University of California, Davis
Department of Economics
Microeconomics

Date: June 21, 2022
Time: 5 hours
Reading Time: 20 minutes

PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

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Please answer **all three** questions

QUESTION 1

Consider the following game. A referee picks a number from the set {1, 2, …, *n*} with equal probability (that is, each number is chosen with probability $\frac{1}{n}$). She writes the number on a piece of paper and puts it in an envelope. She then repeats the operation: picks a number (possibly the same as before) with equal probability, writes it down and puts it in another envelope. She then gives the first envelope to Player 1 and the second envelope to Player 2. Each player opens the envelope given to her and looks at her own number. Neither player gets to look at the content of the envelope given to the other player. Then the two players must *simultaneously* and *independently* choose one of two actions: "Exit" or "Play". A player who chooses "Exit" gets \$10, no matter what the other player chooses. A player who chooses "Play" when the other chooses "Exit" gets \$*C*. If both players choose "Play" then they show their numbers and the one with the higher number gets \$100 while the other gets nothing; if they have the same number then both get nothing. Assume that each player is selfish and greedy (i.e. cares only about her own wealth and prefers more money to less) and is risk neutral.

(a) Consider first the case where $n = 2$ and $C = 0$.
(a.1) Draw an extensive-form game to represent this situation, making Player 1 move before Player 2.
(a.2) Write the corresponding strategic-form game.
(a.3) Does player 1 have any dominated strategies? If Yes, state which strategies and whether it is weak or strict dominance.
(a.4) Find all the Nash equilibria of this game.

(b) Continuing to assume that $C = 0$, with the help of the intuition you obtained from case **(a)**, find a Nash equilibrium of the game for any integer $n \geq 2$. Explain why that is a Nash equilibrium.

(c) Let $n = 2$ and consider now the case where $C > 0$. For what value of *C* is it sequentially rational for Player 2 to choose a completely mixed behavioral strategy at each of her information sets if she thinks that Player 1 is employing the pure strategy "If given a 2, Play, otherwise Exit"?

(d) Continuing to assume that $n = 2$, let *C* be equal to the value you found in part **(c)**. Find all the weak sequential equilibria where Player 1 chooses the pure strategy described in part (c) and Player 2 chooses a completely mixed behavioral strategy at each of her information sets.

(e) Now consider the case where $n = 4$. Define a state as a pair (x,y) where *x* is the number given to Player 1 and *y* is the number given to Player 2 (thus $x \in \{1,2,3,4\}$ and $y \in \{1,2,3,4\}$).
(e.1) Represent, by means of partitions, the states of knowledge of the players when each has looked at (and only at) her own number.
(e.2) Find the common knowledge partition.
(e.3) Consider the state (3,2). Find the smallest event *E* such that, at (3,2), both players know *E,* but neither player knows that the other player knows *E*.

Question 2

I always wondered why some people buy ugly but expensive handbags. It is as if the price, as a special characteristic of the commodity, contributes to the utility of the commodity.

Consider a utility function of the form

$$u(\boldsymbol{x}, \boldsymbol{p}) \quad = \quad \prod_{\ell=1}^{L} x_\ell^{\sqrt{p_\ell}}. \tag{1}$$

where $\boldsymbol{x} = (x_1, ..., x_L) \in \mathbb{R}_+^L$ is the consumption vector and $\boldsymbol{p} = (p_1, ..., p_L) >> 0$ is the price vector. This utility function resembles the Cobb-Douglas utility function except that its parameters are the prices.

Denote the consumer's wealth by w and assume $w > 0$. Let's consider the problem of maximizing this utility function subject to the budget constraint, i.e.,

$$\max_{\boldsymbol{x} \in \mathbb{R}^L} u(\boldsymbol{x}, \boldsymbol{p}) \quad = \quad \prod_{\ell=1}^{L} x_\ell^{\sqrt{p_\ell}} \tag{2}$$

subject to

$$\boldsymbol{p} \cdot \boldsymbol{x} \quad \leq \quad w \tag{3}$$

$$\boldsymbol{x} \quad \geq \quad \boldsymbol{0} \tag{4}$$

a. Use the Lagrangian or the Kuhn-Tucker-Lagrangian approach to solve the constrained maximization problem. Derive step-by-step the Walrasian demand functions.

b. Show that Walrasian demand functions are homogeneous of degree zero.

c. State the indirect utility function and simplify as much as you can.

d. Check whether the indirect utility function is homogeneous of degree zero.

e. We all noticed that inflation is back. For the sake of concreteness, consider inflation as a proportional rise of all prices and wealth. Would consumers with such kind of utility functions be happy about inflation?

f. Now consider the expenditure minimization problem

$$\min_{\boldsymbol{x} \in \mathbb{R}^L} \boldsymbol{p} \cdot \boldsymbol{x} \tag{5}$$

subject to

$$u(\boldsymbol{x}, \boldsymbol{p}) \quad = \quad \prod_{\ell=1}^{L} x_\ell^{\sqrt{p_\ell}} \geq \bar{u} \tag{6}$$

$$\boldsymbol{x} \quad \geq \quad \boldsymbol{0} \tag{7}$$

for $\bar{u} > 0$. Solve step-by-step for the Hicksian demand functions.

g. Is the Hicksian demand function homogeneous of degree zero in prices? Explain.

h. Write down the expenditure function.

i. Is it homogeneous of degree one in prices? Explain.

Question 3: Existence of Equilibrium

The existence proof that we saw in class required a complicated definition of the Walrasian auctioneer for boundary prices, remember? In this exercise, you are going to give an alternative proof where, instead of *one complicated* auctioneer, we consider a *sequence of simpler* ones.

Consider an exchange economy $\{\mathcal{I}, \mathrm{u}, \mathrm{w}\}$ where all the individuals' utility functions $u^i : \mathbb{R}^L_+ \to \mathbb{R}$ are continuous, strictly monotone and strictly quasi-concave, and all endowments $w^i \in \mathbb{R}^L_{++}$. Let $\Delta = \{p \in \mathbb{R}^L_+ \mid \sum_\ell p_\ell = 1\}$ denote the simplex of prices and $\Delta^\circ = \Delta \cap \mathbb{R}^L_{++}$ its interior. Note that the excess demand function of individual i can be written as

$$z^i(p) = \arg\max_{z \in \mathbb{R}^L} \left\{ u^i(w^i + z) : w^i + z \geq 0 \text{ and } p \cdot z = 0 \right\}$$

for any $p \in \Delta^\circ$. On the same domain, the aggregate excess demand function is $Z(p) = \sum_i z^i(p)$. *(So far, everything is as in class.)*

Define the *truncated* individual excess demand functions $z^{i,n} : \Delta \to \mathbb{R}^L$, for each $n \in \mathbb{N}$ and each i, by

$$z^{i,n}(p) = \arg\max_{z \in \mathbb{R}^L} \left\{ u^i(w^i + z) : w^i + z \geq 0, p \cdot z = 0 \text{ and } \forall \ell, |z_\ell| \leq n \right\}. \tag{1}$$

The n-th *truncated* aggregate excess demand is $Z^n(p) = \sum_i z^{i,n}(p)$.

Also, for each $n \in \mathbb{N}$ define the "hypercube"

$$\mathbb{C}^n = [-n \cdot I, n \cdot I]^L = \left\{ z \in \mathbb{R}^L \mid \forall \ell, |z_\ell| \leq n \cdot I \right\}$$

and the n-th *auctioneer* $P^n : \mathbb{C}^n \twoheadrightarrow \Delta$ by

$$P^n(z) = \arg\max_p \{p \cdot z : p \in \Delta\}.$$

Finally, consider the correspondence $\Gamma^n : \Delta \times \mathbb{C}^n \twoheadrightarrow \mathbb{R}^L \times \mathbb{R}^L$ defined by

$$\Gamma(p, z) = P^n(z) \times \{Z^n(p)\}.$$

Before proceeding, you can take for granted that

1. All the individual truncated excess demand functions are well defined[1] and continuous. Each truncated aggregate excess demand function Z^n is well defined and continuous.

2. Each truncated aggregate excess demand Z^n maps Δ into $\mathbb{C}^n$.

3. Each auctioneer $P^n : \mathbb{C}^n \twoheadrightarrow \Delta$ is non-empty-, compact- and convex-valued, and upper hemicontinuous.

4. Each correspondence $\Gamma^n : \Delta \times \mathbb{C}^n \twoheadrightarrow \mathbb{R}^L \times \mathbb{R}^L$ is non-empty, compact- and convex-valued, maps $\Delta \times \mathbb{C}^n$ into itself, and is upper hemicontinuous.

The following steps will show that $Z(\bar{p}) = 0$ for some $\bar{p} \in \Delta^\circ$:

(a) Argue that for each n, there exists $(\bar{p}^n, \bar{z}^n) \in \Delta \times \mathbb{C}^n$ such that $(\bar{p}^n, \bar{z}^n) \in \Gamma(\bar{p}^n, \bar{z}^n)$.[2]

(b) Taking $(\bar{p}^n, \bar{z}^n)$ from part (a), argue that for all n, $\bar{p}^n \cdot \bar{z}^n = 0$ and $\bar{z}^n \leq 0$.[3]

(c) For simplicity, denote $\bar{z}^{i,n} = z^{i,n}(\bar{p}^n)$.[4] Argue that $\bar{z}^{i,n} \geq -w^i$, and hence that $\bar{z}^n \geq -\sum_i w^i$ and

$$\bar{z}^{i,n} \leq -\textstyle\sum_{j \neq i} \bar{z}^{j,n} \leq \sum_{j \neq i} w^j.$$

(d) Argue that the sequence $(\bar{p}^n, \bar{z}^n, \bar{z}^{1,n}, \ldots, \bar{z}^{I,n})$ has a convergent subsequence.

(e) Suppose, for simplicity of notation, that the sequence $(\bar{p}^n, \bar{z}^n, \bar{z}^{1,n}, \ldots, \bar{z}^{I,n})$ itself converges to $(\bar{p}, \bar{z}, \bar{z}^1, \ldots, \bar{z}^I)$. Argue that $\bar{p} \in \Delta$, $\bar{p} \cdot \bar{z} = 0$ and $\bar{z} \leq 0$.

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[1] That is, that for each $p \in \Delta$ there exists one and only one z that satisfies the definition of the function. Critically, note that we are *not* excluding boundary prices from the domain of these functions.

[2] Do you remember Kakutani's fixed point theorem? If not, no worries: look at the appendix.

[3] *Hint:* Suppose by way of contradiction that $\bar{z}^n_1 > 0$. Then, perturb the first price in $\bar{p}^n$ to construct some $p \in \Delta$ for which $p \cdot \bar{z}^n > 0 = \bar{p}^n \cdot \bar{z}^n$.

[4] Note that this implies that $\sum_i \bar{z}^{i,n} = \bar{z}^n$.

(f) Argue that there exists $n^* \in \mathbb{N}$ such that for all $n \geq n^*$, all ℓ and all i, $|\bar{z}^{i,n}_\ell| < n$.

(g) Argue that for all $n \geq n^*$ and all i,

$$\bar{z}^{i,n} = \arg\max_{z \in \mathbb{R}^L} \left\{ u^i(w^i + z) : w^i + z \geq 0 \text{ and } \bar{p}^n \cdot z = 0 \right\},$$

where n^* comes from part (f).[5]

(h) Argue that, moreover, for all $n \geq n^*$, $\bar{z}^n = Z(\bar{p}^n)$ and hence $\bar{z} = Z(\bar{p}) \leq 0$.

(i) Taking for granted that $\bar{p} \in \Delta^\circ$, argue that $\bar{z} = Z(\bar{p}) = 0$.[6]

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[5] *Hint:* To develop intuition look at the figure in the appendix, and suppose that

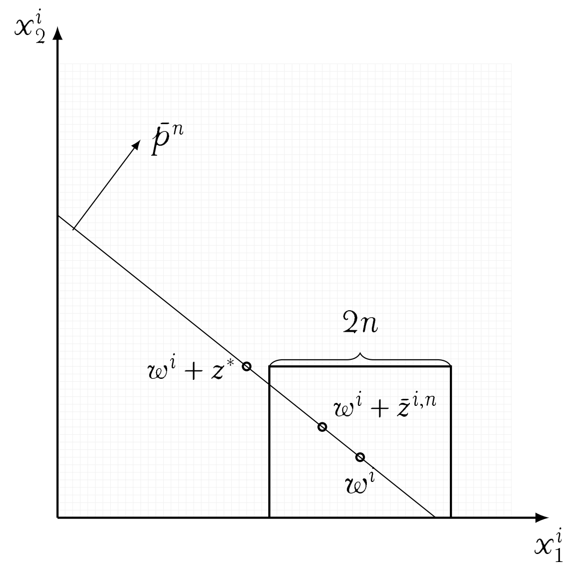
$$u^i(w^i + z^*) > u^i(w^i + \bar{z}^{i,n}).$$

Using strict quasi-concavity of u^i, argue that $\bar{z}^{i,n}$ cannot be optimal in its corresponding truncated problem.

[6] *Hint:* Suppose not: $\bar{z} < 0$. Then, define $z = \bar{z}^1 - \bar{z} > \bar{z}^1$ and argue that $u^1(w^1 + z) > u^1(w^1 + \bar{z}^1)$ and that this is impossible.

KAKUTANI'S FIXED POINT THEOREM: Let $X \subseteq \mathbb{R}^K$ be non-empty and let $\Gamma : X \twoheadrightarrow X$ be a non-empty and compact-valued correspondence. If X is compact and convex, and if Γ is convex-valued and upper hemicontinuous, then there exists $x \in X$ such that $x \in \Gamma(x)$.

Hint for part (g): In the following graph, suppose that $u^i(w^i + z^*) > u^i(w^i + \bar{z}^{i,n})$:



Using strict quasi-concavity of u^i, argue that $\bar{z}^{i,n}$ cannot be optimal in its corresponding truncated problem.