University of California, Davis
Department of Economics
Microeconomic Theory

Date: August 25, 2011
Time: 5 hours
Reading Time: 20 minutes

PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

Please answer four questions (out of five)

QUESTION 1. THE MARGINAL UTILITY OF WEALTH

Let $u:\Re^L_{++}\to\Re$ be a differentiable, strictly concave, locally nonsatiated utility function. We restrict ourselves in what follows to a price-wealth domain P of vectors $(p,w)\in\Re^{L+1}_{++}$ for which the UMAX[p, w] problem has a solution.

1.1. Briefly argue that the solution to the UMAX[p, w] problem, denoted $\tilde{x}(p,w)$ is unique and defined by the first order equalities.

1.2. Verbally define the marginal utility of wealth, to be denoted $\tilde{\lambda}(p,w)$. State and prove its relation to the Lagrange multiplier.

1.3. We now partition the set $\{1,\ldots,L\}$ of goods into the three subsets: $\{1,\ldots,L_1\}$, $\{L_1+1,\ldots,L_2\}$, and $\{L_2+1,\ldots,L\}$. A vector $x=(x_1,\ldots,x_L)$ is similarly partitioned into the three subvectors $x^1\equiv(x_1,\ldots,x_{L_1}), x^2\equiv(x_{L_1+1},\ldots,x_{L_2})$ and $x^3\equiv(x_{L_2+1},\ldots,x_L)$, $x=(x^1;x^2;x^3)$. The price vector p is similarly partitioned into $p^1\equiv(p_1,\ldots,p_{L_1}), p^2\equiv(p_{L_1+1},\ldots,p_{L_2})$ and $p^3\equiv(p_{L_2+1},\ldots,p_L)$, $p=(p^1;p^2;p^3)$. We assume that the utility function u is separable in the form

$$u(x)=u^1(x^1)+u^2(x^2)+u^3(x^3),$$

(for $J = 1, 2, 3$, u^J is differentiable, strictly concave and locally nonsatiated) and adopt the following interpretation: our household, comprised of a husband and a wife, consumes three types of goods: x^1 is a basket of goods consumed in Vacation Island 1, x^2 is a basket of goods consumed in Vacation Island 2, and x^3 is the basket of all other goods consumed by the household. Our household has solved the UMAX[p, w] problem at some point in the past, but they have forgotten some details.

1.3(a). Before engaging in their two-island vacation, the husband says: all we have to remember are our planned expenditures in Island 1, w^1, and in Island 2, w^2. Then when we are in Island 1, we observe its prices p^1 and solve

$$\max u^1(x^1) \text{ subject to } p^1 \bullet x^1 \le w^1,$$

with solution denoted $\tilde{\tilde{x}}^1(p^1, w^1)$, and when we are in Island 2, we observe its prices p^2 and solve

$$\max u^2(x^2) \text{ subject to } p^2 \bullet x^2 \le w^2,$$

with solution denoted $\tilde{\tilde{x}}^2(p^2, w^2)$. Then, the husband claims, our solutions will coincide with what we had originally planned, i. e., $\tilde{\tilde{x}}^J(p^J, w^J) = \tilde{x}^J(p, w)$, J = 1, 2.

Is the husband right? Argue your answer.

1.3(b). The wife interjects: what about just remembering the marginal utility of wealth λ? Then when we are in Island 1 we unconstrainedly solve

$$\max_{x^1} (u^1(x^1) - \lambda p^1 \bullet x^1),$$

with solution $\hat{x}^1(p^1, \lambda)$, and when we are in Island 2 we unconstrainedly solve

$$\max_{x^2} (u^2(x^2) - \lambda p^2 \bullet x^2),$$

with solution $\hat{x}^2(p^2, \lambda)$, and, she claims, we shall also get $\hat{x}^J(p^J, \lambda) = \tilde{x}^J(p, w)$, J = 1, 2.

Is the wife right? Argue your answer. Verbally interpret the maximization problems proposed by the wife.

1.4. Let there be a large number of islands. Do the arguments of husband and wife in 1.3 carry over? Compare the two approaches.

1.5. Check the husband's and wife's approaches for Island 1 in the following example.

$$u: \Re^L_{++} \to \Re : u(x) = \sum_{i=1}^{L} \alpha_i \ln x_i, \sum_{i=1}^{L} \alpha_i = 1, \alpha_i > 0, \forall i, L_1 = 2.$$

QUESTION 2. MONOPOLY REGULATION

Let there be L goods, and write $x_{-L} = (x_1, ..., x_{L-1})$. Consider a consumer with utility function

$$u : \Re_+^{L-1} \times \Re : u(x_1, ..., x_{L-1}, x_L) = \sum_{j=1}^{L-1} u_j(x_j) + x_L ,$$

where, for j = 1,…, L-1, u_j is strictly concave with u_j (0) = 0, strictly increasing and twice differentiable, and satisfies the condition $\lim_{x_j \to 0} u_j\,'(x_j) = \infty$. In what follows, we set p_L = 1 and restrict our attention to strictly positive prices $p_j > 0$, j = 1,…, L-1.

2. 1. Show that the Walrasian demand for good j, j = 1,…, L-1, can be written $\hat{x}_j(p_j)$, and inverted, with inverse denoted $\hat{p}_j(x_j)$.

A firm supplies goods 1,…, L-1 to our consumer. Its cost function is

$$C(x_1, ..., x_{L-1}) = \begin{cases} 0, & \text{if} \quad x_j = 0,\ j = 1, ..., L-1 \\ F + \sum_{j=1}^{L-1} c_j x_j , & \text{otherwise} \end{cases} ,$$

where $F > 0$ and $c_j > 0$, j = 1,…, L-1.

Define: $S : \Re_{++}^{L-1} \to \Re : S(x_{-L}) = \sum_{j=1}^{L-1} u_j(x_j) - C(x_{-L})$,

$\Pi : \Re_{++}^{L-1} \to \Re : \Pi(x_{-L}) = \sum_{j=1}^{L-1} [\hat{p}_j(x_j) - c_j] x_j - F$.

2.2. Interpret S and Π in words.

We consider the following optimization problems, and postulate that any solution to any of them is strictly positive.

<u>*Problem*</u> *SU* (unconstrained) $\text{Max}_{x_{-L}} S(x_{-L})$.

<u>*Problem*</u> *SC*[$\overline{\Pi}$] (constrained) $\text{Max}_{x_{-L}} S(x_{-L})$ subject to $\Pi(x_{-L}) \geq \overline{\Pi}$.

<u>*Problem*</u> *MC*[$\overline{S}$] (constrained) $\text{Max}_{x_{-L}} \Pi(x_{-L})$ subject to $S(x_{-L}) \geq \overline{S}$.

<u>*Problem*</u> *MU* (unconstrained) $\text{Max}_{x_{-L}} \Pi(x_{-L})$.

2.3. Write the first-order conditions of Problem *SU*, express them in the format of the Lerner equation (i. e., MARKUP = DEGREE OF MONOPOLY), and verbally interpret. What can be said about the markups, and about the level of profits, at a solution to Problem *SU*?

2.4. For each of the Problems *SU*, *SC*, *MC* and *MU*, verbally interpret the problem and the agent who may face it.

2.5. Write the first-order conditions of Problem *MU*, express them in the format of the Lerner equation. How are the markups for two different goods related to each other?

2.6. For each of the Problems *SC* and *MC*, write its first-order conditions, and express them in the format of the Lerner equation. Compare across the four problems, and interpret.

2.7. Assume now that the functions u_j are quadratic, of the form:

$u_j : [0, a_j / b_j) : u_j(x_j) = a_j x_j - (1/2) b_j (x_j)^2$. Prove that Problems *SC* and *MC* are "dual" in the following sense:

(i) If $\bar{x}_{-L}$ solves Problem $SC[\bar{\Pi}]$, and $\bar{S}$ is the value of Problem $SC[\bar{\Pi}]$, then $\bar{x}_{-L}$ solves Problem $MC[\bar{S}]$.

(ii) If $\bar{\bar{x}}_{-L}$ solves Problem $MC[\bar{S}]$ and $\bar{\bar{\Pi}}$ is the value of Problem $MC[\bar{S}]$, then $\bar{\bar{x}}_{-L}$ solves Problem $SC[\bar{\bar{\Pi}}]$.

2.8. Consider the special case with $L = 2$ and $u(x_1, x_2) = ax_1 - (1/2)(x_1)^2 + x_2$. Compute and graph the functions S and Π in the same figure, where you are also asked to illustrate the solutions to problems *SU* and *MU*, as well as the "duality theorem" of the previous section.

QUESTION 3 Financing a public good

Consider an economy with a private good and a public good. The economy consists of two types of consumers with utility functions

$$u_1(x_1, y) = \frac{1}{2}\log(x_1) + \frac{1}{2}\log(y), \qquad u_2(x_2, y) = \frac{1}{3}\log(x_2) + \frac{2}{3}\log(y),$$

where y is the amount of public good available to all agents, and x_i ($i = 1, 2$) is the amount of private good consumed by a consumer of type i.

There are n_1 consumers of type 1 with utility function u_1, each endowed with 4 units of private good. There are n_2 consumers of type 2 with utility function u_2, each endowed with 1 unit of private good. Let $w = 4n_1 + n_2$ denote the aggregate endowment of private good for the economy. The public good can be produced from the private good according to the linear technology $y = z$ where $z \geq 0$ is the number of units of private good used as input. Consider only allocations in which all consumers of the same type consume the same bundle of goods.

3(a) Write a (parameterized) constrained maximum problem whose solutions are the Pareto optimal allocations of the economy and write the associated first-order conditions (do not eliminate the multipliers).

3(b) Find the set of Pareto optimal allocations (it will be convenient to parameterize them either by the multiplier associated to the constraint that agents of type 2 have a guaranteed utility level or by the relative weight of the agents of type 2 in the social welfare function). Explain analytically and intuitively how the level of public good varies with the parameter. Find the interval in which the Pareto optimal levels of public good lie.

3(c) Suppose that $n_2 > n_1$ and the public good is financed by taxing agents' endowments of private good, at the tax rate $\tau \in (0, 1)$. The level of public good (or equivalently the tax rate) is determined by majority voting. Find the voting equilibrium. Show that despite the fact that the level of public good is at the limit of the interval found in (b), the voting equilibrium is not Pareto optimal. Analyze the source of the inefficiency.

QUESTION 4

Consider the following four-player game. The players are Bob, Connor, Donna and Emily. Bob and Connor are identical twins; Donna can tell them apart while Emily is not able to tell which one is Bob and which one is Connor. The interaction is going to be among Donna, Emily and one of the twins, whom we will call Player 2. First a Referee rolls a die. If the die shows the face 1 then the referee privately informs Bob that he is going to be Player 2, while if the die shows either a 2 or a 3 or a 4 or a 5 or a 6 then the referee privately informs Connor that he is going to be Player 2. The twin who does not get to play does nothing and receives a payoff of zero, no matter what the other players do. Note that, **while Donna is always able to tell whether Player 2 is Bob or Connor, Emily cannot tell.** The way in which the twin who is to act as Player 2 is chosen is common knowledge among everybody, as are the following details of the game to be played.

The game starts at date 1 with Donna (who is playing face-to-face with Player 2) choosing between *A* (for avoidance) and *C* (for challenge). Player 2 sees Donna's choice and if that choice was *A* then he does nothing, while if that choice was *C* then he chooses between *S* (for soft) and *T* (for tough). The outcomes are w_1 if *A*, w_2 if (*C*,*S*) and w_3 if (*C*,*T*).

At date 2 Emily is informed of the choices made at date 1 by Donna and Player 2 and plays (face-to-face with Player 2) the same game: she chooses between *A* and *C*; Player 2 sees Emily's choice and if that choice was *A* then he does nothing, while if that choice was *C* then he chooses between *S* (for soft) and *T* (for tough); the outcomes are, as before, w_1 if *A*, w_2 if (*C*,*S*) and w_3 if (*C*,*T*).

Payoffs are "collected" at the end of date 2 (that is, after the interaction between Emily and Player 2) and there is no discounting (that is, the discount factor is 1 for every player). **For Player 2 the payoff from the two interactions is equal to the sum of the payoffs from the individual interactions.**
It is common knowledge among everybody that Bob, Connor, Donna and Emily have the following von Neumann-Morgenstern preferences.

Donna and Emily have the same preferences: $w_2 \succ w_1 \succ w_3$ (where $\succ$ denotes strict preference and $\sim$ denotes indifference) and $w_1 \sim \begin{pmatrix} w_2 & w_3 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$.

Bob has the following preferences: $w_1 \succ w_3 \succ w_2$ and $w_3 \sim \begin{pmatrix} w_1 & w_2 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$.

Connor has the following preferences: $w_1 \succ w_2 \succ w_3$ and $w_2 \sim \begin{pmatrix} w_1 & w_3 \\ \frac{3}{10} & \frac{7}{10} \end{pmatrix}$.

4(a). Draw an extensive-form game that represents the interaction described above and explain how you obtained the payoffs [hint.: it is best to compute the normalized payoffs].

4(b). For each of Bob, Connor, Donna and Emily specify how many strategies that player has and describe in words one possible strategy.

4(c). Choose one strategy profile and write the payoffs for each of the four players, that is, fill in one of the cells of the strategic form associated with this extensive game.

4(d). Find two pure-strategy subgame-perfect equilibria which are also weak sequential equilibria. They must be non-trivially different, in the sense that the outcome is different. Give enough details to show that they are weak sequential equilibria.

4(e). If also Emily could tell the twins apart, what would be the backward induction solution of this game?

4(f). Explain how the game of part 4(a) could be interpreted as representing a situation of incomplete information, by providing a set of states, information partitions and the association of a game with every state (give the details of such games).

QUESTION 5

Consider a market for loans to finance investment projects. All investment projects require an outlay of $\$X$. There are 2 types of projects: good and bad. A bad project has probability p_B of yielding a revenue of $\$R$ and a probability $(1-p_B)$ of yielding zero revenue. A good project yields $\$R$ with probability p_G and zero with probability $(1-p_G)$, with $0 \leq p_B < p_G \leq 1$. Assume that $\boldsymbol{R > X}$. Each entrepreneur has one project, and the fraction of entrepreneurs with **bad** projects is λ, with $0 < \lambda < 1$.

Entrepreneurs go to banks to borrow the cash to make the initial outlay (assume for now that they must borrow the entire amount). A **loan contract** specifies the total amount w that is supposed to be repaid to the bank, where $w \leq R$. Entrepreneurs and banks are *risk neutral*. Each entrepreneur knows the type of his project, but the banks do not. *In the event that a project yields zero revenue, the entrepreneur defaults on her loan contract, and the bank receives nothing.* Let r be the risk-free rate of interest (the rate that banks must pay to their depositors in order to obtain funds to be used for loans). There are many competing banks and in equilibrium **each bank will earn zero expected profit**.

5(a)

5(a.1) For what values of the parameters is it the case that efficiency requires that *all and only* the good projects be financed?
5(a.2) For these values of the parameters, what would happen in the absence of asymmetric information, i.e., if banks could distinguish good and bad entrepreneurs? What would the expected utility of each entrepreneur be?

From now on assume that the parameter values satisfy the restrictions of part 5(a)

5(b) Now assume asymmetric information: banks cannot distinguish good and bad entrepreneurs. Find the value of w and the parameter restrictions that yield a pooling equilibrium, that is, an equilibrium where all types of entrepreneurs apply for a loan. Calculate the equilibrium expected utility of each entrepreneur of type G and of type B at the equilibrium.

Now suppose that an entrepreneur can offer to contribute $\$c$ toward the initial outlay of $\$X$, where $c \in [0, X]$. If she contributes $\$c$, she only needs to borrow $\$(X - c)$ from a bank. Now a loan contract is a pair $C = (c, w)$ specifying the amount w that is supposed to be repaid if the entrepreneur contributes c [and thus borrows $(X - c)$]. Also assume that the entrepreneur is liquidity constrained, so her effective cost of contributing c is $(1+\rho)c$, where $\rho > r$ (that is, the entrepreneur has to borrow that money at an interest rate ρ that exceeds the risk-free rate r). As before, in the event that a project yields zero revenue, the entrepreneur defaults on her loan contract, and the bank receives nothing; however, the loan for the amount c is fully guaranteed by the entrepreneur's illiquid assets (his house) and therefore the repayment of $(1+\rho)c$ cannot be avoided.

5(c) Let $U_\theta(c, w)$ denote the expected payoff of an entrepreneur of type $\theta \in \{G,B\}$ from the contract $C = (c, w)$. Find $U_G(c, w)$ and $U_B(c, w)$.

5(d) Suppose that all the banks use the following strategy: offer a loan of $\$(X - c)$ with repayment $\$w$ to all those entrepreneurs who offer to contribute $\$c$ with $c \geq c_0$ and refuse a loan to all those entrepreneurs who offer to contribute $\$c$ with $c < c_0$. Find conditions on the values of c_0 and w that would lead to all and only the G-type entrepreneurs applying for (and obtaining) a loan and each bank making zero expected profits. (That is, find values of c_0 and w that yield a separating equilibrium.) Assume that each entrepreneur will apply for a loan if and only if her expected utility is positive.

5(e) From now on assume that $X = 50$, $R = 100$, $p_G = 1$, $p_B = \frac{1}{10}$, $r = 0$, $\lambda = \frac{1}{3}$, $\rho = \frac{1}{10}$.

5(e.1) Give the answers to 5(b)-5(d) for these values of the parameters.

5(e.2) Compare the payoffs of the two types in the equilibrium without asymmetric information of part (a), in the pooling equilibrium of part 5(b) and in the separating equilibrium of part 5(d) with the lowest value of c_0. Rank them in terms of efficiency. [Hint: calculate average expected utility.]