MICROECONOMIC THEORY PRELIM FALL 2005 ANSWER KEY

QUESTION 1

Throughout this question we assume that consumers have preferences over the quantities of two market goods, $x_1 \geq 0$ and $x_2 \geq 0$, as well as the size $g \geq 0$ of a park. A consumer has no control over g, and therefore takes the value g as given, but he or she spends all his of her (after-tax) wealth w, also given to him or her, buying x_1 and x_2 in the market at the given prices (p_1, p_2). In summary, in his or her Walrasian demand the consumer takes (p_1, p_2, w, g) as given, and chooses x_1 and x_2 subject to the budget constraint $p_1x_1 + p_2x_2 \leq w$. Throughout this question we assume that wealth is always large enough so that at the chosen point the amount of x_2 is strictly positive.

1.1. Consumer Karl's preference relation on $\Re^3_+$ can be represented by the utility function $\hat{u} : \Re^3_+ \to \Re_+ : \hat{u}(x_1, x_2, g) = 2\sqrt{(g+1)x_1} + x_2$.

1.1(a). Compute Karl's Walrasian demand functions $\tilde{x}_1(p_1, p_2, w, g)$ and $\tilde{x}_2(p_1, p_2, w, g)$, and indirect utility function. Comment.

ANSWER. Given p_1, p_2, w and g, choose x_1 and x_2 in order to maximize $2\sqrt{(g+1)x_1} + x_2$ subject to $p_1 x_1 + p_2x_2 = w$, or, equivalently, choose x_1 in order to maximize $2\sqrt{(g+1)x_1} + \frac{w - p_1x_1}{p_2}$, with FOC $\frac{\sqrt{g+1}}{\sqrt{x_1}} = \frac{p_1}{p_2}$, i.e.,

$$\tilde{x}_1(p_1, p_2, w, g) = \frac{p_2^2}{p_1^2}(g+1),$$

and therefore $\tilde{x}_2(p_1, p_2, w, g) = \frac{w}{p_2} - \frac{p_1\tilde{x}_1(p_1, p_2, w, g)}{p_2} = \frac{w}{p_2} - \frac{p_1\frac{p_2^2}{p_1^2}(g+1)}{p_2} = \frac{w}{p_2} - \frac{p_2}{p_1}(g+1)$.

Indirect utility: $2\sqrt{(g+1)\frac{p_2^2}{p_1^2}(g+1)} + \frac{w}{p_2} - \frac{p_2}{p_1}(g+1) = (g+1)\frac{p_2}{p_1} + \frac{w}{p_2}$.

1.1(b). Let $(p_1, p_2) = (1, 1)$, and let Karl's initial wealth be 10. The government is considering a program that would improve g from 0 to 3, and would require Karl to pay a tax of 2. The prices (p_1, p_2) would not change as a result of the program. Would Karl benefit from the program?

ANSWER. Karl's indirect utility without the program $= (0+1)\frac{1}{1} + \frac{w}{1} = 1 + w$.

Karl's indirect utility with the program $= (3+1)\frac{1}{1} + \frac{w-2}{1} = 4 + w - 2 > 1 + w$.

Thus, Karl benefits from the program (his willingness to pay for the improvement in g is 3, greater than the tax).

1.2. Karl has a cousin, Priscilla, whose preference relation on $\Re^3_+$ can be represented by the utility function

$$\frac{2\sqrt{(g+1)x_1}+x_2}{g+1}.$$

1.2(a). Is Priscilla's preference relation on $\Re^3_+$ identical to Karl's?

ANSWER. No way. Notice that for Karl the park is a good, whereas for Priscilla it is a bad (perhaps the park's deer eat her vegetable garden). Note that Priscilla's utility function is NOT an increasing transformation of Karl's.

1.2(b). Compute Priscilla's Walrasian demand functions for x_1 and x_2.

ANSWER. Same as Karl's, because she treats $g + 1$ as a constant in her UMAX problem

1.2(c). Would Priscilla benefit from the program of 1.2 above? (Her initial wealth is 15, and her tax contingent to the implementation of the program would also be 2). Comment

ANSWER. No: the program would add insult to injury, because for her the park is a bad. Her "willingness to pay" for the park is actually negative.

1.3. More generally, let a consumer's preferences be represented by a utility function of the form $\hat{\hat{u}}:\Re^3_+\to\Re:\hat{\hat{u}}(x_1,x_2,g)=\varphi(\hat{u}(x_1,x_2,g),g)$, where $\varphi:\Re^2_+\to\Re$ is a differentiable function with strictly positive first partial derivative, and $\hat{u}$ is defined in 1.1 above. Without further information:

1.3(a). What can you say about the Walrasian demand functions for x_1 and x_2 obtained in the UMAX problem for $\hat{\hat{u}}$?

ANSWER. Note that Priscilla's is of the $\hat{\hat{u}}$ form, because $\frac{2\sqrt{(g+1)x_1}+x_2}{g+1}=\frac{\hat{u}(x_1,x_2,g)}{g+1}$. By the same reason as in 1.2(b), the functions $\hat{u}$ and $\hat{\hat{u}}$ yield the same Walrasian demand functions for x_1 and x_2 . Therefore, these functions are exactly the ones computed in 1.1(a) above.

1.3(b). What can you say about the consumer's willingness to pay for g when his or her utility function is of the $\hat{\hat{u}}$ form?

ANSWER. Nothing can be said, as illustrated by the Karl vs. Priscilla example.

1.4. Suppose now that the function $\hat{\hat{u}}$, as defined in 1.3 above, satisfies the following assumption: For all $(x_2,g)\in\Re^2_+,\ \frac{\partial\hat{\hat{u}}(0,x_2,g)}{\partial g}=0,$ i. e., the consumer does not care about g whenever he or she does not consume good 1.

1.4(a). Argue that now $\hat{\hat{u}}(x_1,x_2,g)=\xi(\hat{u}(x_1,x_2,g))$ for some increasing function ξ.

ANSWER. Because, for all $(x_2, g) \in \Re^2_+, \dfrac{\partial \hat{\hat{u}}(0, x_2, g)}{\partial g} = 0$, we have that

$$\hat{\hat{u}}(0, x_2, g^0) = \hat{\hat{u}}(0, x_2, g^1), \forall (x_2, g^0, g^1) \in \Re^3_+,$$

or by the definition of the form $\hat{\hat{u}}$ in 1.3,

$$\varphi(\hat{u}(0, x_2, g^0), g^0) = \varphi(\hat{u}(0, x_2, g^1), g^1), \forall (x_2, g^0, g^1) \in \Re^3_+,$$

which, by the definition of $\hat{u}$ in (1) yields

$$\varphi(x_2, g^0) = \varphi(x_2, g^1), \forall (x_2, g^0, g^1) \in \Re^3_+.$$

This implies that $\varphi(u, g)$ is constant with respect to its second argument, whereas by assumption it is increasing in its first argument. Thus, $\varphi(u, g)$ can be written as $\xi(u)$, an increasing transformation.

1.4(b). What can you now say about the willingness to pay for *g*?

ANSWER. Because, under the assumption that for all $(x_2, g) \in \Re^2_+, \dfrac{\partial \hat{\hat{u}}(0, x_2, g)}{\partial g} = 0$, $\hat{\hat{u}}$ represents Karl's preference relation, the willingness to pay for *g* of a consumer with utility function $\hat{\hat{u}}$ is the same as Karl's.

Answer Key : Prelim Micro - Fall 2005 - Question 3

(1)

(1.) Lindahl equilibrium

Agents buy the public good with personalized prices p_i. Agent i solves: (l is leisure, L is labor)

$$\max \ln x_i + \ln l_i + \ln y_i$$

subject to $x_i + p_i y_i + w l_i = w \alpha_i$ (no profit because of CR) constant returns implies that maximization of profit by the firms with the wage rate must be $w = 1$.

solution is

$$x_i = \frac{\alpha_i}{3} \quad y_i = \frac{\alpha_i}{3 p_i} \quad l_i = \frac{\alpha_i}{3} \Rightarrow L_i = \frac{2\alpha_i}{3}$$

equilibrium conditions:

- $\sum_i p_i = 1$ (public firm) (I)
- $\exists y \quad y_i = y \quad \forall i$ (public good) (II)
- $\sum_i x_i + y = \sum_i L_i$ (market clearing for good). (III)

By Walras Law, the labor market clears.

(II) $\Rightarrow p_i = k \alpha_i$. (I) $\Rightarrow k \sum \alpha_i = 1 \Rightarrow k = \frac{1}{\sum \alpha_i} = \frac{1}{3I}$

Thus $y_i = y = \dfrac{\alpha_i}{3 \frac{\alpha_i}{3I}} = I$ and (III) is satisfied

(Walras Law).

The Lindahl alloc is

$$\boxed{x_i = \frac{\alpha_i}{3} \quad L_i = \frac{2\alpha_i}{3} \quad y = I}$$

(2a) (1) All P.O. alloc can be found by varying the weights of the agents in the social welfare functn

$$\max \quad \sum \mu_i \left(\ln(x_i) + \ln(\alpha_i - L_i) + \ln y \right)$$

subject to $\quad \sum x_i + y = \sum_{i=1}^{I} L_i \qquad \lambda$

(We are interested only in allocations in which $L_i > 0$ for all i, so that we solve for allocations in which no non-negativity constraint is binding). The FOCs are

$$\frac{\mu_i}{x_i} = \lambda \qquad \frac{\mu_i}{\alpha_i - L_i} = \lambda \qquad \frac{\sum_i \mu_i}{y} = \lambda$$

w. l. o. g $\quad \sum_i \mu_i = 1$

From the resource constraint : $\frac{1}{\lambda} + \frac{1}{\lambda} = 3I - \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{I}$

so that $\boxed{y = \frac{1}{\lambda} = I}$

(2b) Uniform taxation

The government chooses $\eta = I$. The cost is I (since the wage must be equal to 1). The lump sum tax for each agent is 1.

agent i maximizes $\ln x_i + \ln l_i$

subject to $x_i + l_i = \alpha_i - 1$

which gives $x_i = \frac{\alpha_i}{2} - \frac{1}{2}$ $l_i = \frac{\alpha_i}{2} - \frac{1}{2}$ $\Rightarrow L_i = \frac{\alpha_i}{2} + \frac{1}{2}$

Richer agents work and consume more than poorer ones. Intuitively the allocation must be farther than the Lindahl equilibrium from being egalitarian

Since the taxation is uniform while it is progressive in the Lindahl equilibrium: p_i is proportional to α_i. In the Lindahl equilibrium wealthier agents want "more" public good so they pay more.

(2c) To see that, let us take a simple example with 2 agents, 1 poor 1 rich ($\alpha_1 < \alpha_2$), α_1
Note that since $\alpha_1 < \alpha_2$ and $\alpha_1 + \alpha_2 = 6$, it must be that $\alpha_1 < \frac{6}{2} = 3$ and $\alpha_2 > \frac{6}{2} = 3$

$\frac{\alpha_i}{2} - 1/2 < \frac{\alpha_i}{3} \Leftrightarrow 3\alpha_i - 3 < 2\alpha_i \Leftrightarrow \alpha_i < 3$. So in the equilibrium with uniform taxation the poor agent consumes less than in the Lindahl equilibrium while the rich agent consumes more.

$$\frac{\alpha_i}{2} + \frac{1}{2} < \frac{2\alpha_i}{3} \iff 3\alpha_i + 3 < 4\alpha_i \iff \alpha_i > 3$$

In the equilibrium with uniform taxation the rich agent works less than in the Lindahl equilibrium while the poor agent works more.

Microeconomics Prelim September 2005
Answer Keys for Questions 4 and 5

4.

(a) A necessary and sufficient condition for cars of quality $1{,}000i$ to be *offered* for sale at price P is $P > 1{,}000i$. Fix $s \in \{1,2,\ldots,n\}$. Thus *all and only* cars of quality $i \in \{1,\ldots,s\}$ will be offered for sale if and only if $1{,}000s < P < 1{,}000(s+1)$. Let m be the number of cars of each quality level and M the total number of cars. Then $M = n\,m$ and the fraction of cars of any quality is $\frac{m}{m\,n} = \frac{1}{n}$ (uniform distribution). A buyer who purchases a car at a price P with $1{,}000s < P < 1{,}000(s+1)$ faces a lottery where he gets a car of quality $i \in \{1,\ldots,s\}$ with probability $\frac{m}{s\,m} = \frac{1}{s}$ (uniform distribution). Thus her expected utility if she buys is

$$\omega - P + \sum_{i=1}^{s}\left[\frac{1}{s}\alpha(1{,}000i)\right] = \omega - P + \frac{1{,}000\alpha}{s}\sum_{i=1}^{s} i = \omega - P + \frac{1{,}000\alpha}{s}\left(\frac{s(s+1)}{2}\right) = \omega - P + \frac{1{,}000\alpha(s+1)}{2}$$

(where ω is her initial endowment of money), whereas her utility if she does not buy is ω. Thus she will buy if and only if $P < \frac{1{,}000\alpha(s+1)}{2}$. Hence necessary and sufficient conditions for *all and only* cars of quality $i \in \{1,\ldots,s\}$ to be traded are

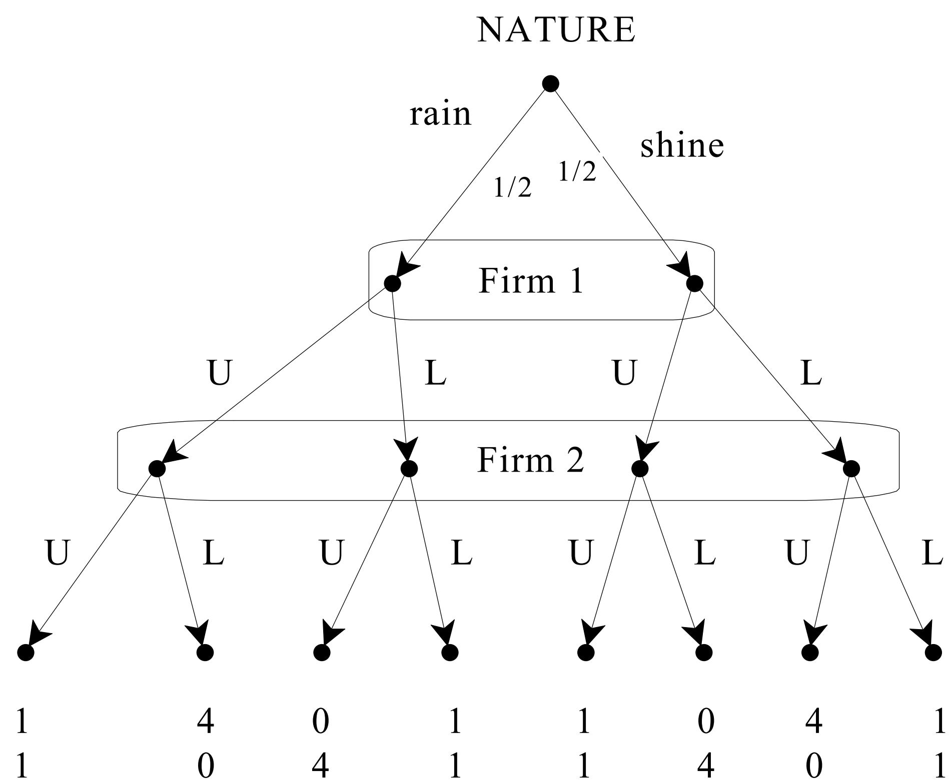
(1) $1{,}000s < P < 1{,}000(s+1)$ and (2) $P < \frac{1{,}000\alpha(s+1)}{2}$. There is a P that satisfies both inequalities if and only if $1{,}000s < \frac{1{,}000\alpha(s+1)}{2}$ i.e. if and only if $\boxed{\alpha > \frac{2s}{s+1}}$

(b) Since Group 2 individuals value cars more than Group 1 individuals, Pareto efficiency requires that all cars be traded. By the previous analysis, this can happen if and only if $\alpha > \frac{2n}{n+1}$.

(c) When $n = 9$ (so that the quality levels are 1,000, 2,000, …, 9,000) and $\alpha = 1.76$, the condition $\alpha > \frac{2s}{s+1}$ is satisfies if and only if $s \le 7$. Thus there is an equilibrium where all cars of quality up to 7,000 are traded, so that the maximum number of cars that can be traded in equilibrium is 2,000(7) = 14,000. Such an equilibrium requires P to be such that $7{,}000 < P < \frac{1{,}000(1.76)(8)}{2} = 7{,}040.$

5.

(a) The extensive form is as follows: (U stands for umbrellas and L for lotion; the top payoff is firm 1's profit and the bottom payoff is firm 2's profit).

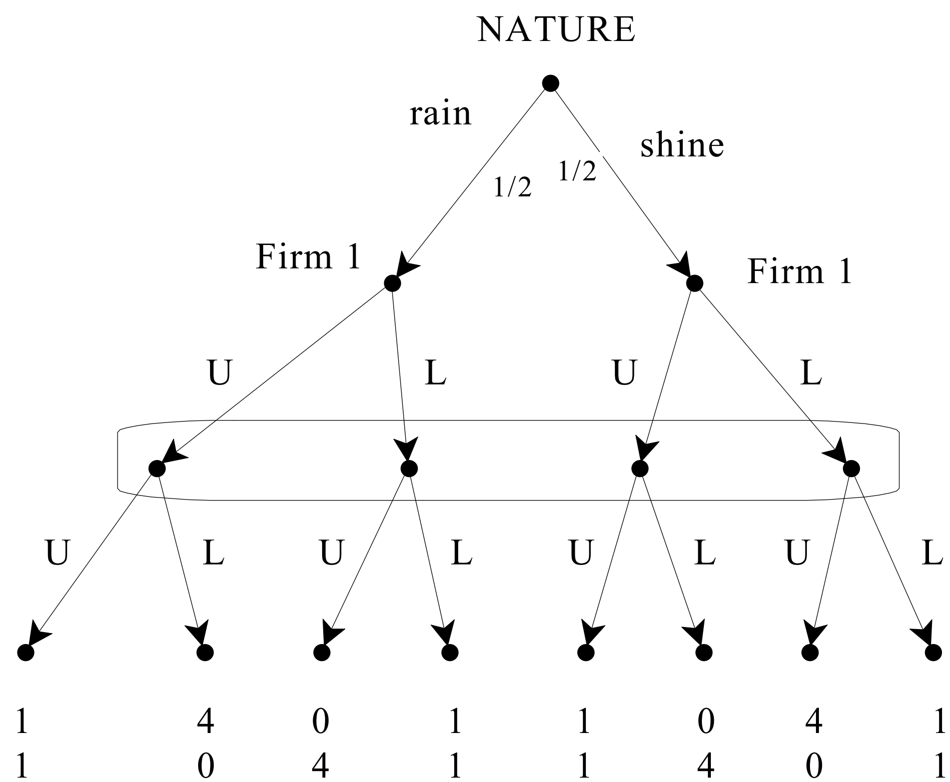


(b) The normal form is as follows:

			Firm 2
		U	L
Firm 1	U	**1 , 1**	**2 , 2**
	L	**2 , 2**	**1 , 1**

There are two pure-strategy Nash equilibria: (L,U) and (U,L). The two equilibria are identical in terms of expected payoffs: each firm has an expected payoff of 2 in both equilibria.

(c) The extensive form is as follows:

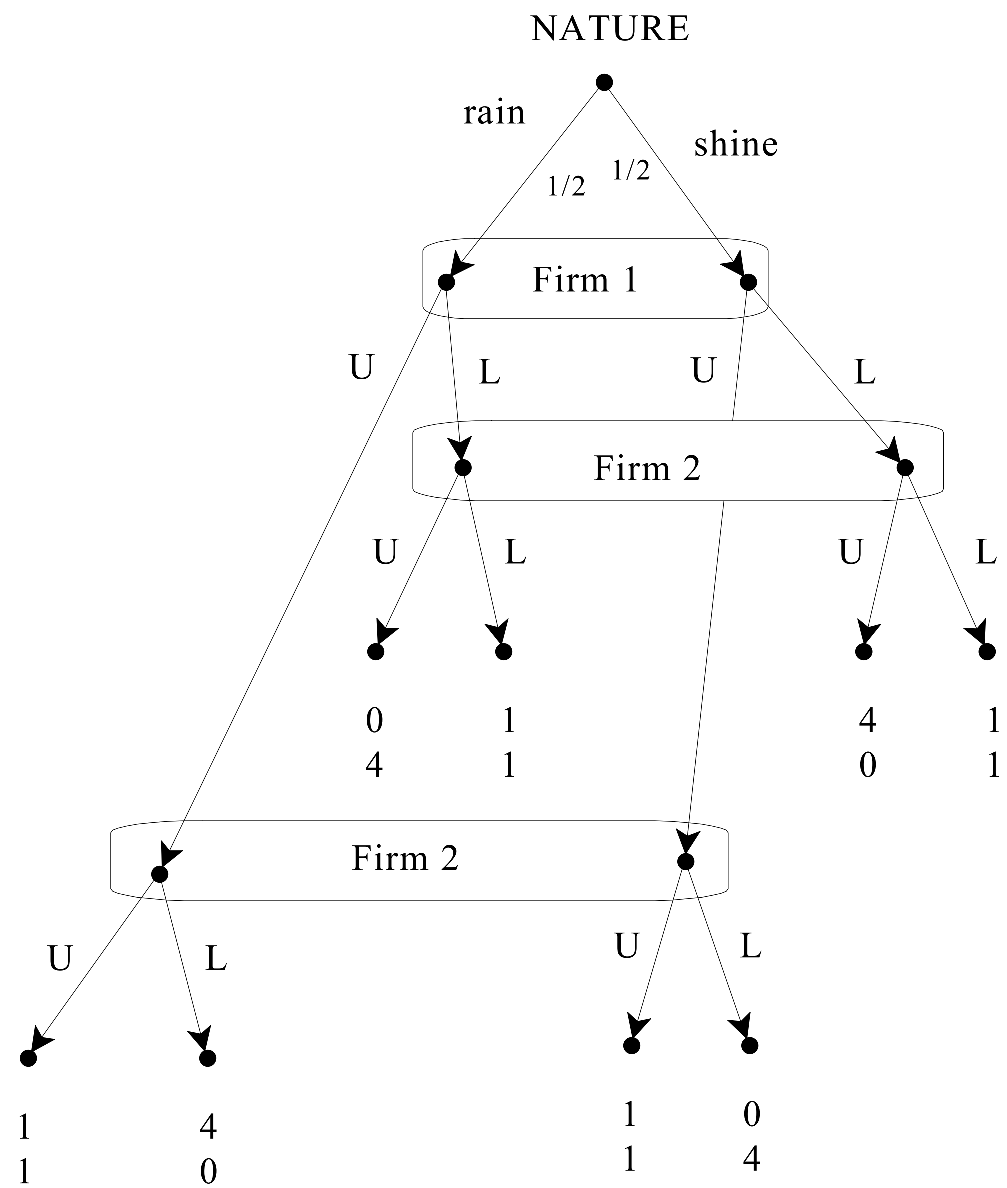


(d) Firm 1 now has two information sets and thus has four strategies. The normal form is as follows:

		Firm 2	
		U	L
	U always	**1 , 1**	**2 , 2**
Firm 1	U if rain, L if shine	**2.5 , 0.5**	**2.5 , 0.5**
	U if shine, L if rain	**0.5 , 2.5**	**0.5 , 2.5**
	L always	**2 , 2**	**1 , 1**

There are two pure-strategy Nash equilibria: (U if rain/L if shine, U) and (U if rain/L if shine, L). The two equilibria are identical in terms of expected payoffs: firm 1 has an expected payoff of 2.5 in both equilibria and firm 2 has an expected payoff of 0.5 in both equilibria.

(e) The extensive game is as follows:

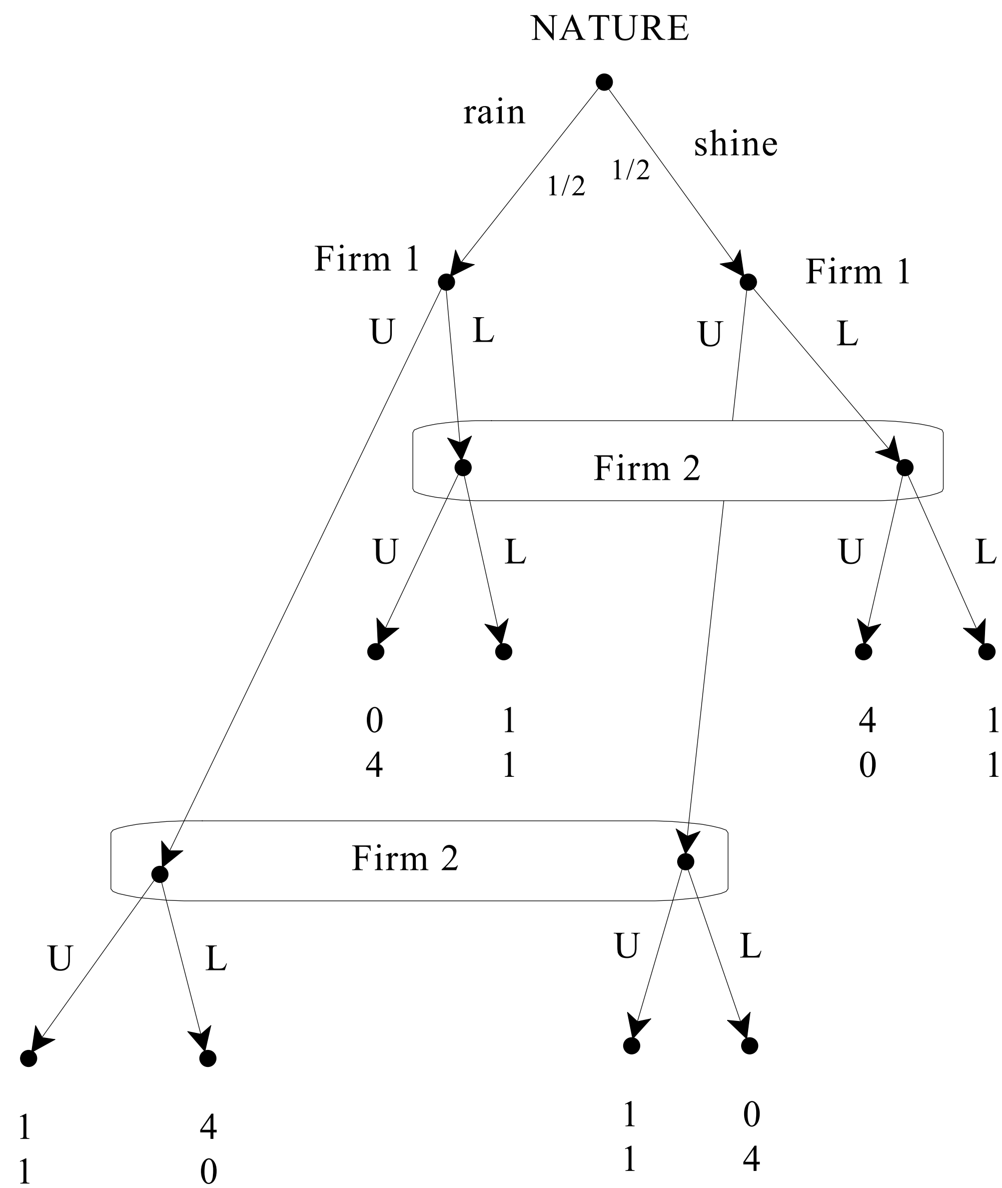


(f) Now firm 2 has two information sets and therefore four strategies.

		Firm 2			
		U always	L always	U if U, L if L	U if L, L if U
Firm 1	U	**1 , 1**	**2 , 2**	**1 , 1**	**2 , 2**
	L	**2 , 2**	**1 , 1**	**1 , 1**	**2 , 2**

There are four pure-strategy Nash equilibria: (U, L always), (U, U if L/L if U), (L, U always) and (L, U if L/L if U). The four equilibria are identical in terms of expected payoffs: both firms have an expected payoff of 2.

(g)



(h) Now each firm has two information sets and thus four strategies.

		Firm 2			
		U always	L always	U if U, L if L	U if L, L if U
Firm 1	U always	**1 , 1**	**2 , 2**	**1 , 1**	**2 , 2**
	U if rain, L if shine	**2.5 , 0.5**	**2.5 , 0.5**	**1 , 1**	**4 , 0**
	U if shine, L if rain	**0.5 , 2.5**	**0.5 , 2.5**	**1 , 1**	**0 , 4**
	L always	**2 , 2**	**1 , 1**	**1 , 1**	**2 , 2**

Now there is only one pure-strategy equilibrium, namely (U if rain/L if shine, U if U/L if L) with expected payoff of 1 each.

(i) Comparing Case 2 to Case 1, the additional information given to firm 1 in Case 2 (the state of the weather) is advantageous to firm 1: its expected equilibrium payoff is 2.5 while it was only 2 in Case 1. On the other hand, firm 2 is hurt by the extra information given to firm 1, because its expected equilibrium payoff is reduced from 2 to 0.5.

(j) Comparing Case 4 to Case 3 we get the surprising result that the additional information given to firm 1 in Case 4 (the state of the weather) hurts *both* firms, in particular also firm 1: the expected payoff of each firm is reduced from 2 to 1. This may seem paradoxical, since in one-person contexts more information is always advantageous. In interactive situations this is no longer true, as the above example shows: more information may be detrimental. The intuitive reason why this is happening here is as follows. Once firm 1 is informed about the weather, it becomes a dominant strategy for it to produce umbrellas if it rains and lotion if it shines. Since firm 2 knows that firm 1 knows what the weather will be like and observes firm 1's choice, firm 2 can infer from firm 1's choice what the weather will be like and the best reply to firm 1's choice is also to produce umbrellas if it rains (that is, if firm 1 chose to produce umbrellas) and to produce lotion if it shines (that is, if firm 1 chose to produce lotion). Thus both firms end up doing the same and making a profit of 1.

[The difference between Case 4 and Case 2 is that in Case 4 firm 2 can adjust its choice to firm 1's choice, which it observes, while in Case 2 firm 2 does not observe firm 1's choice.]