University of California, Davis
Department of Economics
Microeconomic Theory

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PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

Please answer four parts (out of five)

ANSWER KEY

QUESTION 1. THE MARGINAL UTILITY OF WEALTH

Let $u:\Re^{L}_{++}\to\Re$ be a differentiable, strictly concave, locally nonsatiated utility function. We restrict ourselves in what follows to a price-wealth domain *P* of vectors $(p,w)\in\Re^{L+1}_{++}$ for which the UMAX[*p*, *w*] problem has a solution.

1.1. Briefly argue that the solution to the UMAX[*p*, *w*] problem, denoted $\tilde{x}(p,w)$ is unique and defined by the first order equalities.

ANSWER. Uniqueness follows from strict quasiconcavity. The FOC equalities characterize the solution because the domain of *u* is an open set.

1.2. Verbally define the marginal utility of wealth, to be denoted $\tilde{\lambda}(p,w)$. State and prove its relation to the Lagrange multiplier.

ANSWER. The marginal utility of wealth is the partial derivative of the indirect utility function with respect to wealth, and it coincides with the Lagrangian multiplier of the UMAX[*p*, *w*] problem, as can be proved by a standard application of the Envelope Theorem.

1.3. We now partition the set $\{1,\ldots,L\}$ of goods into the three subsets: $\{1,\ldots,L_1\}$, $\{L_1+1,\ldots,L_2\}$, and $\{L_2+1,\ldots,L\}$. A vector $x=(x_1,\ldots,x_L)$ is similarly partitioned into the three subvectors $x^1\equiv(x_1,\ldots,x_{L_1}), x^2\equiv(x_{L_1+1},\ldots,x_{L_2})$ and $x^3\equiv(x_{L_2+1},\ldots,x_L)$, $x=(x^1;x^2;x^3)$. The price vector *p* is similarly partitioned into $p^1\equiv(p_1,\ldots,p_{L_1}), p^2\equiv(p_{L_1+1},\ldots,p_{L_2})$ and $p^3\equiv(p_{L_2+1},\ldots,p_L)$, $p=(p^1;p^2;p^3)$. We assume that the utility function *u* is separable in the form

$$u(x)=u^1(x^1)+u^2(x^2)+u^3(x^3),$$

(for J = 1, 2, 3, u^J is differentiable, strictly concave and locally nonsatiated) and adopt the following interpretation: our household, comprised of a husband and a wife, consumes three types of goods: x^1 is a basket of goods consumed in Vacation Island 1, x^2 is a basket of goods consumed in Vacation Island 2, and x^3 is the basket of all other goods consumed by the household. Our household has solved the UMAX[p, w] problem at some point in the past, but they have forgotten some details.

1.3(a). Before engaging in their two-island vacation, the husband says: all we have to remember are our planned expenditures in Island 1, w^1, and in Island 2, w^2. Then when we are in Island 1, we observe its prices p^1 and solve

$$\max u^1(x^1) \text{ subject to } p^1 \bullet x^1 \leq w^1,$$

with solution denoted $\tilde{\tilde{x}}^1(p^1, w^1)$, and when we are in Island 2, we observe its prices p^2 and solve

$$\max u^2(x^2) \text{ subject to } p^2 \bullet x^2 \leq w^2,$$

with solution denoted $\tilde{\tilde{x}}^2(p^2, w^2)$. Then, the husband claims, our solutions will coincide with what we had originally planned, i. e., $\tilde{\tilde{x}}^J(p^J, w^J) = \tilde{x}^J(p, w)$, J = 1, 2.

Is the husband right? Argue your answer.

ANSWER. Yes, he is right, as long as he correctly remembers w^1 as $p^1 \bullet \tilde{x}^1(p, w)$ and w^2 as $p^2 \bullet \tilde{x}^2(p, w)$. Consider Island 1. Suppose as contradiction hypothesis that $\bar{x}^1 \neq \tilde{x}^1(p, w)$ solves $\text{Max}_{x^1} u^1(x^1)$ subject to $p^1 \bullet x^1 \leq w^1$. Then $u^1(\bar{x}^1) \geq u^1(\tilde{x}^1(p, w))$, because $\tilde{x}^1(p, w)$ clearly satisfies the constraint $p^1 \bullet x^1 \leq w^1$, i. e., $u(\bar{x}^1, \tilde{x}^2(p, w), \tilde{x}^2(p, w)) \underline{\geq} u(\tilde{x}(p, w))$, while $(\bar{x}^1, \tilde{x}^2(p, w), \tilde{x}^3(p, w))$ satisfies the constraint $p \bullet x \leq w$, contradicting the fact that $\tilde{x}(p, w)$ is the unique solution to UMAX[p, w].

1.3(b). The wife interjects: what about just remembering the marginal utility of wealth λ? Then when we are in Island 1 we unconstrainedly solve

$$\max_{x^1} (u^1(x^1) - \lambda p^1 \bullet x^1),$$

with solution $\hat{x}^1(p^1, \lambda)$, and when we are in Island 2 we unconstrainedly solve

$$\max_{x^2} (u^2(x^2) - \lambda p^2 \bullet x^2),$$

with solution $\hat{x}^2(p^2, \lambda)$, and, she claims, we shall also get $\hat{x}^J(p^J, \lambda) = \tilde{x}^J(p, w)$, J = 1, 2.

Is the wife right? Argue your answer. Verbally interpret the maximization problems proposed by the wife.

Answer. Yes, she is right too, as long as she correctly remembers λ as $\tilde{\lambda}(p,w)$. Consider Island 1: when she solves the system of L_1 equations in L_1 unknowns

$$\frac{\partial u^1(x^1)}{\partial x_j} = \lambda p_j,\ j = 1,\ldots,L_1, \quad (1.1)$$

for $\lambda = \tilde{\lambda}(p,w)$ given, she must come up with the solution $\bar{x}^1 = \tilde{x}^1(p,w)$, because $\tilde{x}^1(p,w)$ does satisfy (1.1) for the same $\lambda p_j,\ j = 1,\ldots,L_1$, i. e., $\nabla u^1(\bar{x}^1) = \nabla u^1(\tilde{x}^1(p,w))$. Note that $\bar{x}^1 \neq \tilde{x}^1(p,w)$ is impossible given the strict concavity of u^1: writing $\nabla x^1 \equiv \tilde{x}^1(p,w) - \bar{x}^1$, if $\nabla x^1 \neq 0$, then the strict concavity of u^1 implies $\nabla u^1(\bar{x}^1)\bullet\Delta x^1 > u^1(\tilde{x}^1(p,w)) - u^1(\bar{x}^1)$

and $\nabla u^1(\tilde{x}^1(p,w))\bullet[-\Delta x^1] > u^1(\bar{x}^1) - u^1(\tilde{x}^1(p,w))$,

which yields the contradiction $0 > 0$ when $\nabla u^1(\bar{x}^1) = \nabla u^1(\tilde{x}^1(p,w))$.

In words, the wife's approach equates the marginal utility of the expenditure spent in good j to the marginal utility of wealth.

1.4. Let there be a large number of islands. Do the arguments of husband and wife in 1.3 carry over? Compare the two approaches.

Answer. Yes, they do carry over, as long as the utility function is separable island by island. But with one million islands the husband's approach requires keeping track of the one million numbers w^j, whereas the wife's approach only requires to keep track of λ.

1.5. Check the husband's and wife's approaches for Island 1 in the following example.

$$u : \Re^L_{++} \to \Re : u(x) = \sum_{i=1}^{L} \alpha_i \ln x_i, \sum_{i=1}^{L} \alpha_i = 1, \alpha_i > 0, \forall i,\ L_1 = 2.$$

Answer. The Walrasian demands in the Cobb-Douglas case are

$$\tilde{x}_i(p,w) = \alpha_i \frac{w}{p_i},\ i = 1,\ldots, L. \quad (1.2)$$

The Lagrange multiplier can be obtained from the first-order equality $\frac{\partial u}{\partial x_i} = \lambda p_i$, i. e., $\frac{\alpha_i}{x_i} = \lambda p_i$,

which by (1.2) yields $\frac{\alpha_i}{\alpha_i \frac{w}{p_i}} = \lambda p_i$, or

$$\tilde{\lambda}(p,w) = \frac{1}{w}. \qquad (1.3)$$

(Alternatively, we can compute the indirect utility function $v(p,w) = \sum_{i=1}^{L} \alpha_i \ln \alpha_i \frac{w}{p_i}$, with

$$\tilde{\lambda}(p,w) = \frac{\partial v}{\partial w} = \sum_{i=1}^{L} \alpha_i \frac{1}{\alpha_i \frac{w}{p_i}} \frac{\alpha_i}{p_i} = \frac{1}{w} \sum_{i=1}^{L} \alpha_i = \frac{1}{w}.)$$

Husband. Solves $\max \alpha_1 \ln x_1 + \alpha_2 \ln x_2$ subject to $p_1 x_1 + p_2 x_2 = w^1$, with solution $\tilde{\tilde{x}}_i(p_1, p_2, w^1) = \frac{\alpha_i}{\alpha_1 + \alpha_2} \frac{w^1}{p_i}, i = 1, 2.$ If he correctly recalls that $w^1 = p_1 \tilde{x}_1(p,w) + p_2 \tilde{x}_2(p,w) = p_1 \alpha_1 \frac{w}{p_1} + p_2 \alpha_2 \frac{w}{p_2} = w(\alpha_1 + \alpha_2)$, then $\tilde{\tilde{x}}_i(p_1, p_2, w^1) = \frac{\alpha_i}{\alpha_1 + \alpha_2} \frac{w(\alpha_1 + \alpha_2)}{p_i} = \alpha_i \frac{w}{p_i} = \tilde{x}_i(p,w), i = 1, 2.$ Hence, he gets the Walrasian demands for Island 1 right.

Wife. Solves $\max(\alpha_1 \ln x_1 + \alpha_2 \ln x_2 - \lambda[p_1 x_1 + p_2 x_2])$, with FOC $\frac{\alpha_i}{x_i} = \lambda p_i$, i. e., $x_i = \frac{\alpha_i}{\lambda p_i}$ i = 1, 2. If she correctly recalls that $\lambda = \tilde{\lambda}(p,w) = \frac{1}{w}$ (from (1.3)), then she computes $\hat{x}_i(p_1, p_2, \lambda) = \frac{\alpha_i}{p_i} w = \tilde{x}_i(p,w)$. Hence, she also gets the Walrasian demands for Island 1 right.

QUESTION 2. MONOPOLY REGULATION

Let there be L goods, and write $x_{-L} = (x_1,\ldots,x_{L-1})$. Consider a consumer with utility function

$$u:\Re_+^{L-1}\times\Re : u(x_1,\ldots,x_{L-1},x_L)=\sum_{j=1}^{L-1}u_j(x_j)+x_L\,,$$

where, for $j = 1,\ldots, L$-1, u_j is strictly concave with $u_j(0) = 0$, strictly increasing and twice differentiable, and satisfies the condition $\lim_{x_j\to 0} u_j'(x_j)=\infty$. In what follows, we set $p_L = 1$ and restrict our attention to strictly positive prices $p_j > 0$, $j = 1,\ldots, L$-1.

2. 1. Show that the Walrasian demand for good j, $j = 1,\ldots, L$-1, can be written $\hat{x}_j(p_j)$, and inverted, with inverse denoted $\hat{p}_j(x_j)$.

ANSWER. From the FOC we obtain

$$u_j'(x_j) = p_j, \qquad (2.1)$$

with $u_j'(x_j)$ decreasing, and hence invertible.

A firm supplies goods 1,…, L-1 to our consumer. Its cost function is

$$C(x_1,\ldots,x_{L-1})=\begin{cases}0, & \text{if } \ x_j=0,\ j=1,\ldots,L-1\\[2ex] F+\sum_{j=1}^{L-1}c_jx_j & \text{otherwise}\end{cases},$$

where $F > 0$ and $c_j > 0$, $j = 1,\ldots, L$-1.

Define: $S:\Re_{++}^{L-1}\to\Re : S(x_{-L})=\sum_{j=1}^{L-1}u_j(x_j)-C(x_{-L})$,

$$\Pi:\Re_{++}^{L-1}\to\Re:\Pi(x_{-L})=\sum_{j=1}^{L-1}[\hat{p}_j(x_j)-c_j]x_j-F\,.$$

2.2. Interpret S and Π in words.

ANSWER. S is the social surplus function, equal to

CONSUMER VALUATION - COST,

which can also be expressed as

CONSUMER SURPLUS + PROFITS,

or as

CONSUMER SURPLUS + PRODUCER SURPLUS – FIXED COST.

We consider the following optimization problems, and postulate that any solution to any of them is strictly positive.

Problem SU (unconstrained) $\text{Max}_{x_{-L}} S(x_{-L})$.

Problem SC[$\bar{\Pi}$] (constrained) $\text{Max}_{x_{-L}} S(x_{-L})$ subject to $\Pi(x_{-L}) \geq \bar{\Pi}$.

Problem MC[$\bar{S}$] (constrained) $\text{Max}_{x_{-L}} \Pi(x_{-L})$ subject to $S(x_{-L}) \geq \bar{S}$.

Problem MU (unconstrained) $\text{Max}_{x_{-L}} \Pi(x_{-L})$.

2.3. Write the first-order conditions of Problem *SU*, express them in the format of the Lerner equation (i. e., MARKUP = DEGREE OF MONOPOLY), and verbally interpret. What can be said about the markups, and about the level of profits at a solution to Problem *SU*?

ANSWER.

$$\text{Max}_{x_{-L}} S(x_{-L}) = \text{Max}_{x_{-L}} \sum_{j=1}^{L-1} [u_j(x_j) - c_j x_j] - F,$$

with FOC: $u'(x_j) = c_j$, $j = 1,\ldots, L-1$,

i. e., using (2.1) above, $\hat{p}(x_j) = c_j$. Hence, prices must equal marginal costs, and

$$\frac{\hat{p}(x_j) - c_j}{\hat{p}(x_j)} = 0,$$

i. e., the markups are zero.

It follows that at the solution to Problem *SU*, profits are given by

$$\sum_{j=1}^{L-1} [\hat{p}(x_j) - c_j] x_j - F = -F,$$

and the firm is suffering losses by the amount of the fixed cost *F*.

2.4. For each of the Problems *SU*, *SC*, *MC* and *MU*, verbally interpret the problem and the agent who may face it.

ANSWER.

Problem *SU*, the unconstrained maximization of social surplus, can be interpreted as the problem of a social planner interested in economic efficiency.

Problem *MU*, the unconstrained maximization of profits, can be interpreted as the problem of a profit maximizing, pure monopoly.

Problems *SC*[$\bar{\Pi}$] and *MC*[$\bar{S}$] can be interpreted as those of a regulated monopoly. Problem *SC*[$\bar{\Pi}$], $\text{Max}_{x_{-L}} S(x_{-L})$ subject to $\Pi(x_{-L}) \geq \bar{\Pi}$, can be viewed as the second-best problem of a social planner who wishes to maximize social surplus but must guarantee a minimal level of profits to the firm. For instance, if $\overline{\Pi} = 0$, then Problem *SC*[0] maximizes social surplus subject to the condition that the firm must at least break even, a condition which, as just seen, is not satisfied at the solution to Problem *SU*.

As will be seen below, Problem *MC* is formally similar to problem *SC*. Now the regulator imposes a minimum level of social surplus (perhaps via minimum quantities, or maximum prices), and lets the firm maximize profits subject to this constraint.

2.5. Write the first-order conditions of Problem *MU*, express them in the format of the Lerner equation. How are the markups for two different goods related to each other?

ANSWER.

$$\text{Max}_{x_{-L}} \Pi(x_{-L}) = \text{Max}_{x_{-L}} \sum_{j=1}^{L-1} [\hat{p}_j(x_j) - c_j]x_j - F,$$

with FOC: $\hat{p}'(x_j)x_j + \hat{p}_j = c_j$, $j = 1,\ldots, L\text{-}1$,

which can be written in the Lerner-equation form

$$\frac{\hat{p}_j - c_j}{\hat{p}_j} = \frac{-\hat{p}'(x_j)x_j}{\hat{p}_j} = \frac{1}{-\varepsilon_j(p_j)}, \quad j = 1,\ldots, L\text{-}1,$$

where $\varepsilon_j(x_j)$ is the elasticity of the (direct) demand for good *j*. The monopolist's markups on the various goods are inversely related to their (absolute value of the direct) elasticity of demand.

2.6. For each of the Problems *SC* and *MC*, write its first-order conditions, and express them in the format of the Lerner equation. Compare across the four problems, and interpret.

ANSWER.

Problem *SC*. The Lagrangian is $\sum_{j=1}^{L-1}[u_j(x_j) - c_j x_j] - F - \lambda[\overline{\Pi} - \sum_{j=1}^{L-1}[\hat{p}_j(x_j) - c_j]x_j + F]$,

with FOC: $u'(x_j) - c_j + \lambda[\hat{p}'(x_j)x_j + \hat{p}_j - c_j] = 0$, $j = 1,\ldots, L\text{-}1$,

or, using (2.1), $\hat{p}_j - c_j + \lambda[\hat{p}'(x_j)x_j + \hat{p}_j - c_j] = 0$, $j = 1,\ldots, L\text{-}1$,

i. e., $[\hat{p}_j - c_j][1+\lambda] = -\lambda\hat{p}'(x_j)x_j$, $j = 1,\ldots, L\text{-}1$,

which can be written in the Lerner-equation form

$$\frac{\hat{p}_j - c_j}{\hat{p}_j} = \frac{\lambda}{1+\lambda}\frac{[-\hat{p}'(x_j)x_j]}{\hat{p}_j} = \frac{\lambda}{[1+\lambda]}\frac{1}{[-\varepsilon_j(p_j)]}, \quad j = 1,\ldots, L\text{-}1.$$

Again, the markups on the goods are inversely related to their (absolute value of the direct) elasticity of demand.

When the constraint is not binding, $\lambda = 0$ and we are back to 2.3. At the other extreme, as λ becomes large, we approach 2.5.

Problem *MC*. The Lagrangian is now

$$\sum_{j=1}^{L-1}[\hat{p}_j(x_j) - c_j]x_j - F - \mu[\bar{S} - \sum_{j=1}^{L-1}[u_j(x_j) - c_j x_j]]$$

with FOC: $\hat{p}'(x_j)x_j + \hat{p}_j - c_j + \mu[\hat{p}_j - c_j] = 0$, $j = 1,\ldots, L\text{-}1$,

where (2.1) has been used, i. e.,

$$\frac{\hat{p}_j - c_j}{\hat{p}_j} = \frac{1}{[1+\mu]}\frac{1}{[-\varepsilon_j(p_j)]}, \quad j = 1,\ldots, L\text{-}1.$$

Now, when the constraint is not binding, $\mu = 0$ and we are back to 2.5. At the other extreme, as μ becomes large, we approach 2.3.

2.7. Assume now that the functions u_j are quadratic, of the form:

$u_j : [0, a_j / b_j) : u_j(x_j) = a_j x_j - (1/2) b_j (x_j)^2$. Prove that Problems *SC* and *MC* are "dual" in the following sense:

(i) If $\bar{x}_{-L}$ solves Problem *SC*[$\bar{\Pi}$], and $\bar{S}$ is the value of Problem *SC*[$\bar{\Pi}$],

then $\bar{x}_{-L}$ solves Problem *MC*[$\bar{S}$].

(ii) If $\bar{\bar{x}}_{-L}$ solves Problem *MC*[$\bar{S}$] and $\bar{\bar{\Pi}}$ is the value of Problem *MC*[$\bar{S}$],

then $\bar{\bar{x}}_{-L}$ solves Problem *SC*[$\bar{\bar{\Pi}}$].

ANSWER. Note that the functions S and Π are strictly concave, which in particular implies that their upper level sets are convex sets. Hence, problems *SC*[$\bar{\Pi}$] and *MC*[$\bar{S}$] are concave problems, problem *SC*[$\bar{\Pi}$] has at most one solution, and so does problem *MC*[$\bar{S}$].

(i). Let $\bar{x}_{-L}$ solve Problem *SC*[$\bar{\Pi}$], with value $\bar{S}$, i. e., $S(\bar{x}_{-L}) = \bar{S}$, and $\Pi(\bar{x}_{-L}) \geq \bar{\Pi}$. By the necessity of the Kuhn-Tucker conditions, there is a nonnegative multiplier λ such that

$$\nabla S(\bar{x}_{-L}) + \lambda \nabla \Pi(\bar{x}_{-L}) = 0. \qquad (2.2)$$

First note that, because $S(\bar{x}_{-L}) = \bar{S}$, $\bar{x}_{-L}$ satisfies the constraint in Problem *MC*[$\bar{S}$].

If $\lambda = 0$, then $\nabla S(\bar{x}_{-L}) = 0$, i. e., $\bar{x}_{-L}$ is the unique maximizer of S, which implies that the set $\{x_{-L} : S(x_{-L}) \geq \bar{S}\}$ is the singleton $\{\bar{x}_{-L}\}$. But this is the constraint set of Problem $MC[\bar{S}]$. Hence, $\bar{x}_{-L}$ solves Problem $MC[\bar{S}]$.

If, on the contrary, $\lambda > 0$, then, defining $\mu \equiv 1/\lambda > 0$, (2.2) implies that $\nabla\Pi(\bar{x}_{-L}) + \mu\nabla S(\bar{x}_{-L}) = 0$, which is the Kuhn-Tucker condition of Problem $MC[\bar{S}]$. The sufficiency of these conditions in the concave case guarantees that $\bar{x}_{-L}$ solves Problem $MC[\bar{S}]$.

(ii) is proved in a parallel manner.

2.8. Consider the special case where $L = 2$ and $u(x_1, x_2) = ax_1 - (1/2)(x_1)^2 + x_2$. Compute and graph the functions S and Π in the same figure, where you are also asked to illustrate the solutions to problems *SU* and *MU*, as well as the "duality theorem" of the previous section.

Answer. $S(x_1) = ax_1 - (1/2)(x_1)^2 - F - cx_1$, with $\nabla S(x_1) = a - x_1 - c$.

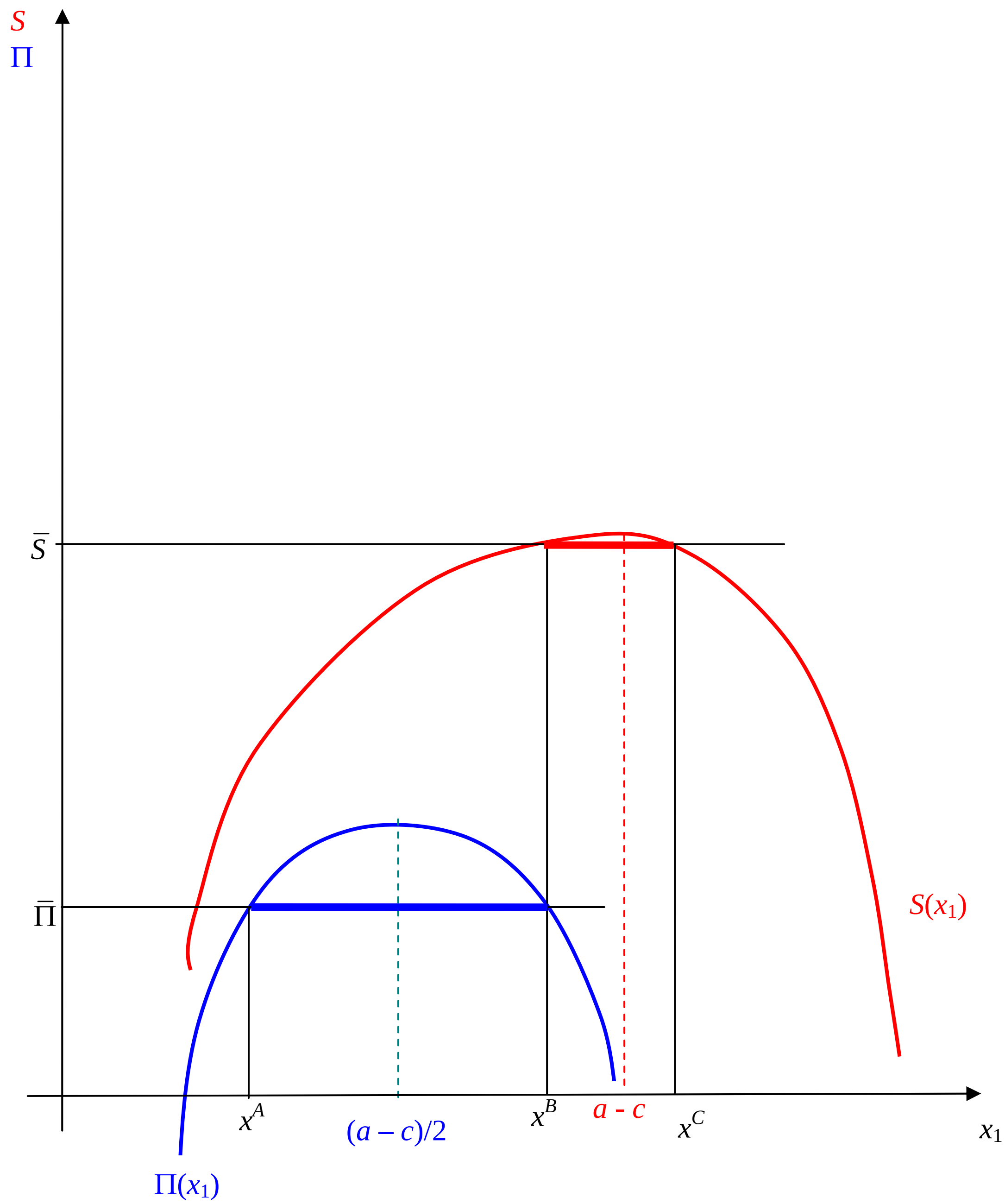
$\Pi(x_1) = [a - x_1]x_1 - F - cx_1$, with $\nabla\Pi(x_1) = a - 2x_1 - c$.

Problem *SU*. Solution at $x_1 = a - c$.

Problem *MU*. Solution at $x_1 = \dfrac{a-c}{2}$.

"Duality Theorem." Let $\bar{\Pi}$ be given, as illustrated in the Figure. Consider Problem $SC[\bar{\Pi}]$. The constraint set is $\{x_1 : \Pi(x_1) \geq \bar{\Pi}\} = [x^A, x^B]$, and S is maximized at x^B, with value $S(x^B) = \bar{S}$.

Consider now the "dual" problem $MC[\bar{S}]$.The constraint set is now $\{x_1 : S(x_1) \geq \bar{S}\} = [x^B, x^C]$, and Π is maximized at x^B, with value $\Pi(x^B) = \bar{\Pi}$.


S
Π
$\bar{S}$
$\bar{\Pi}$
$S(x_1)$
x^A
$(a - c)/2$
x^B
$a - c$
x^C
x_1
$\Pi(x_1)$

Question 3 - Answer Key

(a) Pareto optima:

$$\max \frac{1}{2}\log x_1 + \frac{1}{2}\log y \qquad \text{subject to}$$

$$\frac{1}{3}\log x_2 + \frac{2}{3}\log y \geq v_2 \qquad \alpha_2$$

$$n_1 x_1 + n_2 x_2 + y = w \qquad \lambda$$

no need to write non-negativity constraints because of the log utility functions

FOCs: $\frac{1}{2x_1} = \lambda n_1 \qquad \frac{1}{2y} + \alpha_2 \frac{2}{3y} = \lambda \qquad \frac{\alpha_2}{3x_2} = \lambda n_2$

(b) Solving the FOCs:

$$x_1 = \frac{1}{2\lambda n_1} \qquad x_2 = \frac{\alpha_2}{3\lambda n_2} \qquad y = \frac{1}{\lambda}\left(\frac{1}{2} + \frac{2}{3}\alpha_2\right)$$

replacing in feasibility constraint:

$$\frac{1}{2\lambda} + \frac{\alpha_2}{3\lambda} + \frac{1}{\lambda}\left(\frac{1}{2} + \frac{2}{3}\alpha_2\right) = w$$

$$\frac{1}{\lambda}\left(1 + \alpha_2\right) = w \quad \Rightarrow \quad \lambda = \frac{1+\alpha_2}{w}$$

$$x_1 = \frac{w}{2n_1}\frac{1}{1+\alpha_2} \qquad x_2 = \frac{\alpha_2}{1+\alpha_2}\frac{w}{3n_2} \qquad y = \frac{\frac{1}{2} + \frac{2}{3}\alpha_2}{1+\alpha_2} w$$

α_2 is related to v_2 by the equation

$$\frac{1}{3}\log \frac{\alpha_2}{1+\alpha_2}\frac{w}{3n_2} + \frac{2}{3}\log \frac{\frac{1}{2} + \frac{2}{3}\alpha_2}{1+\alpha_2} w = v_2$$

When $v_2 \to -\infty$ $(x_2 \to 0)$ α_2 tends to zero

When $v_2 \to \frac{1}{3}\log\frac{w}{3n_2} + \frac{2}{3}\log\frac{2w}{3}$ (which is the optimal allocation for the agents of type 2 if they can confiscate all the endowments of the agents of type 1), α_2 tends to infinity.

α_2 indicates the relative weight of agents of type 2 in a social welfare function. $u_1(x_1, y) + \alpha_2 u_2(x_2, y)$.

$$\frac{\partial y}{\partial \alpha_2} = \frac{\frac{2}{3}(1+\alpha_2) - \left(\frac{1}{2} + \frac{2}{3}\alpha_2\right)}{(1+\alpha_2)^2} = \frac{1/6}{(1+\alpha_2)^2} > 0$$

Since the agents of type 2 have a larger relative preference for the public good than the agents of type 1 (weights $\left(\frac{1}{3}, \frac{2}{3}\right)$ instead of $\left(\frac{1}{2}, \frac{1}{2}\right)$ in the Cobb Douglas utility function), the level of public good increases with the weight of the agents of type 2 in the social welfare.

When $\alpha_2 \to 0$ $y \to \frac{w}{2}$. When $\alpha_2 \to \infty$ $y \to \frac{2}{3}w$

The P.O. levels of public good lie in $\left(\frac{w}{2}, \frac{2w}{3}\right)$.

(c) voting equilibrium. Agents of type 2 have the majority so their optimal tax rate will be adopted in a voting equilibrium. Their optimal tax rate solves

$$\max_{\tau \geq 0} \frac{1}{3}\log(1-\tau) + \frac{2}{3}\log \tau w$$

FOC

$$-\frac{1}{1-\tau} + 2\frac{1}{\tau} = 0 \iff \frac{2}{\tau} = \frac{1}{1-\tau} \iff 1-\tau = \frac{\tau}{2}$$

$$\iff \frac{3\tau}{2} = 1 \implies \boxed{\tau = \frac{2}{3}}.$$

allocation $x_1 = \frac{4}{3}$ $x_2 = \frac{1}{3}$ $y = \frac{2}{3}w$

The level of public good is at the upper bound of the set of P.O. levels of public good. To check whether the allocation is PO, let us check whether the Samuelson condition holds

$$n_1 \frac{\frac{\partial u_1}{\partial y}(x_1, y)}{\frac{\partial u}{\partial x_1}(x_1, y)} + n_2 \frac{\frac{\partial u_2}{\partial y}(x_2, y)}{\frac{\partial u_2}{\partial x_2}(x_2, y)} = n_1 \frac{\frac{1}{2}\frac{1}{2/3\,w}}{\frac{1}{2}\frac{1}{4/3}} + n_2 \frac{\frac{2}{3}\frac{1}{2/3\,w}}{\frac{1}{3}\frac{1}{1/3}}$$

$$= \frac{2n_1 + n_2}{w} < 1 \quad \text{since} \quad w = 4n_1 + n_2$$

The sum of the propensities to pay is less than the marginal cost: because the tax rate is uniform and is chosen by the agents of type 2 who "like" the public good, the agents end up contributing too much toward the production of the public good. Reducing the public good by Δy, compensating each agent of type 1 by $\Delta x_1 = 2\Delta y$, each agent of type 2 by $\Delta x_2 = \frac{1}{w}\Delta y$ leaves a surplus

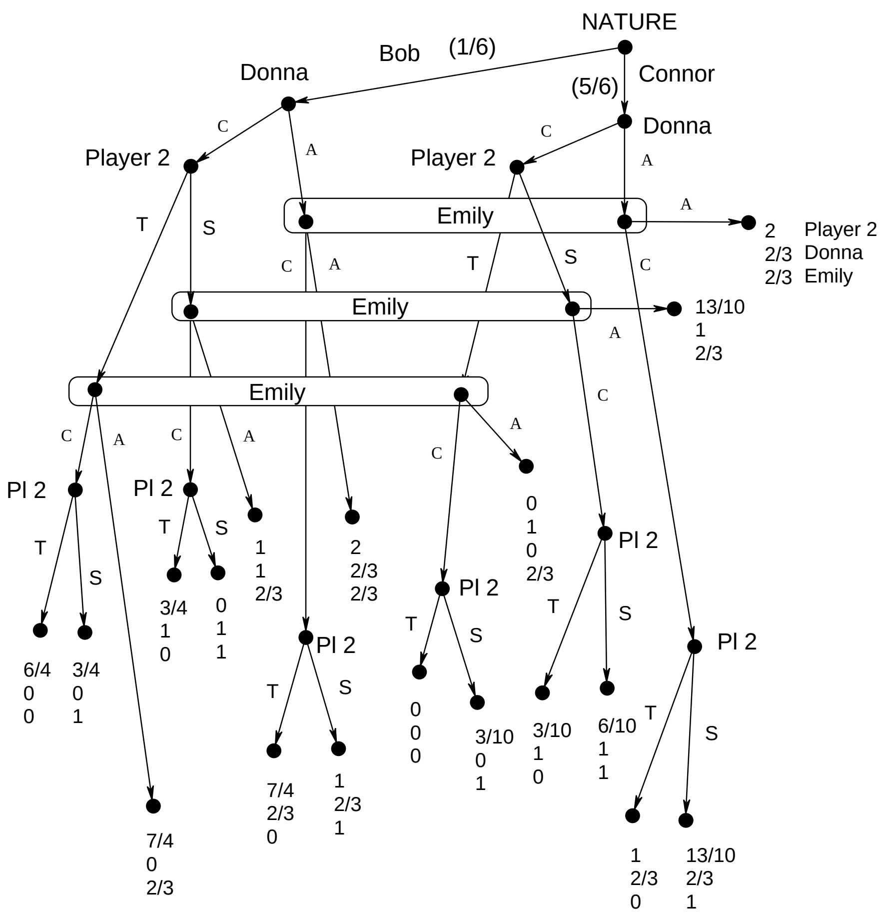
$\Delta y - \frac{2n_1 + n_2}{w}\Delta y > 0$ which can be distributed among the agents to increase the welfare of all agents. Note however that the basic compensation is different for the agents of different types, so that such marginal improvement cannot be achieved by decreasing the uniform marginal tax rate.

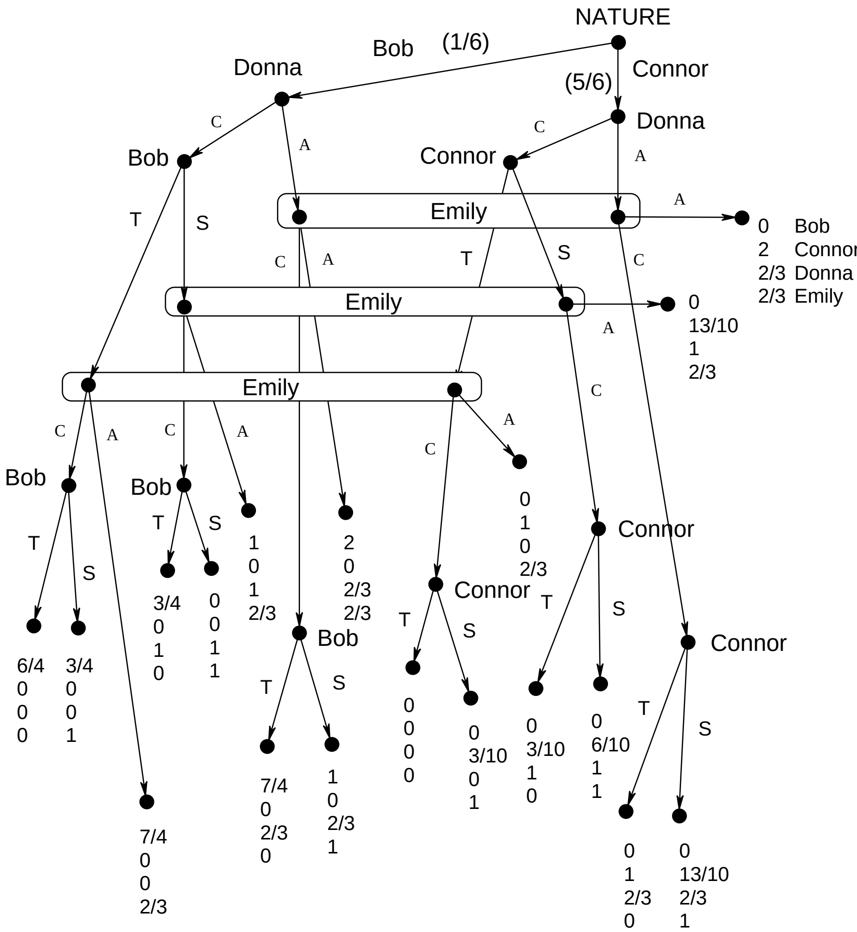
Micro Prelim. August 2011. Answer keys for Questions 4 and 5

QUESTION 4. (a) The von Neumann-Morgenstern normalized utility functions are as follows:

	Bob	*Connor*	*Donna*	*Emily*
(if A) w_1	1	1	$\frac{2}{3}$	$\frac{2}{3}$
(if (C,S)) w_2	0	$\frac{3}{10}$	1	1
(if (C,T)) w_3	$\frac{3}{4}$	0	0	0

For the extensive game there are two choices, one with 3 players (Donna, Emily, Player 2) as shown in the first figure below and the other with 4 players, as shown in the second figure. In some of the questions it is implicitly assumed that the representation is in terms of 4 players.

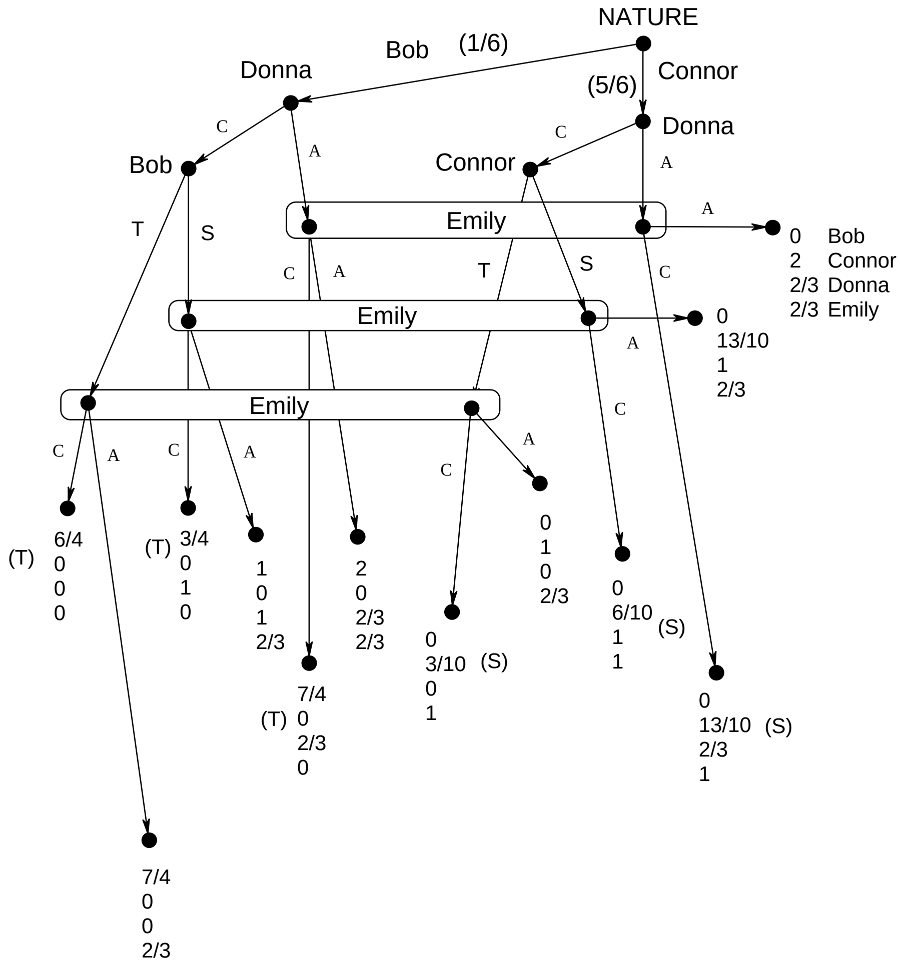




(b) Bob has 2^4 = 16 strategies (one of them is "if chosen as Player 2, I choose (T,S,S,T)" where the choices apply to Bob's nodes top first and then left to right), Connor has 2^4 = 16 strategies (one of them is "if chosen as Player 2, I choose (T,S,S,T)" where the choices apply to Bob's nodes top first and then left to right), Donna has 4 strategies (one of them is "against Bob I play *A*, against Connor I play *C*") and Emily has 2^3 = 8 strategies (one of them is (C,A,C) where the choices apply to Emily's information sets from top to bottom).

(c) Consider the strategy profile $\left((T,T,T,T),(S,S,S,S),(A,A),(A,A,A)\right)$. Then the payoffs are $\left(\frac{2}{6},\frac{10}{6},\frac{2}{3},\frac{2}{3}\right)$.

(d) At every subgame-perfect equilibrium Bob and Connor choose the payoff-maximizing action at each of the singleton nodes. Thus the game can be simplified to:

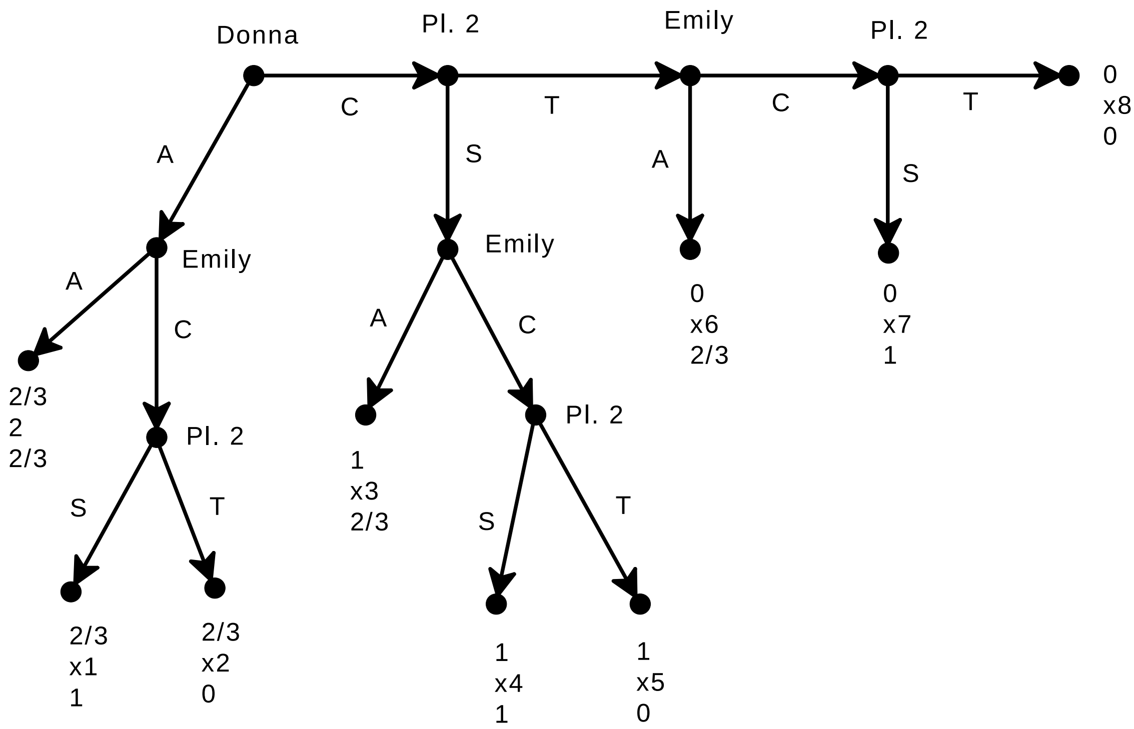


One weak sequential equilibrium of this game is: $\left(T,T,(A,A),(C,C,A)\right)$ with beliefs for Emily (from top to bottom) $\left(\left(\frac{1}{6},\frac{5}{6}\right),(0,1),(1,0)\right)$

Another weak sequential equilibrium is $\left(T,S,(A,C),(A,C,C)\right)$ with beliefs for Emily (from top to bottom) $\left((1,0),(0,1),(0,1)\right)$

(e) Then we would have a perfect-information game (the information sets of Emily given in the figure above would be eliminated; hence Emily's strategy now has to specify a choice at each of **six** nodes, so that she has $2^6 = 64$ strategies). Both Donna and Emily then would know that against Bob *C* would be followed by *T*, while against Connor *C* would be followed by *S*. Thus the backward induction solution would be $\left((T,T,T,T),(S,S,S,S),(A,C),(A,C,A,C,A,C)\right)$ (Emily's strategy is read as follows: top-left, top-right, middle left, middle-right, bottom left, bottom-right).

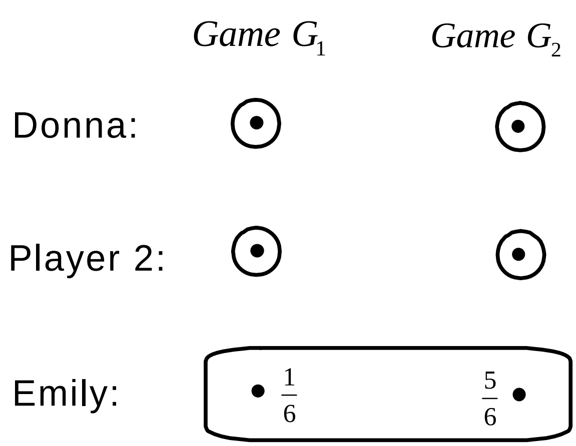
(f) The game of part (a) can be viewed as a situation of one-sided incomplete information among three players: Donna, Player 2 and Emily, where Emily is uncertain about the payoffs of Player 2. The game is one of perfect information, as follows:



Call G_1 the above game with $x_1 = 1, x_2 = \frac{7}{4}, x_3 = 1, x_4 = 0, x_5 = \frac{3}{4}, x_6 = \frac{7}{4}, x_7 = \frac{3}{4}, x_8 = \frac{6}{4}$. (These are the payoffs of Bob in the game of part a.)

Call G_2 the above game with $x_1 = \frac{13}{10}, x_2 = 1, x_3 = \frac{13}{10}, x_4 = \frac{6}{10}, x_5 = \frac{3}{10}, x_6 = 1, x_7 = \frac{3}{10}, x_8 = 0$. (These are the payoffs of Connor in the game of part a.)

The situation of incomplete information can then be represented using two states, as follows:



QUESTION 5.

(a) **(a.1)** Whenever $p_G R > (1+r)X > p_B R$ (so that there does not exist a $w \le R$ such that $p_B w - (1+r)X > 0$ but there exists a $w \le R$ such that $p_G w - (1+r)X > 0$ and t).

(a.2) Each bank would offer any type *G* entrepreneur a loan of $*X* with repayment $w = \frac{X(1+r)}{p_G}$. No bank would offer a loan to type *B* entrepreneurs. The expected utility of a type *G* entrepreneur would be $U_G = p_G\left(R - \frac{X(1+r)}{p_G}\right) = p_G R - X(1+r) > 0$ and the expected utility of a type *B* entrepreneur would be $U_B = 0$.

(b) A bank's expected profit from a loan is $w\left[p_B\lambda + (1-\lambda)p_G\right] - X(1+r)$. The zero profit condition yields $w\left[p_B\lambda + (1-\lambda)p_G\right] - X(1+r) = 0$, that is, $w = \frac{X(1+r)}{p_B\lambda + (1-\lambda)p_G}$. So $U_G = p_G\left(R - \frac{X(1+r)}{p_B\lambda + (1-\lambda)p_G}\right)$ and $U_B = p_B\left(R - \frac{X(1+r)}{p_B\lambda + (1-\lambda)p_G}\right)$. In order for this to be an equilibrium it must be that $U_B \ge 0$ (which implies $U_G \ge 0$), that is, $\lambda \le \frac{R - X(1+r)}{R(p_G - p_B)}$: the fraction of high-risk borrowers should not be too high.

(c) $U_G(c, w) = p_G(R-w) - (1+\rho)c$. $U_B(c, w) = p_B(R-w) - (1+\rho)c$

(d) First of all, note that nobody would offer a collateral $c > c_0$. Furthermore, the zero-profit condition requires $w = \hat{w} \equiv \frac{(X - c_0)(1+r)}{p_G}$. The *B* types will not apply if and only if $p_B(R - \hat{w}) - (1+\rho)c_0 \le 0$. The *G* types will apply if and only if $p_G(R - \hat{w}) - (1+\rho)c_0 > 0$. Thus we need $\frac{p_B(R-\hat{w})}{1+\rho} \le c_0 < \frac{p_G(R-\hat{w})}{1+\rho}$ (♦)

(e) **(e.1)** (b) $w = 500/7 = 71.429$, $U_G = 200/7 = 28.57$, $U_B = 20/7 = 2.857$.

(c) $U_G(c, w) = 100 - w - \frac{11}{10}c$; $U_B(c, w) = \frac{1}{10}(100 - w) - \frac{11}{10}c = 10 - \frac{1}{10}w - \frac{11}{10}c$.

(d) $w = 50 - c_0$, $5 \le c_0 \le X = 50$ $(< R = 100)$ [the RHS of (♦) would give $c_0 < 500$, thus the binding constraint becomes $c_0 \le X$].

(e.2) The lowest value of c_0 is 5. In the separating equilibrium $w = 50 - 5 = 45$ and thus $U_G = 100 - 45 - (1 + \frac{1}{10})5 = 49.5$ and $U_B = 0$. So, relative to pooling, signaling helps the *G* types and hurts the *B* types. It is also more efficient: under pooling the expected average utility is $\lambda(2.857) + (1-\lambda)28.57 = 20$ while with signaling it equals $\lambda(0) + (1-\lambda)49.5 = 33$, close to the average expected utility without asymmetric information, which is $(1-\lambda)(100-50) = \frac{2}{3}50 = 33.33$.