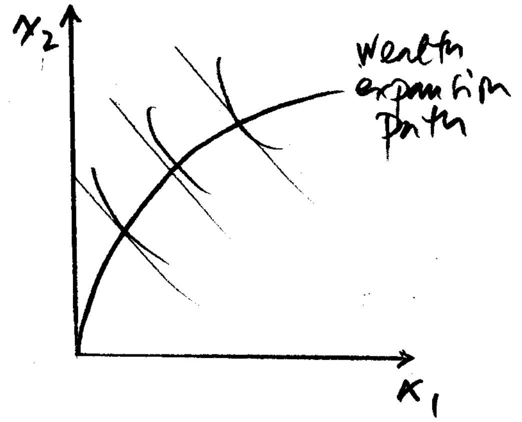
Question 1

1(a) NO. Example: $u(x_1, x_2) = \sqrt{x_1} + \ln x_2$

FOC: $$\left.\begin{aligned} \frac{1}{2\sqrt{x_1}} &= \lambda p_1 \\ \frac{1}{x_2} &= \lambda p_2 \end{aligned}\right\} \frac{x_2}{2\sqrt{x_1}} = \frac{p_1}{p_2},$$



i.e., the wealth expansion path, given by the equation $x_2 = 2\frac{p_1}{p_2}\sqrt{x_1}$, is not a straight line.

Thus, no representative consumer exists.

Note. Several popular preferences (such as Stone-Geary, Cobb-Douglas, quasilinear for $L=2$, CES) can be represented by utility functions of the form (1.1), and also by indirect utility functions of the Gorman form, in which case a positive representative consumer exists. But (1.1) by itself is insufficient to guarantee the existence of a positive representative consumer, even with identical utility functions for all consumers.

1(b). $$\left.\begin{aligned} u_1'(\tilde{x}_1(p,w)) &= \lambda(p,w)p_1 \\ &\vdots \\ u_j'(\tilde{x}_j(p,w)) &= \lambda(p,w)p_j \\ &\vdots \\ u_L'(\tilde{x}_L(p,w)) &= \lambda(p,w)p_L \end{aligned}\right\} L \text{ equations} \quad (1.5)$$

$$p \cdot \tilde{x}(p,w) = w \qquad 1 \text{ equation} \quad (1.6)$$

Note that (1.5) implies that $\lambda(p,w) > 0$.

1(c) Differentiating equations (1.5) w.r.t. $p_1, \dots, p_L$ and w, we obtain

$$u''_j \cdot \frac{\partial \tilde{x}_j}{\partial p_k} = \frac{\partial \lambda}{\partial p_k} \cdot p_j, \quad \left.\begin{array}{l} j = 1, \dots, L \\ k = 1, \dots, L \\ j \neq k \end{array}\right\} \left(L(L-1) \text{ equations}\right) \tag{1.7}$$

$$u''_k \cdot \frac{\partial \tilde{x}_k}{\partial p_k} = \frac{\partial \lambda}{\partial p_k} \cdot p_k + \lambda, \quad k = 1, \dots, L \quad (L \text{ equations}) \tag{1.8}$$

$$u''_j \cdot \frac{\partial \tilde{x}_j}{\partial w} = \frac{\partial \lambda}{\partial w} \cdot p_j, \quad j = 1, \dots, L \quad (L \text{ equations}) \tag{1.9}$$

while differentiating (6) w.r.t. $p_1, \dots, p_L$ and w, we obtain

$$\sum_{j=1}^{L} p_j \frac{\partial \tilde{x}_j}{\partial p_k} = -\tilde{x}_k, \quad k = 1, \dots, L \quad (L \text{ equations}) \tag{1.10}$$

$$\sum_{j=1}^{L} p_j \frac{\partial \tilde{x}_j}{\partial w} = 1 \quad (1 \text{ equation}) \tag{1.11}$$

Total number of equations: $L^2 - L + L + L + L + 1 = L^2 + 2L + 1 = (L+1)^2$.

1(d) $v(p,w) = \sum_j u_j(\tilde{x}_j(p,w))$.

Thus
$$\frac{\partial v}{\partial w} = \sum_j u'_j \cdot \frac{\partial \tilde{x}_j}{\partial w} = \sum_j \lambda \cdot p_j \frac{\partial \tilde{x}_j}{\partial w} \quad [\text{by } (1.5)]$$
$$= \lambda, \quad [\text{by } (1.11)]$$

(alternatively, use Envelope Theorem), i.e., as usual, the Lagrange multiplier is the marginal (indirect) utility of wealth.

1(e). By (1.11) $\frac{\partial \tilde{x}_k}{\partial w} > 0$ for at least one good k. Then (1.2) requires the RHS of (1.9) for good k to be negative, proving (1.3). Thus, for any $j \in \{1, \dots, L\}$, the RHS of (1.9) is negative, i.e., $\frac{\partial \tilde{x}_j}{\partial w} > 0$ for all $j \in \{1, \dots, L\}$.

Inequality (1.3) says that, for the utility index (1.1) satisfying (1.2) the marginal utility of wealth is decreasing.

1(f). For completeness sake, here is the derivation of (1.4) (not required in the exam). Using (1.2) we obtain from (1.7)

$$\frac{\partial \tilde{x}_j}{\partial p_k} = \frac{\partial \lambda}{\partial p_k} \frac{p_j}{u''_j} \quad , \quad j \neq k \quad ,$$

and from (1.8)

$$\frac{\partial \tilde{x}_k}{\partial p_k} = \frac{\partial \lambda}{\partial p_k} \frac{p_k}{u''_k} + \frac{\lambda}{u''_k} \quad ,$$

which substituted into (1.10) yield

$$\sum_{j=1}^{L} p_j \left[\frac{\partial \lambda}{\partial p_k} \frac{p_j}{u''_j} \right] + p_k \frac{\lambda}{u''_k} = - \tilde{x}_k \, ,$$

or

$$\frac{\partial \lambda}{\partial p_k} \sum_{j=1}^{L} \frac{p_j^2}{u''_j} = - \tilde{x}_k - p_k \frac{\lambda}{u''_k} \quad . \qquad (1.12)$$

Similarly, from (1.9)

$$\frac{\partial \tilde{x}_j}{\partial w} = \frac{\partial \lambda}{\partial w} \frac{p_j}{u''_j} \quad ,$$

which substituted into (1.11) yields

$$\sum_{j=1}^{L} p_j \frac{\partial \lambda}{\partial w} \frac{p_j}{u''_j} = 1 \, , \quad \text{or} \quad \frac{\partial \lambda}{\partial w} \sum_{j=1}^{L} \frac{p_j^2}{u''_j} = 1 \qquad (1.13) .$$

From (1.12) and (1.13) we obtain

$$\frac{\frac{\partial\lambda}{\partial p_k}}{-\tilde{x}_k - p_k\frac{\partial\lambda}{u''_k}} = \frac{\partial\lambda}{\partial w}, \text{ i.e., } \frac{\partial\lambda}{\partial p_k} = \frac{\partial\lambda}{\partial w}\left[-\tilde{x}_k - p_k\frac{\lambda}{u''_k}\right],$$

or, using (9), $$\frac{\partial\lambda}{\partial p_k} = \frac{\partial\lambda}{\partial w}\left[-\tilde{x}_k - \lambda\frac{\frac{\partial\tilde{x}_k}{\partial w}}{\frac{\partial\lambda}{\partial w}}\right],$$

which is the desired (1.4).

Next we answer the question in 1(f)

From (1.3), (1.7) and (1.9), we obtain

$$\frac{p_j}{u''_j}\cdot\frac{\partial\tilde{x}_j}{\partial w} = \frac{p_j}{u''_j}\cdot\frac{\partial\tilde{x}_j}{\partial w}\left[-\tilde{x}_k - \lambda\frac{\frac{\partial\tilde{x}_k}{\partial w}}{\frac{\partial\lambda}{\partial w}}\right], \qquad j \neq k$$

i.e., $$\frac{\partial\tilde{x}_j}{\partial w} = -\tilde{x}_k\frac{\partial\tilde{x}_j}{\partial w} - \frac{\lambda}{\frac{\partial\lambda}{\partial w}}\frac{\partial\tilde{x}_j}{\partial w}\frac{\partial\tilde{x}_k}{\partial w},$$

which, recalling the definition of the Slutsky matrix, yields

$$S_{jk}(p,w) \equiv \frac{\partial\tilde{x}_j}{\partial p_k} + \tilde{x}_k\frac{\partial\tilde{x}_j}{\partial w} = -\frac{\lambda}{\frac{\partial\lambda}{\partial w}}\frac{\partial\tilde{x}_j}{\partial w}\cdot\frac{\partial\tilde{x}_k}{\partial w}, \quad j \neq k, \qquad (1.14)$$

a positive expression given the results in 1(e). Thus, all goods are net substitutes.

1(g). The example does not satisfy condition (1.2), because $u_2''(x_2) = 0$. Its FOC include

$$a\,\tilde{x}_1(p,w) = \lambda(p,w)\,p_1$$
$$1 = \lambda(p,w)\,p_2,$$

i.e., $\lambda(p,w) = \frac{1}{p_2}$, with $\frac{\partial \lambda}{\partial w} = 0$, contrary to (1.3).

Moreover, $\tilde{x}_1(p,w) = a - \frac{p_1}{p_2}$ with $\frac{\partial \tilde{x}_1}{\partial w} = 0$, contrary to the normality of 1(e).

<u>Note</u>: Expression (1.14) is not valid here, because $\frac{\partial \lambda}{\partial w}$ appears in its denominator, while $\frac{\partial \lambda}{\partial w} = 0$ here. In fact, the product $\frac{\partial \tilde{x}_1}{\partial w}\frac{\partial \tilde{x}_2}{\partial w}$ is zero, while the off-diagonal Slutsky entry is

$$s_{12}(p,w) \equiv \frac{\partial \tilde{x}_1}{\partial p_2} + \tilde{x}_2 \frac{\partial \tilde{x}_1}{\partial w} = \frac{\partial \tilde{x}_1}{\partial p_2} \quad \left[\text{because } \frac{\partial \tilde{x}_1}{\partial w} = 0\right]$$
$$= \frac{p_1}{p_2^2} > 0.$$

Answer Key, Question 2

2(a). There is a production externality, because the physical cost of firm *i* depends on the aggregate output $q = \sum_{j=1}^{N} q_j$ through the term $(q_i + q_{-i})^{\alpha}$. The parameter α reflects the externality. If $\alpha > 0$, then the externality is negative, because an increase in aggregate output increases the cost of firm *i*: perhaps all these firms are using an exhaustible natural resource, or there is some form of congestion among them. If, on the contrary, $\alpha < 0$, then the externality is positive (because, say, knowledge spillovers). Note that firm *i* contributes to these effects not only on the other firms but also on itself, and that the externality across firms depends only on aggregate output *q*, and not in the manner in which *q* is allocated among firms.

The value $\alpha = 0$ corresponds to the traditional case of absence of production externalities.

When $\alpha = 0$, the parameter β depicts the scale economies within the firm, with $\beta > 1$ corresponding to decreasing returns to scale (the cost function is strictly convex), and $\beta < 1$ corresponding to increasing returns to scale (cost function strictly concave). When $\alpha \neq 0$ this interpretation is not exact, because q_i appears in the term $(q_i + q_{-i})^{\alpha}$, and needs to be made conditional to keeping aggregate output constant.

2(b). $C(q)$ is the value of the problem

$$\min_{(q_1,\ldots,q_N)} \sum_{i=1}^{N} (q_1 + \ldots + q_N)^{\alpha} (q_i)^{\beta} \text{ subject to } \sum_{i=1}^{N} q_i \geq q, \quad (2.2)$$

i.e., $\min_{(q_1,\ldots,q_N)} q^{\alpha} \sum_{i=1}^{N} (q_i)^{\beta}$ subject to $\sum_{i=1}^{N} q_i \geq q$.

Its solutions are those of the problem

$$\min_{(q_1,\ldots,q_N)} \sum_{i=1}^{N} (q_i)^{\beta} \text{ subject to } \sum_{i=1}^{N} q_i \geq q. \quad (2.3)$$

Case 1: $\beta < 1$. For fixed *q*, $\beta < 1$ entails economies of scale within each plant. Intuitively, the merged firm will then concentrate its production in one plant. Formally, the objective function in Problem (2.3) is strictly quasiconcave, with corner solutions, see Figure 2.1, i. e., any vector of the form $q_i = q$, $q_j = 0$ for $j \neq i$ solves Problem (2.3), and hence, Problem 2.2. The value function of Problem (2.2), or, in other words, the physical cost function of the merged firm is then $C(q) = q^{\alpha}(q + 0 + \ldots + 0)^{\beta} = q^{\alpha+\beta}$.

Case 2: $\beta = 1$. Problem (2.2) reduces now to the minimization of $\sum_{i=1}^{N} q_i$ subject to $\sum_{i=1}^{N} q_i = q$, i.e., any allocation of the output q among the N firm minimizes costs: see Figure 2.2. The physical cost function of the merged firm is then $C(q) = q^{\alpha} q = q^{\alpha+\beta}$.

Case 3: $\beta > 1$. This case corresponds to diseconomies of scale within each firm and, intuitively, the merged firm will then allocate output equally among the (identical) plants. Formally, the objective function in Problem (2.3) is strictly quasicovex, with a unique interior solution. (see Figure 2.3). Because of symmetry, $q_i = \frac{q}{N}$ (the FO equalities of Problem 2.2 are $\beta(q_i)^{\beta-1} = \lambda$).

The physical cost function of the merged firm is then

$$C(q) = q^{\alpha} N \left(\frac{q}{N}\right)^{\beta} = N^{1-\beta} q^{\alpha+\beta}.$$

PROFITMAX Problem of the merged firm: $\max_q pq - C(q)$, i.e.,

$\max_q pq - q^{\alpha+\beta}$ in cases (1) and (2)

$\max_q pq - N^{1-\beta} q^{\alpha+\beta}$ in case (3).

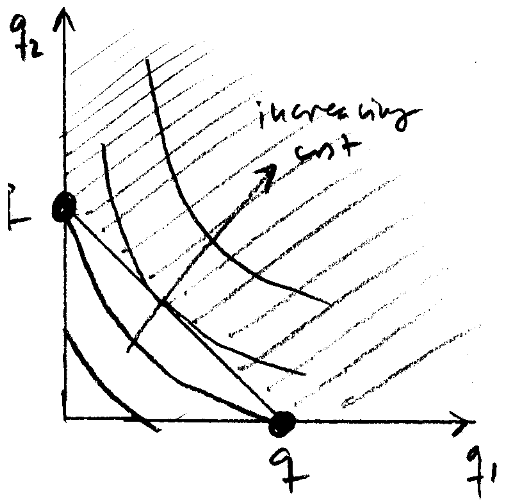


Fig. 2.1
Case 1
$\beta < 1$

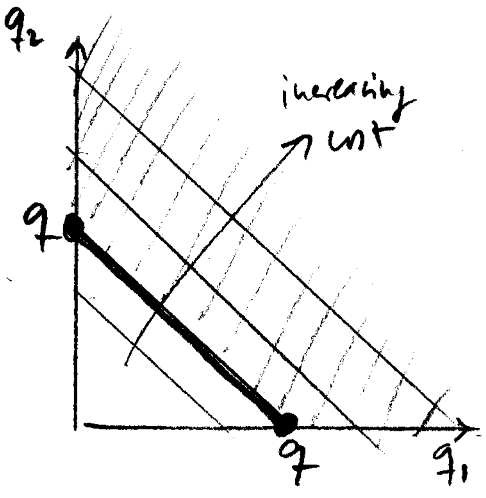


Fig. 2.2
Case 2
$\beta = 1$

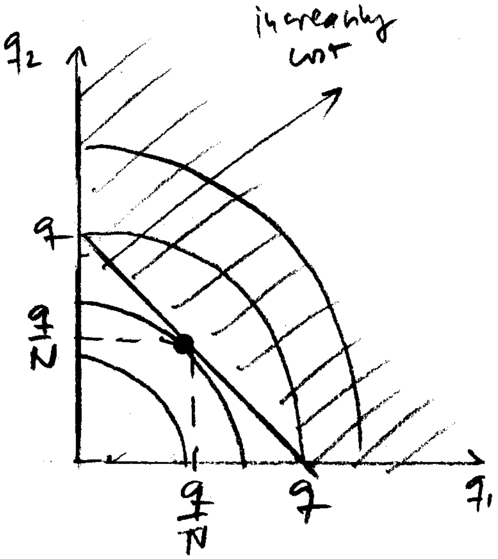


Fig. 2.3
Case 3
$\beta > 1$

Existence of solutions to the PROFITMAX Problem

Note that the (physical) average cost of the merged firm is $C^A(q) = \begin{cases} q^{\alpha+\beta-1} \textit{ in cases } 1,2, \\ N^{1-\beta} q^{\alpha+\beta-1} \textit{ in case } 3, \end{cases}$

and the (physical) marginal cost of the merged firm is $C^M(q) = \begin{cases} (\alpha+\beta) q^{\alpha+\beta-1} \textit{ in cases } 1,2, \\ (\alpha+\beta) N^{1-\beta} q^{\alpha+\beta-1} \textit{ in case } 3, \end{cases}$.

* If $\alpha+\beta<1$, then the average cost of the merged firm is decreasing, and its technology displays increasing returns to scale, which are incompatible with profit-maximizing, price-taking behavior. Thus, no solution to the PROFITMAX Problem of the merged firm exists in this case.

* If $\alpha+\beta=1$, then the average cost of the merged firm is constant, and its technology displays constant returns to scale. Price-taking profit maximization then requires $p \le C^A(q)$, i.e., either $p < C^A(q)$ and $q = 0$, or $p = C^A(q)$ and $q \in \Re_+$.

* If $\alpha+\beta>1$, then the average cost of the merged firm is increasing, and its technology displays decreasing returns to scale. Because, in this case, the marginal cost is strictly increasing and satisfies $C^M(0)=0$ and $\lim_{q\to\infty} C^M(q) = \infty$, the equation "price = marginal cost" has a unique solution. In cases 1 and 2, the equation is $p = (\alpha+\beta)q^{\alpha+\beta-1}$, with solution $q^M = \left(\frac{p}{\alpha+\beta}\right)^{\frac{1}{\alpha+\beta-1}}$, whereas in Case 3 the equation is

$$p = (\alpha+\beta) N^{1-\beta} q^{\alpha+\beta-1}, \qquad (2.4)$$

with solution $q^M = \left(N^{\beta-1}\frac{p}{\alpha+\beta}\right)^{\frac{1}{\alpha+\beta-1}}$. See Figure 2.4 for $\alpha+\beta>1$, $\beta>1$.

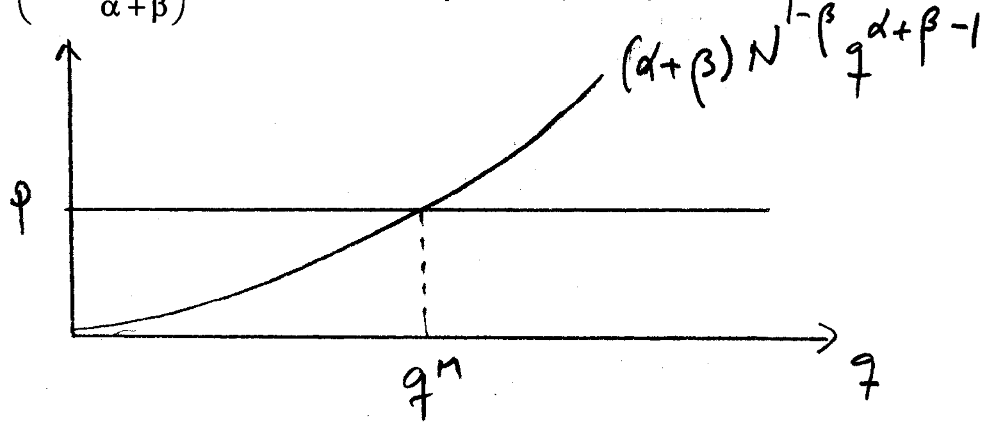


Figure 2.4.
The solution to the PROFITMAX problem of the merged firm
$\alpha+\beta>1$
$\beta>1$

2(c). The PROFITMAX Problem of the i firm is $\max_{q_i} pq_i - (q_i + q_{-i})^{\alpha} q_i^{\beta}$, with first-order equality $p = \alpha(q_i + q_{-i})^{\alpha-1} q_i^{\beta} + (q_i + q_{-i})^{\alpha} \beta q_i^{\beta-1}$, which under symmetry becomes

$$p = \alpha(q)^{\alpha-1}\left(\frac{q}{N}\right)^{\beta} + (q)^{\alpha}\beta\left(\frac{q}{N}\right)^{\beta-1} = q^{\alpha+\beta-1}(\alpha N^{-\beta} + \beta N^{1-\beta}) = (\frac{\alpha}{N} + \beta)N^{1-\beta} q^{\alpha+\beta-1}. \qquad (2.5)$$

Equation (2.5), with solution $\hat{q} = \left(N^{\beta-1}\dfrac{p}{\frac{\alpha}{N}+\beta}\right)^{\frac{1}{\alpha+\beta-1}}$, looks like (2.4) except that $\dfrac{\alpha}{N}$ substitutes α in its RHS.

2(d). Comparing (2.4) and (2.5), we see that the output of the merged firm equals the aggregate output of the N independent firms *if and only if* $\alpha = 0$. This is the externality-free case, where the "decentralization theorem" holds: the institutional organization of production in firms does not affect aggregate supply.

If $\alpha > 0$ (negative externality), then the RHS of (2.5) is below that of (2.4) (see Figure 2.5): $\hat{q} > q^M$, and the independent firms overproduce relative to the merged firm. If, on the contrary, $\alpha < 0$ (positive externality), then the RHS of (2.5) is above that of (2.4) (see Figure 2.6): $\hat{q} < q^M$, and the independent firms underproduce relative to the merged firm.

Fig. 2.5
$\alpha > 0$
(negative externality)

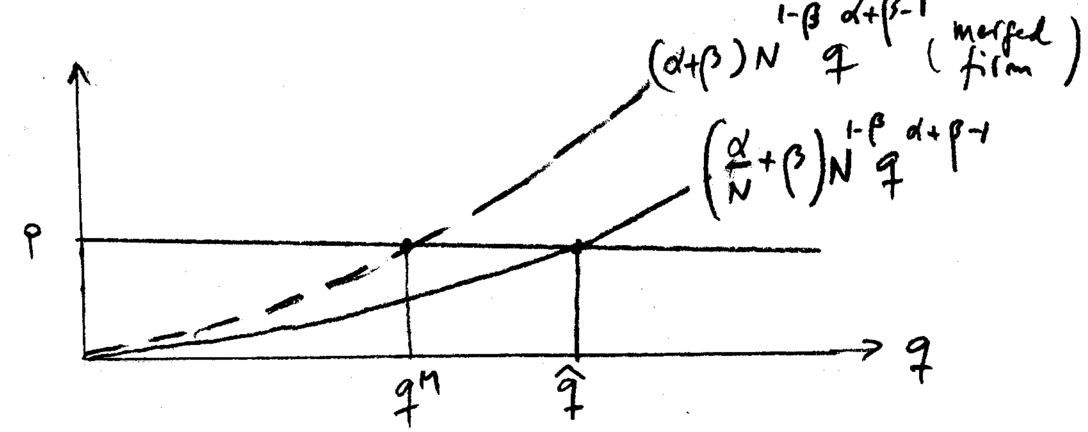
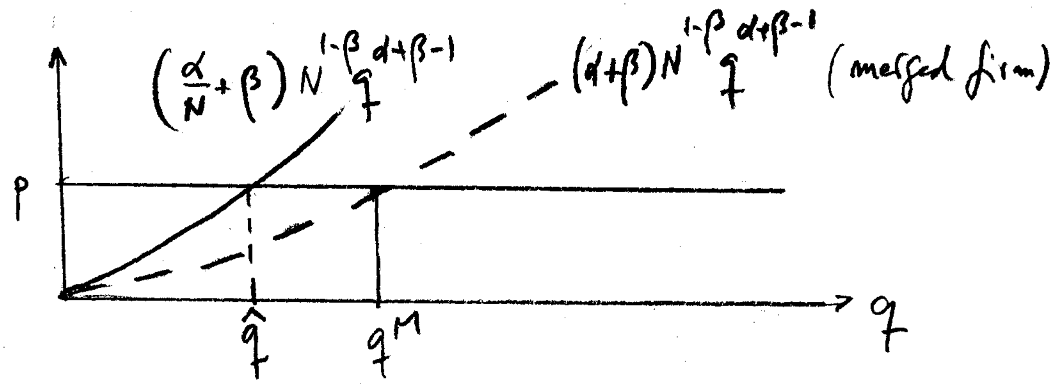


Fig. 2.6
$\alpha < 0$
(positive externality)



Answer Key

3(a) the problem of the planner is : max $\sum_i \alpha_i u_i(x_i, l_i, x_i - \frac{1}{I}\sum_{i'} x_{i'})$ subject to

$$\sum_i x_i \leq \sum_i \bar{L}_i - l_i \Longleftrightarrow \sum_i x_i + \sum_i l_i \leq \sum_i \bar{L}_i \tag{1}$$

Let λ denote the multiplier associated to the resource constraint. The FOCs are

$$\left.\begin{array}{l} \alpha_i(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial s_i}) - \frac{1}{I}\sum_{i'} \alpha_{i'}\frac{\partial u_{i'}}{\partial s_{i'}} = \lambda \\ \alpha_i \frac{\partial u_i}{\partial l_i} = \lambda \end{array}\right. \qquad i = 1, \ldots, I$$

where, for simplicity, the arguments (x_i, l_i, s_i) of the functions have been omitted. Eliminating the multiplier gives the condition

$$\frac{\frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial s_i}}{\frac{\partial u_i}{\partial l_i}} - \frac{1}{I}\sum_{i'} \frac{\frac{\partial u_{i'}}{\partial s_{i'}}}{\frac{\partial u_{i'}}{\partial l_{i'}}} = 1, \quad i = 1, \ldots, I \tag{2}$$

Given that the maximum problem of the planner is convex, (1) and (2) characterize the interior PO allocations.

(b) The equilibrium is such that

- For all j, firm j choose (l_j, y_j) so as to maximize $py_j - wl_j$subject to $y_j \leq l_j$.
- For all i, consumer i chooses (x_i, l_i) so as to maximize $u_i(x_i, l_i, x_i - \bar{x})$ subject to $px_i - wl_i \leq 0$.
- $\bar{x} = (1/I)\sum_i x_i$
- Markets clear: $\sum_j l_j = \sum_i l_i$ and $\sum_j y_j = \sum_i x_i$.

(c) Given the assumptions on the utility functions there must be positive production at equilibrium. The firms' maximum problem has a positive solution only if $p = w$. We thus nomalize prices so that $p = w = 1$.

It is easy to see that the FOCs for the representative consumer imply after elimination of the multiplier that

$$\frac{\frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial s_i}}{\frac{\partial u_i}{\partial l_i}} = 1$$

which is incompatible with (2) as soon as $\frac{\partial u_{i'}}{\partial s_{i'}} > 0$ for some agent i'. Thus an equilibrium is not Pareto optimal. The inefficiency comes from the fact that in equilibrium agents do not take into account that their consumption create an externality on the other agents through its effect on the average consumption. A high consumption for agent i increases the average consumption and thus tends to decrease the utility of the other agents.

(d) Using (1) and (2) it is easy to see that the symmetric Pareto optimal allocation is such that each agent works and consumes $(3/4)\bar{L}$.

The symmetric equilibrium is such that the choice (x, l) of the representative agent satisfies

$$\begin{aligned} \frac{\frac{3}{4}(\frac{l}{x})^{\frac{1}{4}} + \frac{13}{8}}{\frac{1}{4}(\frac{x}{l})^{\frac{3}{4}}} &= 1 \qquad (3) \\ Ix + Il &= I\bar{L} \qquad (4) \end{aligned}$$

(3) can be written as the equation

$$\frac{3}{4}\frac{1}{X} + \frac{13}{8} = \frac{1}{4}X^3$$

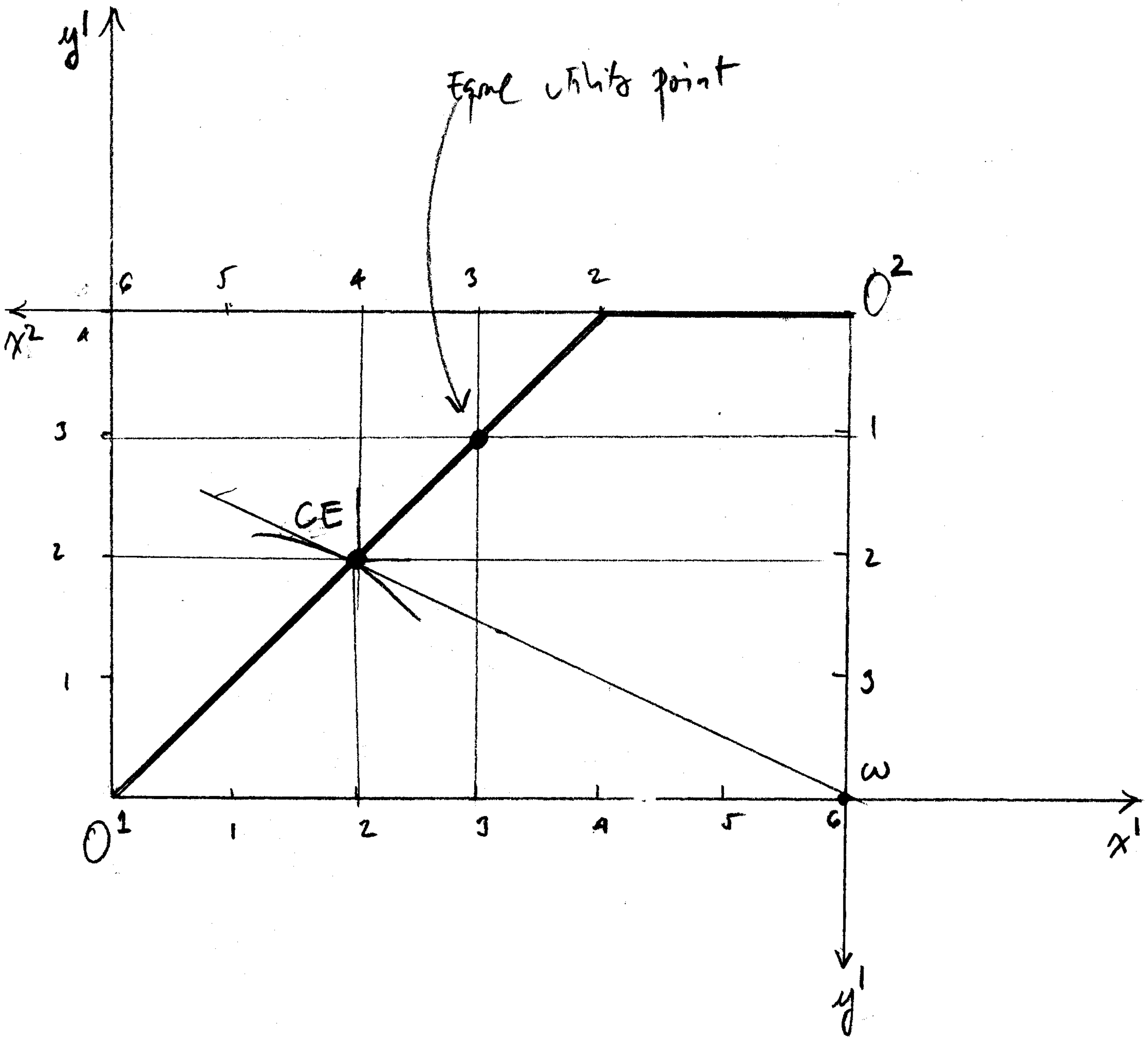
in the variable $X = \left(\frac{x}{l}\right)^{1/4}$, or

$$\frac{1}{4}X^4 - \frac{13}{8}X - \frac{3}{4} = 0$$

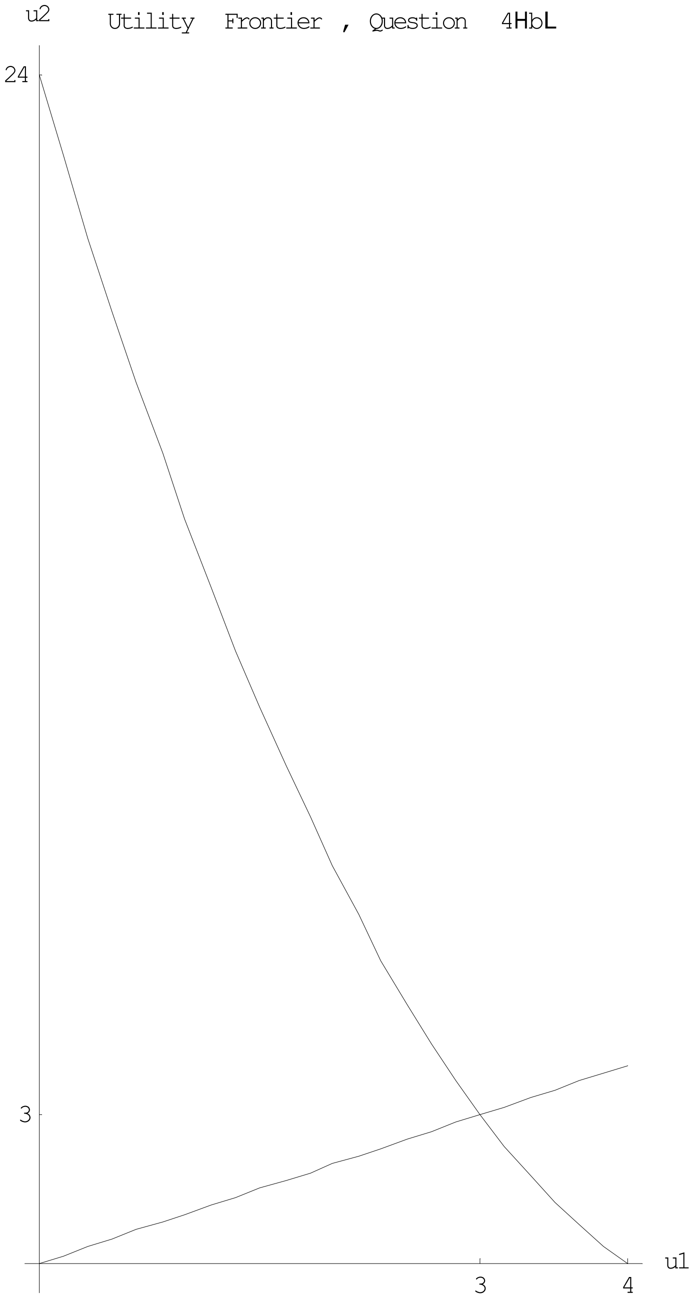
with obvious root $X = 2$. Thus in equilibrium, agents work and consume $(16/17)\bar{L}$, which is much larger than the efficient consumption.

4. (a) I do not have a simple way of drawing the Edgeworth box. The P.O allocations are such that $x^1 = y^1, x^2 = 6 - x^1, y^2 = 4 - y^1$, for $0 \leq y_1 \leq 4$, and $x^1 \geq y^1, x^2 = 6 - x^1, y^2 = 0$ for $y^1 = 4$.

(b) The utility frontier is : $v^2 = (6 - v^1)(4 - v^1)$, for $0 \leq v^1 \leq 4$. The point where $v^1 = v^2 = v$ is such that $v = (6 - v)(4 - v)$ or $v^2 - 11v + 24 = 0$, i.e. $v = 3$. The P.O allocation such that the two agents have equal utilities is $(\bar{x}^1, \bar{y}^1) = (3, 3)$, $(\bar{x}^2, \bar{y}^2) = (3, 1)$.



Question 4(a)
Edgeworth Box


Utility Frontier , Question 4HbL
u2
24
3
u1
3
4

(c) Normalize $p_x = 1$ and let $p_y = p$. The demand of agent 1 is $x^1(p) = y^1(p) = \frac{6}{1+p}$. The demand of agent 2 is $x^2(p) = 2p$, $y^2(p) = 2$. The equilibrium price is thus $\tilde{p} = 2$ and the equilibium allocation is $(\tilde{x}^1, \tilde{y}^1) = (2, 2)$, $(\tilde{x}^2, \tilde{y}^2) = (4, 2)$. The corresponding utilities are $(\tilde{v}^1, \tilde{v}^2) = (2, 8)$.

(d) By the Second Welfare Theorem, we know that by redistributing initial endowments in goods or income and letting markets work, a planner can achieve any Pareto optimal allocation. The allocation of question (b) is an equilibrium allocation if the price p is such that

$$p = \frac{\frac{\partial u^2}{\partial y^2}(3,1)}{\frac{\partial u^2}{\partial x^2}(3,1)} = 3$$

The income needed by agent 1 to afford the allocation (3,1) when $p = 3$ is 12, so that agent 1 needs a transfer of 2 units of good 2. By transferring 2 units of good 2 from agent 2 to agent 1 and letting markets do the allocation the planner can equalize the utilities.

(d) Utility functions are convenient for representing preferences but, unless special circumstances (same utilities, quasi- linear preferences) utility levels do not have much meaning, and do not permit relevant interpersonal comparisons of welfare. The concept of fairness usually considered is the concept of "envy free" and the allocation of question (b), which equalizes utilities, is not envy free since agent 2 envies agent 1. On the other hand the equilibrium allocation is (c) is envy free, so that in this example no redistribution seems necessary.

ANSWER - Q5 - Fall 04

5. Caught on both horns

(a) $\underline{w}(e) = e$ for all $e \leq 4$. So $\boxed{\underline{w}(e_0) = 2,\ \underline{w}(e_1) = 3,\ \underline{w}(e_2) = 4}$.

(b) Since $Ey(e) = 10p(e)$, $\boxed{g(e_0) = -1,\ g(e_1) = 2,\ g(e_2) = 4}$. So working hard is socially efficient.

(c) $\boxed{C^0 = (2,2)}$. Notice that a straight wage (i.e., the same wage even if the project fails) automatically leads the agent to choose his least costly action e_0.

(d) Leaving out the '1' superscript:

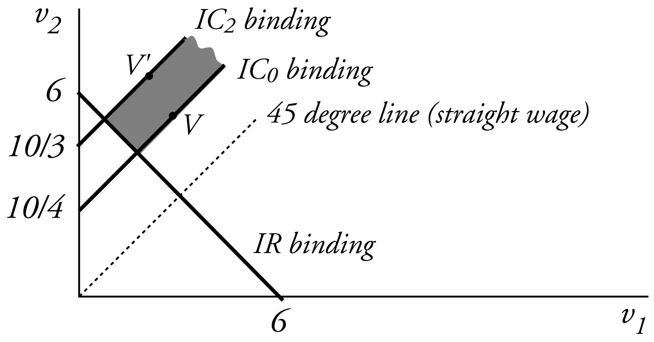
$$U(C, e_1) \geq 0 \qquad (IR)$$
$$U(C, e_1 \geq U(C, e_0) \qquad (IC_0)$$
$$U(C, e_1) \geq U(C, e_2). \qquad (IC_2)$$

Or, substituting and re-arranging, these can be re-written:

$$v_1 + v_2 \geq 6 \qquad (IR)$$
$$v_2 - v_1 \geq \frac{10}{4} \qquad (IC_0)$$
$$v_2 - v_1 \leq \frac{10}{3}. \qquad (IC_2)$$

(e) Intuitively, the greater the spread between w_2 and w_1, the greater the agent's incentive to increase his effort e. Hence IC_0 says the spread must be sufficiently positive — otherwise the agent would prefer shirking. And IC_2 says it cannot be too large, otherwise the agent would prefer e_2 to e_1.

(f) The interesting observation is that working with utility wages rather than money wages makes all 3 constraints linear. The set of feasible contracts is shaded in the GRAPH.



(g) Suppose first that IC_2 is not binding, like contract $V = (v_1, v_2)$ in the FIGURE. Notice by decreasing the wage if the project fails (w_1), the principal loosens IC_0 by increasing the spread $w_2 - w_1$; and further he pays less if the the project fails, so he increases his expected profit $Ey(e_1) - Ew(e_1)$. Graphically, we are moving to the west, toward the IR-binding line.

Suppose instead that IC_2 is binding, like contract V' in the FIGURE. Then by decreasing w_2 and keeping w_1 unchanged (moving southward), the principal still satisfies all constraints and increases his expected profit. Now IC_2 is again loose, so we can again move westward toward the IR-binding line.

(h) If IC_0 were loose, the principal can decrease the spread $w_2 - w_1$ while still keeping IR binding. A smaller spread implies a lower risk premium, that is, a decrease in $Ew(e_1)$, decreasing the principal's cost of implementing work and hence increasing his expected profit. NOTE: To implement e_1, w_2 must be greater than 4, hence risk aversion sets in. This explains the presence of a risk premium in the cost-minimizing contract C^1.

(i) C^1 occurs where both IR and IC_0 bind. Solving the two linear equations $v_1 + v_2 = 6$ and $v_2 - v_1 = 2.5$ shows $\boxed{v_1 = 1.75,\ v_2 = 4.25}$. Using the utility function v we can now convert the utility wages into money wages, to find $\boxed{C^1 = (1.75, 6.5)}$.

(j) From the principal's perspective the cost of implementing any e is $c(e) \equiv Ew(e) \equiv (1-p(e))w_1 + p(e)w_2$, where the expectation is taken over the incentive contract (w_1, w_2) that implements e in the least-cost way. Hence $c(e_0) = 2$, $c(e_1) = .5(1.75) + .5(6.5) = 4.125$, and $c(e_2) = .2(4/3) + .8(32/3) = 8.8$. Since the principal's profit from implementing e is $\pi(e) \equiv Ey(e) - c(e)$, we find

$$\begin{aligned}\pi(e_0) &= 1 - 2 = -1,\\ \pi(e_1) &= 5 - 4.125 = .875,\\ \pi(e_2) &= 8 - 8.8 = -.8.\end{aligned}$$

We conclude that the principal will implement e_1, which from a social point of view leads to inefficient risk sharing (the agent is bearing some risk) *and* inefficient effort (the agent is working too little). So society is caught on both horns.