\Psi(f, s)). \tag{3.5}$$

Given our normalization of U, we have for any state s

$$0 = \sum_{f \in \mathcal{H}} R^0(f) U(f) = \sum_{f \in \mathcal{H}} R^0(f) U^{ca}(\Psi(f, s))$$

since the support of the lottery R^0 only contains constant acts. Letting $\gamma := \beta/\alpha$, we have

$$\begin{aligned}\sum_{f\in\mathcal{H}} P(f)U^{ca}(\Psi(f,s)) &\geq \gamma\sum_{f\in\mathcal{H}} Q(f)U^{ca}(\Psi(f,s)) + (1-\gamma)\sum_{f\in\mathcal{H}} R^0(f)U^{ca}(\Psi(f,s)) \\ &\geq \sum_{f\in\mathcal{H}} \left[\gamma Q(f) + (1-\gamma)R^0(f)\right] U^{ca}(\Psi(f,s)).\end{aligned}$$

We have thus proved that $I(\Psi(P,s)) \geq I(\Psi(\gamma Q + (1-\gamma)R^0, s))$, for each $s \in S$, or, equivalently,

$$\Psi(P,s) \succeq \Psi(\gamma Q + (1-\gamma)R^0, s), \quad \forall s \in S.$$

Extended monotonicity then implies that

$$P \succeq \gamma Q + (1-\gamma)R^0.$$

Using the representation function I, we deduce that

$$I(P) \geq I(\gamma Q + (1-\gamma)R^0) = \gamma I(Q) + (1-\gamma)I(R^0) = \gamma I(Q)$$

since $I(R^0) = 0$ by the normalization choice of U. We then deduce that

$$c(\theta) = \alpha c(P) - \beta c(Q) = \alpha I(P) - \beta I(Q) \geq 0$$

which is the desired result. □

We can now apply a standard convex separation theorem to prove the following weak version of the Fundamental Theorem of Finance.

Lemma 3.3. *There exists a probability measure $\mu \in \mathcal{L}(S)$ such that*

$$c(\theta) = \sum_{s\in S} \mu(s)\Pi(\theta,s), \quad \forall\theta \in \Theta.$$

Proof of Lemma 3.3. Assume, without any loss of generality, that $0 \notin S$ and pose $\Sigma := \{0\} \cup S$. Let $A \subseteq \mathbb{R}^\Sigma$ be the set of financial transfers $\tau = (\tau(\sigma))_{\sigma\in\Sigma} \in \mathbb{R}^\Sigma$ that can be implemented by a portfolio, i.e., there exists $\theta \in \Theta$ such that $\tau(0) = -c(\theta)$ and $\tau(s) = \Pi(\theta, s)$

for each $s \in S$. Since c is a weak no-arbitrage price (see Lemma 3.2), we have

$$A \cap \underbrace{(0,\infty) \times \mathbb{R}^S_+}_{=:\, B} = \emptyset.$$

A is a linear subspace (and therefore convex) and B is convex. However, applying directly the standard Convex Separation Theorem does not allow to get the desired result. We follow an approximation argument. Fix $\varepsilon \in (0,1)$ and let C_ε be the set of all vectors $c = (c(\sigma))_{\sigma \in \Sigma}$ such that

$$c \in [\varepsilon, \infty) \times \mathbb{R}^S_+ \quad \text{and} \quad \sum_{\sigma \in \Sigma} c(\sigma) = 1.$$

The set C_ε is compact and convex. The set A is closed convex. Since $A \cap C_\varepsilon = \emptyset$, we can apply the Strict Convex Separation Theorem, to deduce the existence of two real numbers $\alpha, \beta \in \mathbb{R}$ and some non-zero vector $\xi_\varepsilon = (\xi_\varepsilon(\sigma))_{\sigma \in \Sigma} \in \mathbb{R}^\Sigma$ such that

$$\forall a \in A, \quad \xi_\varepsilon \cdot a \leq \alpha < \beta \leq \xi_\varepsilon \cdot c, \quad \forall c \in C_\varepsilon.$$

Since $0 \in A$, we deduce that

$$\xi_\varepsilon(0) > 0 \quad \text{and} \quad \xi_\varepsilon(s) > -\frac{\varepsilon}{1-\varepsilon}\xi_\varepsilon(0).$$

Since A is a linear space, the inequality $\xi_\varepsilon \cdot a \leq \alpha$ for each $a \in A$ implies that $\xi_\varepsilon \cdot a = 0$ for each $a \in A$. We then deduce that

$$c(\theta) = \sum_{s \in S} \mu_\varepsilon(s)\Pi(\theta, s), \quad \forall \theta \in \Theta \tag{3.6}$$

where $\mu_\varepsilon(s) := \xi_\varepsilon(s)/\xi_\varepsilon(0)$. Observe that $\mu_\varepsilon(s) \geq -\varepsilon/(1-\varepsilon)$ for each $s \in S$. Fix now the portfolio $\theta^\star$ that is the degenerate lottery $\mathbf{1}_{p\mathbf{1}_S}$ on the constant act $s \mapsto p$ for some lottery $p \in \mathcal{L}(Z)$. Observe that

$$\Pi(\theta^\star, s) = U^{ca}(p) = U(p\mathbf{1}_S) = c(\theta^\star).$$

We then deduce from (3.6) that

$$\sum_{s \in S} \mu_\varepsilon(s) = 1.$$

Passing to a subnet if necessary, we can assume that we can pass to the limit in (3.6) and get the existence of some probability measure $\mu \in \mathcal{L}(S)$ such that

$$c(\theta) = \sum_{s \in S} \mu(s)\Pi(\theta, s), \quad \forall \theta \in \Theta$$

□

Fix an arbitrary act $f \in \mathcal{H}$. Choosing $\theta := \delta_f$ to be the degenerate lottery on f, we get that

$$U(f) = \sum_{s \in S} \mu(s) U^{ca}(\Psi(f, s)).$$

Applying Lemma 3.1, we have

$$\begin{aligned} U(f) &= \sum_{s \in S} \mu(s) U^{ca}(\Psi(f, s)) \\ &= \sum_{s \in S} \mu(s) v \circ I^{ca}(\Psi(f, s)) \\ &= \sum_{s \in S} \mu(s) v(\mathbb{E}_{f(s)}(u)) \end{aligned}$$

and we get the desired result. □

Extended monotonicity plays a crucial role in the representation result of Theorem 3.1. It cannot be replaced by the weaker axiom of monotonicity. Indeed, for a given function $u \in \mathbb{R}^Z$, consider an arbitrary function

$$\Theta : W \to \mathbb{R} \quad \text{where } W := [\text{co}(u(X))]^S$$

that is increasing on W and strictly increasing on the certainty line $\{t\mathbf{1}_S \, : \, t \in \text{co}(u(X))\}$. If we pose

$$J(P) := \sum_{f \in \mathcal{H}} P(f) \Theta((\mathbb{E}_{f(s)}(u))_{s \in S}),$$

then the function $J : \mathcal{L}(\mathcal{H}) \to \mathbb{R}$ defines a preference $\succeq$ that satisfies order, mixture continuity, first-stage independence, second-stage independence, and monotonicity. However, we do not necessarily get a SOEU representation. Indeed, a possible choice for the function Θ

is

$$\Theta_{\text{GS}}(w) := \min\{w(s)\ :\ s \in S\}$$

where the subscript GS obviously stands for Gilboa and Schmeidler (1989).

4 Uniqueness

To analyze uniqueness of the representation in Theorem 3.1, we consider the following standard axiom.

Non-degeneracy. There exist $P, Q \in \mathcal{L}(\mathcal{H})$ such that $P \succ Q$.

Before presenting the uniqueness properties, we introduce the following notation. Given a function $u : Z \to \mathbb{R}$, we can define the expected utility function $\mathbb{E}^u$ from lotteries in $\mathcal{L}(Z)$ to $\mathbb{R}$ defined by

$$\mathbb{E}^u(p) := \mathbb{E}_p(u) = \sum_{z \in Z} p(z)u(z).$$

The function $\mathbb{E}^u : \mathcal{L}(Z) \to \mathbb{R}$ is affine. Moreover, any affine function from $\mathcal{L}(Z)$ to $\mathbb{R}$ is of the form $\mathbb{E}^u$ for some utility function $u \in \mathbb{R}^Z$.

Proposition 4.1. Consider a preference $\succeq$ that admits an SOEU representation (μ, u, v). If $\succeq$ also satisfies non-degeneracy, then

(i) the probability measure μ is unique;

(ii) the function u is unique up to a positive affine transformation,[10]

(iii) the function $v \circ \mathbb{E}^u$ is unique up to a positive affine transformation.

Proof. Let $I : \mathcal{L}(\mathcal{H}) \to \mathbb{R}$ be the representation of $\succeq$ associated to (μ, u, v), i.e.,

$$I(P) = \sum_{f \in \mathcal{H}} P(f)U(f) \quad \text{where} \quad U(f) := \sum_{s \in S} \mu(s)v(E_{f(s)}(u)).$$

The function $U : \mathcal{H} \to \mathbb{R}$ is the restriction of I to degenerate lotteries. Denote by $I' : \mathcal{L}(\mathcal{H}) \to \mathbb{R}$ be the representation of $\succeq$ associated to (μ', u', v'), i.e.,

$$I'(P) = \sum_{f \in \mathcal{H}} P(f)U'(f) \quad \text{where} \quad U'(f) := \sum_{s \in S} \mu'(s)v'(E_{f(s)}(u')).$$

[10]Equivalently, the function $\mathbb{E}^u$ is unique up to a positive affine transformation.

Since $\succeq$ satisfies order, mixture continuity and first-stage independence, it follows from Theorem 5.15 in Kreps (1988) that there exists $a > 0$ and $b \in \mathbb{R}$ such that $U' = aU + b$. That is, for every act $f \in \mathcal{H}$,

$$\sum_{s \in S} \mu'(s) v'(\mathbb{E}^{u'}(f(s))) = b + a \sum_{s \in S} \mu(s) v(\mathbb{E}^{u}(f(s))). \tag{4.1}$$

Fix an arbitrary lottery $p \in \mathcal{L}(Z)$. Letting f be the constant act $p\mathbf{1}_S$, we deduce that

$$v'(\mathbb{E}^{u'}(p)) = av(\mathbb{E}^{u}(p)) + b. \tag{4.2}$$

We have thus proved that $v' \circ \mathbb{E}^{u'} = a\,(v \circ \mathbb{E}^{u}) + b$.

The preference $\succeq^{ca}$ is represented by the functions $p \mapsto \mathbb{E}_p(u)$ and $p \mapsto \mathbb{E}_p(u')$. Since $\succeq^{ca}$ satisfies order, mixture continuity and second-stage independence, it follows from Theorem 5.15 in Kreps (1988) that there exists $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u' = \alpha u + \beta$.

Combining (4.1) and (4.2), we deduce that for every act $f \in \mathcal{H}$,

$$\begin{aligned} \sum_{s \in S} \mu'(s) v'(\mathbb{E}^{u'}(f(s))) &= \sum_{s \in S} \mu(s) \left[av(\mathbb{E}^{u}(f(s))) + b \right] \\ &= \sum_{s \in S} \mu(s) v'(\mathbb{E}^{u'}(f(s))). \end{aligned}$$

Letting $\mathbb{U}' := \mathrm{co}(u'(Z))$, we have that

$$\sum_{s \in S} \mu'(s) v'(w(s)) = \sum_{s \in S} \mu(s) v'(w(s))$$

for any $(w(s))_{s \in S} \in [\mathbb{U}']^S$. Since $\succeq$ satisfies non-degeneracy, the interval $\mathbb{U}'$ cannot be reduced to a singleton. Since v' is strictly increasing, we deduce that the set

$$\{(v'(w(s)))_{s \in S} \;:\; (w(s))_{s \in S} \in [\mathbb{U}']^S\}$$

admits a non-empty interior in $\mathbb{R}^S$. Since $\mu - \mu'$ is orthogonal to the above set, we deduce that $\mu = \mu'$. $\square$

To characterize the support of the probability in the SOEU representation, we recall the standard definition of a null set.

Definition 4.1. A set $s \in S$ is null when for all acts $f, g \in \mathcal{H}$ and all lotteries $p, q \in \mathcal{L}(Z)$,

$$f \succeq g \iff p_s f \succeq q_s g.$$

In other words, the state s is null when it does not matter for the ranking of acts.

In a SOEU representation (μ, u, v) of a preference $\succeq$, the support of the probability μ corresponds to non-null states.

Proposition 4.2. Consider a preference $\succeq$ that admits an SOEU representation (μ, u, v). If $\succeq$ also satisfies non-degeneracy, then $\mu(s) = 0$ if, and only if, s is null.

Proof. The "only if" part is obvious. To prove the "if" part, we assume that s is null. Fix an act $f \in \mathcal{H}$ and two lotteries $p, q \in \mathcal{L}(Z)$. Since s is null, we have $p_s f \succeq q_s f$ if, and only if, $p_s f \succeq q_s f$. Using the SOEU representation (μ, u, v), we deduce that

$$\mu(s)v(\mathbb{E}_p(u)) \geq \mu(s)v(\mathbb{E}_q(u)) \iff \mu(s)v(\mathbb{E}_q(u)) \geq \mu(s)v(\mathbb{E}_p(u)).$$

Since $\succeq$ satisfies non-degeneracy, there exists $p, q \in \mathcal{L}(Z)$ such that $p \succ^{ca} q$, or, equivalently, $\mathbb{E}_p(u) > \mathbb{E}_q(u)$. We then deduce that $\mu(s) = 0$. $\square$

5 Continuity

In order to obtain continuity properties of the utility function, we consider a topology on Z. In this section, we assume that Z is a separable metric space. We endow the space $\mathcal{L}(Z)$ with the weak topology defined as the coarsest topology such that every linear mapping $p \mapsto \mathbb{E}_p(u)$ is continuous for every bounded and continuous function u. This implies that a sequence (p_n) of lotteries in $\mathcal{L}(Z)$ weakly converges to another lottery $p \in \mathcal{L}(Z)$ when

$$\lim_{n \to \infty} \sum_{z \in Z} p_n(z)u(z) = \sum_{z \in Z} p(z)u(z)$$

for every bounded and continuous function $u : Z \to \mathbb{R}$. The weak topology on $\mathcal{L}(Z)$ is metrizable and separable. The space $\mathcal{H}$ of acts is identified with the product space $\mathcal{L}(Z)^S$ and is therefore endowed with the product of weak topologies. This topology on $\mathcal{H}$ is also metrizable and separable. We then endow the choice space $\mathcal{L}(\mathcal{H})$ with the weak topology defined as the coarsest topology such that every linear mapping $P \mapsto \sum_{f\in\mathcal{H}} P(f)U(f)$ is continuous for every bounded and continuous function $U : \mathcal{H} \to \mathbb{R}$.

We introduce the following strengthening of the mixture continuity axiom.

Continuity. The preference $\succeq$ is continuous with respect to the weak topology.[11]

We can now characterize the continuity of the functions u and v in a SOEU representation.

Theorem 5.1. *Preference $\succeq$ on $\mathcal{L}(\mathcal{H})$ satisfies order, continuity, first-stage independence, second-stage independence, and extended monotonicity if, and only if, it has an SOEU representation (μ, u, v) where u is continuous and bounded and v is continuous.*

Proof. We start by proving the "if" part. We shall prove that the function $f \mapsto U(f)$ defined on $\mathcal{H}$ by

$$U(f) = \sum_{s\in S} \mu(s)v(\mathbb{E}_{f(s)}(u)) = \sum_{s\in S} \mu(s)(v \circ \mathbb{E}^u)(f(s))$$

is bounded and continuous. Since u is bounded and continuous, the linear functional $\mathbb{E}^u : \mathcal{L}(Z) \to \mathbb{R}$ is weakly continuous. Since v is continuous, the mapping $v \circ \mathbb{E}^u$ is weakly continuous and we get that U is continuous with respect to the product topology on $\mathcal{H}$. Boundedness of U follows from boundedness of u and continuity of v.

The "only if" part corresponds to the proof of Theorem 3.1 replacing the use Theorem 5.15 in Kreps (1988) by Theorem 5.21 in Kreps (1988). We then get that $U : \mathcal{H} \to \mathbb{R}$ is bounded and continuous with respect to the weak topology. Moreover, the weak topology on $\mathcal{L}(Z)$ coincides with the relative topology on $\mathcal{L}(Z)$ induced by $\mathcal{L}(\mathcal{H})$. This implies that the preference $\succeq^{ca}$ on constant acts is continuous with respect to the weak topology. We then get that u is bounded and continuous. Finally, to prove that v is continuous, recall that v is defined on the set $\text{co}(u(X))$ as follows

$$v(t) = U^{ca}(p) \quad \text{for any } p \in \mathcal{L}(Z) \text{ such that } t = \mathbb{E}_p(u),$$

where $U^{ca}(p) := U(p\mathbf{1}_S)$. Since $\mathcal{L}(Z)$ is connected and U^{ca} is continuous, the set $v(\text{co}(u(X))) = U^{ca}(\mathcal{L}(Z))$ is also connected. Since v is strictly increasing, we necessarily have that v is continuous. □

[11] That is, for every $P \in \mathcal{L}(\mathcal{H})$, the set $\{Q \in \mathcal{L}(\mathcal{H}) : Q \succeq P\}$ and $\{Q \in \mathcal{L}(\mathcal{H}) : P \succeq Q\}$ are closed for the weak topology.

6 Uncertainty Aversion

Our general representation results (Theorem 3.1 and Theorem 5.1) do not require any specific assumption regarding the decision maker's attitude towards subjective uncertainty. We consider the following standard axiom introduced by Schmeidler (1989) to represent the decision maker's aversion towards subjective uncertainty.

Uncertainty Aversion. Let $f, g \in \mathcal{H}$ and $\alpha \in (0,1)$.

$$f \sim^a g \Longrightarrow \alpha f + (1-\alpha) g \succeq^a f.$$

This axiom captures the plausible assumption that "smooting out" acts should provide a hedge against subjective uncertainty and consequently should be desirable. That is, we assume that a decision maker who finds two acts equally attractive, will be embarrassed to state that their mixture is worse than both.

Under the basic axioms and extended monotonicity, the axiom of uncertainty aversion translates into the concavity of the function v.

Theorem 6.1. *Preference $\succeq$ on $\mathcal{L}(\mathcal{H})$ satisfies order, continuity, first-stage independence, second-stage independence, extended monotonicity and uncertainty aversion if, and only if, it has an SOEU representation (μ, u, v) where u is bounded continuous and v is concave continuous.*

Proof. We omit the straightforward arguments of the "if" part and focus on the proof of the "only if" part. Let $\succeq$ be a preference on $\mathcal{L}(\mathcal{H})$ satisfying order, continuity, first-stage independence, second-stage independence, extended monotonicity and uncertainty aversion. It follows from Theorem 5.1 that $\succeq$ has an SOEU representation (μ, u, v) where u is bounded continuous and v is continuous. We only have to prove that v is also concave. If u is constant, then the domain of v is a singleton and v is obviously concave. We now assume that u is not constant. This implies that $\text{co}(u(Z))$ is an interval of $\mathbb{R}$ with a non-empty interior denoted by $\text{int}(\text{co}(u(X)))$.

Let $W := [\text{co}(u(Z))]^S$ and define $\Psi : W \to \mathbb{R}$ defined as follows: for every $w = (w(s))_{s \in S} \in W$,

$$\Psi(w) := \sum_{s \in S} \mu(s) v(w(s)).$$

Replacing S by the support of μ is necessary, we can assume without any loss of generality that $\mu(s) > 0$ for each $s \in S$.

Lemma 6.1. *The function Ψ is quasi-concave.*

Proof of Lemma 6.1. Fix $w, \tilde{w} \in W$ and assume, without any loss of generality, that $\Psi(w) \leq \Psi(\tilde{w})$. We analyze two cases.

If $w(s) = \inf \mathrm{co}(u(Z))$ for each s, then $\Psi(w) = \inf \Psi(W)$, and for any $\alpha \in [0, 1]$, we have $\Psi(\alpha w + (1 - \alpha)\tilde{w}) \geq \Psi(w)$.

Assume now that there exists $\underline{u} \in \mathrm{co}(u(Z))$ such that $\underline{w} := \underline{u}\mathbf{1}_S < w$.[12] Replacing $\underline{u}$ by $\min\{\min\{\underline{u}, \hat{w}(s)\} : s \in S\}$ if necessary, we can assume that $\underline{w} \leq \tilde{w}$. Since v is strictly increasing, we deduce that $\Psi(\underline{w}) < \Psi(w)$. Since Ψ is continuous, we can apply the Intermediate Value Theorem to get the existence of some $\gamma \in [0, 1]$ close enough to 1 such that

$$\Psi(\hat{w}) = \Psi(w) \quad \text{where } \hat{w} := \gamma\underline{w} + (1 - \gamma)\tilde{w}.$$

Since $w(s) \in \mathrm{co}(u(Z))$, there exists $f(s) \in \mathcal{L}(Z)$ such that $w(s) = \mathbb{E}_{f(s)}(u)$. Similarly, there exists an act $g \in \mathcal{H}$ such that $\hat{w}(s) = \mathbb{E}_{g(s)}(u)$. Observe that $f \sim^a g$. Uncertainty aversion implies that $\alpha f + (1 - \alpha)g \succeq^a f$. In particular, we get that

$$\Psi(\alpha w + (1 - \alpha)\hat{w}) \geq \Psi(w).$$

Since $\underline{w} \leq \tilde{w}$, we deduce that $\alpha w + (1 - \alpha)\hat{w} \leq \alpha w + (1 - \alpha)\tilde{w}$. Monotonicity of Ψ implies the desired result: $\Psi(\alpha w + (1 - \alpha)\tilde{w}) \geq \Psi(w)$. □

The restriction of the function Ψ to the open set $[\mathrm{int}(\mathrm{co}(u(X)))]^S$ satisfies the assumptions of Theorem 2 in Debreu and Koopmans (1982) and deduce that with at most one exception, every function $\mu(s)v$ is concave on $\mathrm{int}(\mathrm{co}(u(X)))$. This implies that v is concave on the interior of the interval $\mathrm{co}(u(X))$. Since we already proved that v is concave on the whole interval $\mathrm{co}(u(X))$, we can follow a standard limiting argument to deduce that v is concave on the whole interval $\mathrm{co}(u(X))$. □

Grant et al. (2009) also provide an axiomatization of SOEU preferences with a concave function v. In Theorem 6.1, we obtain concavity of v under the mere assumption of uncertainty aversion, while Grant et al. (2009) impose an additional axiom, called weak translation

[12] That is, $\underline{u} \leq w(s)$ for each s with a strict inequality for at least one s.

invariance at certainty, which requires that the supporting hyperplanes are equal along the certainty line whenever they are unique.

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