University of California, Davis
Department of Economics
Microeconomics

Date: June 21, 2004
Time: 4 hours
Reading Time: 20 minutes

PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

This exam is designed to be done in **3 hours**. The extra 1 hour is intended to remove the time constraint, to make the process a little more relaxed. Please try to keep your answers brief and to the point; you are *not* expected to write a 4 hour exam! Good luck.

Please answer <u>four</u> of the five questions

1. Inequality and luxuries

It is sometimes asserted that an increase in the income inequality of a country, other things being equal, will increase the country's demand for luxuries. To scrutinize this claim, we consider a society with I individuals and two goods. The price vector (p_1, p_2) is fixed. All individuals have the same preferences, but their wealth levels may vary as long as they do not fall below a given minimum admissible wealth level $\beta \geq 0$. We postulate that, for each consumer, given (p_1, p_2) and $w_i \geq \beta$, good 2 is a luxury,[1] and both goods are normal.

Denote by $(x_1(w_i), x_2(w_i))$ the (smooth) Walrasian demand function of a consumer with wealth w_i $(w_i \geq \beta)$.

(a). Using the above notation, define "luxury" in the sentence "for each consumer, good 2 is a luxury, given (p_1, p_2) and $w_i \geq \beta$."

(b). Given that good 2 is a luxury and that both goods are normal, what can be said about good 1? Justify your answer.

[1] In the strict sense, i.e., the inequality in the definition is strict.

(c). Suppose that consumer i's wealth increases. What happens to the share $b_1(w_i) \equiv \frac{p_1 x_1(w_i)}{w_i}$ of good 1 in her budget? What happens to the share $b_2(w_i) \equiv \frac{p_2 x_2(w_i)}{w_i}$ of good 2 in her budget?

(d). We define an equiproportional increase in the wealth vector as a move from $(w_1, \ldots, w_I)$ to $(\lambda w_1, \ldots, \lambda w_I)$, where $\lambda > 1$. How does such an equiproportional increase in the wealth vector affect the aggregate budget share of good 2, defined as $B_2(w) \equiv \frac{p_2 \sum_{i=1}^{I} x_2(w_i)}{\sum_{i=1}^{I} w_i}$? Justify your answer.

(e). We define an increase in inequality as a transfer of $\varepsilon > 0$ units of wealth from a person i to another one who is not poorer than i.[2] Does the fact that good 2 is a luxury guarantee that an increase in inequality increases the aggregate demand $\sum_{i=1}^{I} x_2(w_i)$ for good 2 (and hence its budget share)? If not, what condition or conditions on the functions $x_1(w_i)$ and/or $x_2(w_i)$ guarantee it? In either case, prove your claim.

For (f) – (i) below, refer to the following example: $x_1(w_i) = w_i + \frac{1}{1+w_i}$, $x_2(w_i) = \frac{w_i}{4} - \frac{3}{4}\frac{1}{(1+w_i)}$, considering only $(p_1, p_2) = (3/4, 1)$, and $w_i \geq \beta = 3/2$.

(f). Classify the two goods as necessities, luxuries, or borderline.

(g). Does an equiproportional increase in the wealth vector increase the budget share of good 2? Explain.

(h). Does the example satisfy the condition that you propose in (e) above? Explain.

(i). In this example, does an increase in inequality increase the demand for a luxury? Comment

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[2] Of course, we still require that $w_i - \varepsilon \geq \beta$.

2. Labor supply under certainty and uncertainty

We start with the certainty case. There are two goods: leisure (labor) and a consumption good. The wage rate (price of leisure) is $p_1 = p > 0$, and the price of the consumption good is $p_2 = 1$. The consumer is endowed with $\omega > 0$ units of leisure, 0 units of the consumption good and $m > 0$ units of nonlabor wealth. The preferences of the consumer are represented by the utility function

$$a\ln x + (1 - a)\ln c\ ,$$

where x and c are, respectively, the amounts of leisure and consumption good that she enjoys. Writing L for the amount of labor supplied, we have that $x = \omega - L$, whereas the budget equality implies that $c = m + pL$. Accordingly, we write the consumer's problem as the (unconstrained) maximization of the single-variable function

$$a\ln(\omega - L) + (1 - a)\ln(m + pL).$$

(a). Find the supply-of-labor function $\widetilde{L}(p, m)$.

(b). Compute $\dfrac{\partial \widetilde{L}}{\partial m}$ and interpret its sign.

(c). Compute $\dfrac{\partial \widetilde{L}}{\partial p}$ and interpret its sign in terms of the substitution and wealth effects.

Now we move to a world of uncertainty. There are two states of the world: bad and good. In the bad state, which occurs with probability π, $(p, m) = (p^B, m^B)$. In the good state, which occurs with probability $(1 - \pi)$, $(p, m) = (p^G, m^G)$.

The consumer must commit to her labor supply L before the state of the world is known, and thus, she will get the amount $c^B = m^B + p^B L$ of the consumption good in the bad state, and $c^G = m^G + p^G L$ in the good state.

Her preferences satisfy the expected utility hypothesis, with von Neumann-Morgenstern-Bernoulli utility function $a\ln(\omega - L) + (1 - a)\ln c$.

(d). Argue that the consumer is strictly risk averse.

(e). Write the expected utility maximization problem.

(f). Write the first order condition for the expected utility maximization problem.

(g). Show that the second order condition for the expected utility maximization problem is satisfied with strict inequality.

Find the signs of the derivatives requested below by applying the implicit function theorem to the appropriate version of (f) above, taking (g) into account. All epsilons below are positive.

(h). Uncertainty about m: the effect of increased m^B and m^G. Suppose that $p^B = p^G = p$, but $m^G > m^B$. Let m^B and m^G increase to $m^B + \varepsilon_h$ and $m^G + \varepsilon_h$, respectively. What is the sign of $\frac{dL}{d\varepsilon_h}$? Interpret in words, and compare with (b) above.

(i). Uncertainty about m: the effect of increased dispersion at unchanged mean. Suppose again that $p^B = p^G = p$, and that $m^G > m^B$. Let m^B decrease to $m^B - \varepsilon_i$, while m^G increases to $m^G + \frac{\pi}{1-\pi}\varepsilon_i$. What is the sign of $\frac{dL}{d\varepsilon_i}$? Interpret in words, and comment.

(j). Uncertainty about p: the effect of increased p^B and p^G. Suppose that $m^B = m^G = m$, but $p^G > p^B$. Let p^B and p^G increase to $p^B + \varepsilon_j$ and $p^G + \varepsilon_j$, respectively. What is the sign of $\frac{dL}{d\varepsilon_j}$? Interpret in words, and compare with (c) above.

(k). Uncertainty about p: the effect of increased dispersion at unchanged mean. Suppose again that $m^B = m^G = m$, and that $p^G > p^B$. Let p^B decrease to $p^B - \varepsilon_k$, while p^G increases to $p^G + \frac{\pi}{1-\pi}\varepsilon_k$. What is the sign of $\frac{dL}{d\varepsilon_k}$? Interpret in words, and comment.

3. The public production of public goods A newly formed city must choose the bus fleet that it will purchase. It must choose the type of buses that it will buy and how many of them to purchase. Of course more comfortable buses make for more enjoyable rides, but they are more expensive. Let $C(y_1, y_2)$ denote the cost in money of a bus service of frequency y_1 (a proxy for the number of buses) and comfort level y_2. There are I inhabitants in the city. Agent i has a utility function $u_i(x_i, y_1, y_2)$, where x_i is her consumption of the private good (the numeraire, money), and ω_i is her initial endowment of money. For each $i = 1, \ldots, I$, u_i is assumed to satisfy the usual monotonicity, differentiability, and quasi-concavity conditions.

(a) Derive the necessary conditions that an interior allocation $\left((x_i)_{i=1}^{I}, y_1, y_2\right)$ must satisfy in order to be Pareto optimal. Interpret these conditions.

(b) Under what condition(s) on the cost function C are these conditions sufficient for Pareto optimality? Justify your answer.

(c) To decide on the fleet to buy, the city organizes a town meeting. Whoever does not attend will have to pay a very high transportation tax so that all citizens have an incentive to attend the meeting, which has been announced well in advance. During the meeting each citizen i receives a piece of paper with prices (p_1^i, p_2^i) and a transfer T written on it, and must send back a piece of paper indicating the pair (y_1^i, y_2^i) of frequency and comfort level which is optimal for her, knowing that she will have to pay $p_1^i y_1^i + p_2^i y_2^i$ in taxes but will receive the (lump-sum) transfer T. On the other side of the room the chief of the city transportation board receives a piece of paper indicating $\sum_i p_1^i$ as the price of good 1 and $\sum_i p_2^i$ as the price of good 2. The chief is in charge of finding the combination (y_1, y_2) maximizing the board's profit, taking these prices as given. He sends back a piece of paper indicating the profit maximizing choice (y_1, y_2) and the profit that is generated. The mayor, who has a Ph.D. in economics, enters all the responses in the city computer, and changes the unit prices (p_1^i, p_2^i) that i must pay and the transfer T that each citizen will receive, and changes accordingly the prices that he gives to the transportation board. In turn each citizen changes her choice of (y_1^i, y_2^i) to make it optimal with the new prices, the board chief modifies his supply and profit forecast, the mayor registers the new demands, supplies, and profit forecasts and changes the prices and transfers, and so on ... until all the citizens demand the same combination $(\bar{y}_1, \bar{y}_2)$, which is the profit-maximizing supply of the transportation board, and the profit generated by the board exactly pays for the transfers T to the citizens. Write out the definition of the equilibrium which is achieved.

(d) Show that the equilibrium allocation is Pareto optimal. You get maximum credit if you only use the assumption that utility functions satisfy local non-satiation. Otherwise you can use the assumptions given above on the utility functions and the assumption that you mentioned in (b), but then you show a much weaker result.

(e) Actually things did not go too well in the town meeting: the mayor's computer kept crashing, and although everybody accepted staying late, the equilibrium was not found at the end of the meeting. So a citizen proposed a simpler system for the next meeting. There will be a series of votes: the mayor will first propose a combination (y_1, y_2, t) such that, if every citizen pays the fraction t of her income in taxes, the cost of the public good (y_1, y_2) is covered. Then any citizen is allowed to make a counter proposal (y_1', y_2', t'), subject to the condition that (y_1', y_2') must be feasible for the tax rate t'. A citizen's proposal will replace the mayor's proposal if a majority votes for it, if not it will be eliminated. As long as some citizen makes a counter proposal, each counter proposal is voted on against the survivor of the last round of voting. The city turns out to have two types of citizens: there are N_1 rich ones, with per-capita income $\omega_i = a$, who favor comfort over frequency of buses (they rarely take the bus but when they take it they like it to be clean and comfortable). Their utility function is

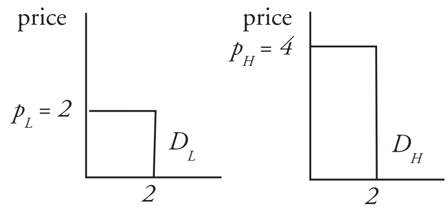
$$u(x, y_1, y_2) = x^{\frac{1}{2}} y_1^{\frac{1}{8}} y_2^{\frac{3}{8}}.$$

There are N_2 poorer citizens with per-capita income $\omega_i = b$, where $b < a$ and $N_2 > N_1$. They use the bus a lot and favor frequency over comfort. Their utility function is

$$u(x, y_1, y_2) = x^{\frac{1}{2}} y_1^{\frac{3}{8}} y_2^{\frac{1}{8}}.$$

The cost function for bus service is $C(y_1, y_2) = (y_1)^2 + (y_2)^2$. Find the combination (y_1^*, y_2^*, t^*) that will be adopted in the next town meeting. Is the resulting allocation Pareto optimal?

4. The incentive to cooperate in a supergame[1] Consider a Bertrand duopoly game. For simplicity, we assume both firms have zero production costs. We also assume that there are only 2 customers, each with unit demand. The wrinkle is that the demand for the product is stochastic: it may be either Low (L) or High (H). In state L each buyer is willing to pay at most $p_L = 2$ for a unit, and in state H each buyer is willing to pay at most $p_H = 4$ for a unit; so the market demand curves are as illustrated below. Notice, under Bertrand competition,



in state L the demand any firm i faces is 2 if $p_i < p_j$ and $p_i \leq 2$, is 1 if $p_i = p_j$ and $p_i \leq 2$, and is 0 if $p_i > p_j$ or $p_i > 2$, where $i, j = 1, 2$ and $i \neq j$. A similar statement holds in state H.

Consider the T-times repeated Bertrand duopoly game based on the above stage game. In each period $t = 1, 2, \ldots, T$, demand may be either High or Low with equal probability. Both firms learn at the beginning of each period t — *before* they set their prices — whether demand will be high or low during t.

(a) Prove that in the one-shot Bertrand game (that is, when $T = 1$), both firms charging the same price $p_i = p_j = 0$ regardless of the state of demand Nature picks is a Nash equilibrium. Also prove that $p_i = p_j > 0$ in either state L or state H is not a Nash equilibrium.

(b) Now consider the infinitely-repeated Bertrand game (that is, $T = \infty$). Notice p_L and p_H are the nondiscriminatory monopoly prices for the two levels of demand. The firms can use trigger strategies to sustain these price levels (i.e., to both play p_s whenever the level of demand is s, for $s = H, L$) in a subgame perfect Nash equilibrium for the supergame, provided firms' common discount factor δ satisfies $\delta \geq \delta^*$.

- Specify each firm's trigger strategy.
- Then calculate the value of the minimum discount factor δ^*, briefly explaining your calculations as you go along.

HINT: You will find the temptation to deviate — i.e., to start a "price war" — is stronger in periods when demand is high.

(c) For each value of δ between .5 and δ^*, find the highest price $p(\delta)$ such that the firms can use trigger strategies to sustain the price $p(\delta)$ when demand is high and p_L when demand is low in a subgame-perfect Nash equilibrium. Again briefly explain your calculations as you go along.

[1] Based on Rotemberg and Saloner, "A Supergame-Theoretic Model of Business Cycles and Price Wars during Booms," *AER*, 1986.

5. Regulating a monopolist with unknown marginal cost A monopolist has constant marginal cost $\theta \geq 0$ and zero fixed cost of production. There is only one consumer, with valuation function

$$v(q) = q - \frac{1}{2}q^2.$$

Thus the gains from trade if q units are produced and the monopolist is of type θ is

$$GFT(q, \theta) \equiv v(q) - \theta q.$$

(a) Calculate the value of q that maximizes the gains from trade as a function of θ. Call it $\hat{q}(\theta)$.

A regulator does not know the monopolist's true type θ, and hence uses a **revelation game** to try to induce efficient production. In the game, the regulator first asks the monopolist what his type is; so the monopolist announces a marginal cost $a \in [0, \infty)$. The regulator understands that the monopolist will tell the truth (that is, choose $a = \theta$) only if it is in his self interest. So the regulator tells the monopolist that if he announces a, he will have to produce $\hat{q}(a)$ units — the efficient output for someone with marginal cost a, — and he will be paid $T(a)$. Notice if the monopolist announces a and his true type is θ, his profit would be

$$\pi(a, \theta) \equiv T(a) - \theta \hat{q}(a).$$

Assume the consumer is taxed to finance the payment to the monopolist; so if the monopolist announces a, the consumer's surplus would be

$$CS(a) \equiv v(\hat{q}(a)) - T(a).$$

(b) Let $a^*(\theta)$ denote the monopolist's profit maximizing announcement given he's a type θ, i.e., $a^*(\theta) \in \arg\max_{a \geq 0} \pi(a, \theta)$. Show that the necessary condition for the monopolist to tell the truth (that is, for $a^*(\theta) = \theta$) is:

$$\frac{\partial \pi(a, \theta)}{\partial a} = \frac{\partial GFT(\hat{q}(a), \theta)}{\partial q} \frac{d\hat{q}(a)}{da} = 0,$$

where all derivatives are evaluated at $a = \theta$. Briefly interpret the condition in Pigovian terms.

Groves and Loeb (G–L) proposed that, to give the monopolist good incentives, the regulator should offer the monopolist

$$T(a) = \int_0^{\hat{q}(a)} (1 - q) dq$$

when he announces a, that is, the area under the market demand from 0 to $\hat{q}(a)$. Let's check that this incentive scheme does the trick.

(c) For *any* nonnegative θ, show that $a^*(\theta) = \theta$ under the G–L incentive scheme. Explain the intuition underlying this fact by calculating the monopolist's Private Benefit (PB) from any announcement a and also the Social Benefit (SB) from any announcement a. Illustrate the intuition by graphing the PB and SB functions (both functions of a) for some arbitrary fixed value of θ, say $\theta = 1/2$.

Consider instead the **Walrasian scheme** in which the regulator again asks the monopolist to announce his marginal cost, again requires him to produce $q = \hat{q}(a)$ if he claims his marginal cost is a, but this time offers him total revenue

$$T(a) = a \times \hat{q}(a),$$

that is, his total revenue as a Walrasian price-taker when his true marginal cost is a.

(d) For any $\theta \geq 0$, calculate the monopolist's profit maximizing announcement $a^*(\theta)$ under the Walrasian incentive scheme. For what values of θ is $a^*(\theta) \neq \theta$? Explain the intuition why truthtelling is typically not optimal under the Walrasian scheme. Hint: Calculate the monopolist's Private Benefit from any announcement a and also the Social Benefit from any announcement a. Compare his MPB from the information he provides—whether it is truthful or false—with the MSB from the information he provides. Illustrate the intuition by graphing the PB and SB functions (both functions of a) for some arbitrary fixed value of θ, say $\theta = 1/2$.