

ADVANCED ECONOMIC THEORY

PRELIMINARY EXAMINATION FOR THE Ph. D. DEGREE

Answer **all** questions

1. Consider a game in which player 1 first chooses between *In* and *Out*. If player 1 selects *Out* then the game ends with payoff of x for player 1 and payoff of 1 for player 2, with $0 < x < 3$. If player 1 selects *In* then this selection is revealed to player 2 and then the players play a game where they simultaneously and independently choose between *A* and *B*. If they both choose *A* then player 1 gets a payoff of 3 and player 2 a payoff of 1. If they both choose *B* then player 1 gets a payoff of 1 and player 2 a payoff of 3. If they make different choices then they both get a payoff of 0.
- (a) Represent this game in extensive form.
- (b) Write the corresponding normal form. Find the pure-strategy Nash equilibria and note how they depend on x .
- (c) Find the pure-strategy subgame-perfect equilibria. [Note that they depend on x .]
- (d) For the case where $1 < x < 3$, find a pure-strategy Nash equilibrium which is not subgame perfect.
- (e) What are the mixed-strategy subgame-perfect equilibria of this game when $x = \frac{1}{2}$? [Your answer should be in terms of behavioral strategies.]

2. Dan and his three daughters Elise, Andrea and Theresa are in the same room. Dan gives a sealed envelope to Elise and tells her (in a loud voice, so that everybody can hear) “in this envelope I put a sum of money; I don’t remember how much I put in it, but I know for sure that it was either \$4 or \$8 or \$12. Now, my dear Elise, go to your room by yourself and open it. Divide the money into two equal amounts, put the two sums in two different envelopes, seal them and give one to Andrea and one to Theresa”. Unbeknownst to her sisters, Elise likes one of them more than the other and decides to disobey her father: after dividing the sum into two equal parts, she takes \$1 from one envelope and puts it in the other envelope. She then gives the envelope with more money to her favorite sister and the envelope with the smaller amount to the other sister. Andrea and Theresa go to their respective rooms and privately open their envelopes, to discover, to their surprise, an odd number of dollars. So they realize that Elise did not follow their father’s instructions. Neither Andrea nor Theresa suspect that Elise kept some money for herself. In fact, it is common knowledge between them that Elise simply rearranged the money, without taking any for herself. Of course, **neither Andrea nor Theresa know in principle how much money Elise took from one envelope** (although in some cases they might be able to figure it out).

- (a) Represent the state of knowledge between Andrea and Theresa using a set of states and information partitions.
- (b) Let E be the event that Andrea is Elise’s favorite sister. Find the events $K_A E$ (the event that Andrea knows it), $K_T E$ (the event that Theresa knows it), $K_A K_T E$ and $K_T K_A E$.
- (c) Find the common knowledge partition.
- (d) At what states is it common knowledge between Andrea and Theresa that Elise’s favorite sister is Andrea?
- (e) The night before Dan was looking through his digital crystal ball and saw what Elise was planning to do. However the Microsoft PredictorTM software was not working properly (the screen kept freezing) and he could not tell whether Elise was trying to favor Andrea or Theresa. He knew he couldn’t stop Elise and wondered what he could do to protect the mistreated sister (whether it be Andrea or Theresa), by making sure that she would not know that she had been mistreated. So he asked the digital mirror on the wall “mirror, mirror on the wall, how much money should I put in the envelope?”. The mirror did not answer because “envelope” does not rhyme with “wall”. So, you should answer (riddles are not acceptable: you need to fully explain your answer).

3. Consider a finance (one good) exchange economy with two periods ($t = 0, 1$) and uncertainty at date 1 represented by S possible states of nature. Suppose that there are J securities traded at date 0: security j has price q_j at date 0 and payoff $V^j \in \mathbb{R}^S$ at date 1. Let $q = (q_1, \dots, q_J)$ the vector of security prices and V the $J \times S$ matrix of security payoffs.

- Explain what it means for the vector q to be a no-arbitrage price vector.
- Prove the following proposition: the price vector q is a no-arbitrage price vector if and only if there exist a vector $\pi \in \mathbb{R}_{++}^S$ such that $q = \pi V$. Give the economic interpretation of this proposition.
- Suppose that $S = 3$, that the first security has payoff $V^1 = (30, 20, 10)$ and price $q_1 = 20$, that the second security has payoff $V^2 = (1, 1, 1)$ and price $q_2 = 0.8$, and that the third security is a call option on security 1 with striking price 15, i.e. its payoff is the difference between the payoff of security 1 and 15 if this payoff is larger than 15, and zero otherwise. Find the necessary and sufficient conditions that the price q must satisfy in order that there is no arbitrage opportunity.

4. Consider a stochastic economy with I agents which lasts for 3 periods, $t = 0, 1, 2$. The uncertainty is represented by an event tree \mathcal{D} . There are J long-lived securities issued at date 0 which pay dividends at each date-event after date 0. Let $D^j(\xi)$ denote the dividend of security j at node ξ , for $t(\xi) = 1, 2$. Financial markets are incomplete ($J < b(\xi)$ for at least one node ξ , where $b(\xi)$ is the branching number).

- Give the definition of a Constrained Feasible allocation and of a Constrained Pareto Optimal allocation (CPO) and explain in words what it means (remember that there are 3 periods.)
- Let $(\bar{x}, \bar{z}, \bar{q})$ be an equilibrium of this economy. A planner considers marginal changes $(dz_0^i)_{i \in I}$ in the agents' portfolios at date 0, accompanied by transfers of income $(dt^i)_{i \in I}$ at the same period. The planner will not intervene after date 0, but he knows that the date 0 changes will affect the prices at date 1. Compute the effect of the planner's intervention on social welfare (i.e. calculate $\sum_i du^i / \bar{\lambda}_0^i$ where $\bar{\lambda}_0^i$ is agent i 's marginal utility of income) and explain why, in general, the planner can increase the social welfare.
- Suppose the I agents have expected, discounted quadratic utilities of the form

$$u^i(x^i) = -\frac{1}{2} E \left(\sum_{t=0}^2 \delta^t (x_t^i - \alpha^i)^2 \right) = -\frac{1}{2} \sum_{\xi \in \mathcal{D}} \delta^{t(\xi)} \rho(\xi) (x^i(\xi) - \alpha^i)^2$$

Compute the equilibrium prices as a function of the aggregate output $w = \sum \omega^i$ and of the parameters $\delta, (\alpha^i)_{i=1}^I$, of the preferences. What is unusual in this example?

- Using the result of question (b), show that the equilibrium of an economy with quadratic utilities as in (c) is CPO. If you were asked to do a proof that generically the equilibrium of a T -period economy with $T > 2$ is not CPO, how would you parameterize the family of economies that you consider?