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PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

Question 1. Fancy scotch in Scotland and in Australia

Is the ratio of fancy scotch relative to ordinary scotch consumed in Australia higher than in Scotland? This is of course an empirical question. But it has been claimed that this should be so on theoretical ground. To fix the ideas, let good 1 be fancy scotch and good 2 be ordinary scotch, and assume that there is a representative consumer in Scotland, as well as one in Australia, and that the two representative consumers have identical preferences and wealth. (All prices and wealth are expressed in US$). We assume that these preferences are represented by a smooth utility function, and that the consumption of all goods is always positive. Let p_1 (resp. p_2) be the price of a bottle of fancy (resp. ordinary) scotch in Scotland, and assume throughout this question that $p_1 > p_2$. Shipping one bottle of scotch from Scotland to Australia costs c. Accordingly, the price of fancy (resp. ordinary) scotch in Australia is $p_1 + c$ (resp. $p_2 + c$). The claim is that the ratio x_1/ x_2 in the consumption of a representative agent is an increasing function of c, and is verbally defended by noting that, because $p_1 > p_2$, an increase in c decreases the price of fancy scotch relative to that of ordinary scotch. We examine the claim in various forms.

1.1. **Walrasian demand, two goods**

The claim here takes the form $\dfrac{d\left(\dfrac{\tilde{x}_1(p_1+c, p_2+c, w)}{\tilde{x}_2(p_1+c, p_2+c, w)}\right)}{dc} > 0$, where $\tilde{x}_j$ is the representative consumer's Walrasian demand for good j, $j = 1,2$. Is the claim correct? A graphical argument suffices.

1.2. **Hicksian demand, two goods**

The claim now takes the form $\dfrac{d\left(\dfrac{h_1(p_1+c, p_2+c, u)}{h_2(p_1+c, p_2+c, u)}\right)}{dc} > 0$, where h_j is the representative consumer's Hicksian demand for good j, $j = 1,2$. Is the claim correct? Argue both graphically and analytically.

1.3. **Hicksian demand, L goods ($L \geq 3$)**

Last, the claim takes the form $\dfrac{d\left(\dfrac{h_1(p_1+c,p_2+c,p_3,\ldots,p_L,u)}{h_2(p_1+c,p_2+c,p_3,\ldots,p_L,u)}\right)}{dc} > 0$, where h_j is the representative consumer's Hicksian demand for good j, $j = 1,2$.

Examine the validity of the claim for the following two cases. Comment, comparing 1.3 with 1.2 above.

Case 1. Quasilinear utility function $\tilde{u}(x_1, x_2,\ldots, x_L) = \frac{a}{b}\sum_{j=1}^{L-1} x_j - \frac{1}{2b}\sum_{j=1}^{L-1} x_j^{\,2} + x_L$, where $a > 0$, and $b > 0$ (defined on the domain where all marginal utilities are positive).

Case 2. CES utility function $\tilde{\tilde{u}}(x_1, x_2,\ldots, x_L) = \left(\sum_{j=1}^{L} x_j^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$, where $\sigma > 0$, $\sigma \neq 1$. Hint. The corresponding expenditure function is $e(p,u) = u\left(\sum_{j=1}^{J} p_j^{\,1-\sigma}\right)^{\frac{1}{1-\sigma}}$.

Question 2. Human capital externalities in production

There are J identical firms, each producing the same good (called output) by using M inputs, which are interpreted as labor of various skills due to varying amounts of human capital. For $m = 1, \ldots, M$, skill type m is defined by a positive real number h_m, interpreted as the human capital embedded in one hour of work by a worker of skill type m.

Firm j ($j = 1,\ldots, J$) chooses the amounts $(z_{j1}, \ldots, z_{jM})$ of hours of work by workers of the various skill types. The firm is a price taker in all the markets where it operates, and faces a price of p for its output, and a price of w_m for a unit of labor of skill level m ($m = 1,\ldots, M$).

2.1. For $j = 1,\ldots, J$, firm j bases its decisions on the following production function,

$$f(z_{j1}, \ldots, z_{jM}) = \left(\sum_{m=1}^{M} h_m z_{jm}\right)^{\beta}\left(\frac{\sum_{m=1}^{M} h_m z_{jm}}{\sum_{m=1}^{M} z_{jm}}\right)^{\theta} A, \qquad (1)$$

where the firm takes $A > 0$ as a constant, and where $\beta > 0$ and $\theta \geq 0$. The term $\frac{\sum_{m=1}^{M} h_m z_{jm}}{\sum_{m=1}^{M} z_{jm}}$ is the *average human capital in the firm's labor force*. The idea is that, when $\theta > 0$, an increase in the average human capital in the firm's labor force increases the productivity of all workers employed by the firm.

2.1 (a). Is f homogeneous of any degree? Does it present decreasing, constant or increasing returns to scale? Justify your answer.

2.1(b). Write firm j's profit maximization problem where the objective function (profits) of the firm is a function of the input quantities (z_{j1}, …, z_{jM}), which are subject to nonnegativity constraints (no other constraints in the problem).

2.1(c). Let $M = 2$, $h_2 < h_1$ and $\theta > 0$. Compute the marginal products of labor of the two skill levels, and argue that If h_1/h_2 and θ are large, then the marginal product of labor of skill level 2 may be negative in a region of the (z_{j1}, z_{j2}) nonnegative quadrant. Interpret.

2.1(d). It turns out that, for $\theta > 0$ and $h_m \neq h_n$, the production function typically fails to be quasiconcave: Assume that f is in fact strictly quasiconvex. Argue that, in the cost minimization problem (for a positive amount of output): **(i)** For some positive w_1 and w_2, firm j chooses $z_{j2} = 0$ and $z_{j1} > 0$, and **(ii)** For some positive w_1 and w_2, firm j chooses $z_{j1} = 0$ and $z_{j1} > 0$.

2.2. Now we introduce a human capital externality across firms. The parameter A, which firm j takes as given, is in fact a function of the *average human capital in the whole industry* of J firms, in particular

$$A(z_{11},...,z_{1M},...,z_{J1},...,z_{JM}) = \left(\frac{\sum_{m=1}^{M} h_m \sum_{k=1}^{J} z_{km}}{\sum_{m=1}^{M} \sum_{k=1}^{J} z_{km}} \right)^{\alpha}, \qquad (2)$$

where the parameter α captures the interfirm externality: if $\alpha > 0$, then an increase in the average human capital in the whole industry improves the technology of each firm. Substituting (2) into (1), the amount of output obtained by firm j ($j = 1$, …, J) can be written as a function in the arguments $(z_{11},...,z_{1M},...,z_{J1},...,z_{JM})$ as

$$\varphi_j(z_{11},...,z_{1M},....,z_{J1},....,z_{JM}) = (\sum_{m=1}^{M} h_m z_{jm})^{\beta} \left(\frac{\sum_{m=1}^{M} h_m z_{jm}}{\sum_{m=1}^{M} z_{jm}} \right)^{\theta} \left(\frac{\sum_{m=1}^{M} h_m \sum_{k=1}^{J} z_{km}}{\sum_{m=1}^{M} \sum_{k=1}^{J} z_{km}} \right)^{\alpha} .$$

2.2(a). Is φ_j homogeneous of any degree? Does it present decreasing, constant or increasing returns to scale? Compare with your answer to 2.1(a) above.

2.2(b). We know that, in the usual case of absence of externalities, the maximization of (price-taking) profits by each individual firm is equivalent to the maximization of profits on the aggregate production set. To check that this is not necessarily true in the case of interfirm externalities, consider the following parameter values and market prices

$h_1 = 2.5$, $h_2 = 1$, $\beta = 1$, $\theta = 0.8$, $\alpha = 1$, $p =$, $w_1 = 8$, $w_2 = 1$.

In turns out that the production function on which the firm bases its profit-maximizing decisions, as defined in 1 above, is indeed strictly quasiconvex (you do not have to prove this). Check that, when if all firms use only Type 2 labor, then firm j solves its profit-maximizing problem by using only Type 2 labor. (Hint. Begin with the cost-minimization problem). What are the profits of each firm?

Argue that aggregate profits would be higher if all firms produced a positive amount of output by using only Type 1 labor. But would firm j be solving its profit-maximizing problem there? Comment.

3. Marginal Cost Pricing Equilibrium

Consider an economy with 2 factors of production that we will call capital and labor, and two consumption goods, goods 1 and 2. Good 1 is produced from capital and labor with the production function $y_1 = f(k_1, l_1)$. The technology has convex inputs requirement sets (i.e. the set $\{(k_1, l_1) \in \mathbb{R}^2_+ \mid f(k_1, l_1) \geq \bar{y}_1\}$ is convex for any level of production $\bar{y}_1 \geq 0$) but exhibits increasing returns to scale. We will thus assume that there is only one firm producing good 1. Good 2 is produced from capital and labor with a concave production function $y_2 = g(k_2, l_2)$ which exhibits constant returns to scale. Thus the number of firms producing good 2 does not matter and, to simplify, we will assume that there is only one firm producing good 2.

There are I consumers who are endowed with capital and labor and have preferences for bundles of good 1 and 2. Let $u^i(x_1^i, x_2^i)$ denote agent i's utility function and $\omega^i = (\omega_l^i, \omega_k^i)$ his endowment, for $i = 1, \ldots, I$. u^i has all the nice properties that we can wish, i.e it is differentiable, quasi-concave, increasing. In all the problem we will consider only interior allocations, in which both goods are produced with positive quantities of capital and labor, and all agents consume positive quantities of the two consumption goods.

(a) Assuming that all the functions are differentiable, write the first-order conditions that a Pareto optimal allocation must satisfy.

(b) Because of the increasing returns, firm 1 cannot behave competitively (i.e. there is no solution to maximizing profit with prices taken as given). If left as an unregulated "natural" monopoly, firm 1 will create inefficiencies by using its monopoly power to sell good 1 at too high a price. So the government decides to nationalize firm 1, instruct its manager to sell at marginal cost and satisfy the demand of the consumers, and the government finances the deficit by (lump-sum) taxes. We then consider an equilibrium, that we will call a *marginal-cost pricing equilibrium*, defined as follows: $\left((\bar{x}_1^i, \bar{x}_2^i)_{i=1}^I,\ (\bar{y}_1, \bar{l}_1, \bar{k}_1),\ (\bar{y}_2, \bar{l}_2, \bar{k}_2),\ (\bar{p}_1, \bar{p}_2, \bar{w}, \bar{r}, (\bar{t}_i)_{i=1}^I)\right)$ is a marginal-cost pricing equilibrium if

(i) $(\bar{l}_1, \bar{k}_1)$ minimize the cost of producing $\bar{y}_1$ at prices $(\bar{w}, \bar{r})$

(ii) $\bar{p}_1$ is the marginal cost of firm 1 at $\bar{y}_1$

(iii) Firm 1 breaks even: $\bar{p}_1\bar{y}_1 - \bar{w}\bar{l}_1 - \bar{r}\bar{k}_1 + \sum_{i=1}^I \bar{t}_i = 0$

(iv) $(\bar{y}_2, \bar{l}_2, \bar{k}_2)$ maximizes the profit of firm 2 at prices $(\bar{p}_2, \bar{w}, \bar{r})$

(v) $(\bar{x}_1^i, \bar{x}_2^i)$ maximizes agent i's utility given his budget constraint, taking into account that agent i has to pay the lump sum tax $\bar{t}_i$

(vi) All markets clear

Show that a marginal cost pricing equilibrium satisfies the first-order conditions for Pareto optimality.

(c) Since firm 1's production set is not convex, the first-order conditions for Pareto optimality are not sufficient to ensure that the allocation is efficient. Although a marginal cost pricing is not always Pareto optimal, let us see on a simple example that the system described above may lead to Pareto optimality. Suppose that the economy has a single representative agent with utility function $u(x_1, x_2) = \sqrt{x_1 x_2}$ and initial resources $\omega = (1, 1)$ in labor and capital, that the production function of firm 1 is $f(k_1, l_1) = (\min\{l_1, k_1\})^2$, and the production function of firm 2 is $f(k_2, l_2) = \sqrt{l_2 k_2}$.

c(i) Compute the Pareto optimal allocation(s) for this economy.

c(ii) Compute the marginal cost pricing equilibrium(a) for this economy and compare to the result in (i).

4. Grading on a curve

Consider a typical class of I first-year graduate students. Assume that the selection process is such that all the students have the same ability. Each student has L hours of time that can be used either for working or for leisure, and a utility function

$$u(g_i, l_i) = \phi(g_i) + l_i, \qquad i = 1, \ldots, I$$

where g_i is the grade (GPA) obtained, l_i is leisure time, and ϕ is a differentiable, concave, increasing function such that $\phi'(g) \to +\infty$ when $g \to 0$. Because all instructors grade "on a curve" the grade obtained by student i is

$$g_i = \frac{h_i}{\bar{h}} \quad \text{where} \quad \bar{h} = \frac{1}{I}(h_1 + \ldots + h_I)$$

and h_i is the time that student i has spent working.

(a) Find the symmetric Nash equilibrium choice $\tilde{h}$ of working time by the students.

(b) Study how $\tilde{h}$ varies with I and explain. Derive intuitively the limit of $\tilde{h}$ when I tends to infinity.

(c) Find the symmetric Pareto optimal choice h^* of working time. Explain intuitively the result.

(d) Change the utility function of the representative student to

$$v(h_i, g_i, l_i) = \psi(h_i) + \phi(g_i) + l_i$$

where ψ is a differentiable, concave, increasing function.

d(i) Interpret the utility function v and explain why it is more realistic than u.

d(ii) Adapt the results of questions (a) and (b) to this case, and comment on the difference between the Nash equilibrium and the Pareto optimal choices of working time.

(e) Which advantages or disadvantages (modeled or unmodeled in the previous questions) do you see to "grading on a curve"?

5. King Solomon's wisdom.

The King James Bible (I Kings 3:24-28) tells the story of two women going to king Solomon with a baby, each claiming to be the mother. The king asked for a sword to be brought to him and announced that he was going to cut the baby into two and give half each. One woman agreed, while the other cried "do not kill the baby: give it to the other woman!". From this the king deduced that she was the true mother and gave the baby to her.

If we analyze this from a game-theoretic point of view, we can find fault with two players. The woman who agreed made a strategic blunder: she should have given the same answer as the other one or else she was sure to reveal herself as the false claimant, since the child's true mother would never have consented to its death. The king was lucky that the false claimant made a strategic mistake. Had she been more sophisticated, the king would have found himself in the embarrassing situation of either having to cut the baby into two or admitting that he was bluffing.

In this problem we ask the question: What strategic device could Solomon have used that would have worked even against a pair of strategically savvy women? Let's call the two women Anna and Bess. Suppose Solomon does the following:

First Solomon announces a fine of $F > 0. Then he asks Anna and Bess to play the following game.

> **Move 1**: Anna has to move first. Either she gives up her claim to the child (in which case Bess gets the child, the game ends and nobody pays the fine) or she asserts her claim, in which case the game proceeds to move 2.
>
> **Move 2**: Bess either accepts Anna's claim (in which case Anna gets the child, the game ends and nobody pays the fine) or challenges her claim. In the latter case, Bess must put in a bid, call it B, and Anna must pay the fine of $*F* to Solomon. The game goes on to move 3.
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> **Move 3**: Anna now either matches Bess's bid (in which case Anna gets the child, Anna pays $*B* to Solomon, − in addition to the fine that she already paid − and Bess pays the fine of $*F*, to Solomon) or chooses not to match (in which case Bess gets the child and pays her bid of $*B* to Solomon and, furthermore, Solomon keeps the fine that Anna already paid).

Denote by C_A the monetary equivalent of getting the child for Anna (that is, getting the child is as good, in Anna's mind, as getting $$C_A$) and C_B the monetary equivalent of getting the child for Bess. Not getting the child is considered by both as good as getting zero dollars.

(a) Draw an extensive game with perfect information to represent the above situation. Assume that there are only two possible bids, B_1 and B_2. Write the payoffs to Anna and Bess next to each terminal node.

(b) Find the backward-induction solution of the game you drew in part (a) for the case where $B_1 > C_A > C_B > B_2 > F > 0$.

Now consider the general case where the bid *B* can be any non-negative number and assume that both Anna and Bess are very wealthy. Assume also that C_A, C_B and F are positive numbers and that C_A and C_B are common knowledge between Anna and Bess . We want to show that, at the backward-induction solution of the game, the child always goes to the real mother. Since we (like Solomon) don't know who the true mother is, we must consider two cases.

Case 1: the true mother is Anna. We can assume that this implies that $C_A > C_B$.

Case 2: the true mother is Bess. We can assume that this implies that $C_B > C_A$.

(c) Find the backward-induction solution for case 1 and show that it implies that the child goes to Anna.

(d) Find the backward-induction solution for case 2 and show that it implies that the child goes to Bess.

(e) How much money does Solomon make in equilibrium? How much money do Anna and Bess end up paying in equilibrium? (By 'equilibrium' we mean 'backward-induction solution'.)