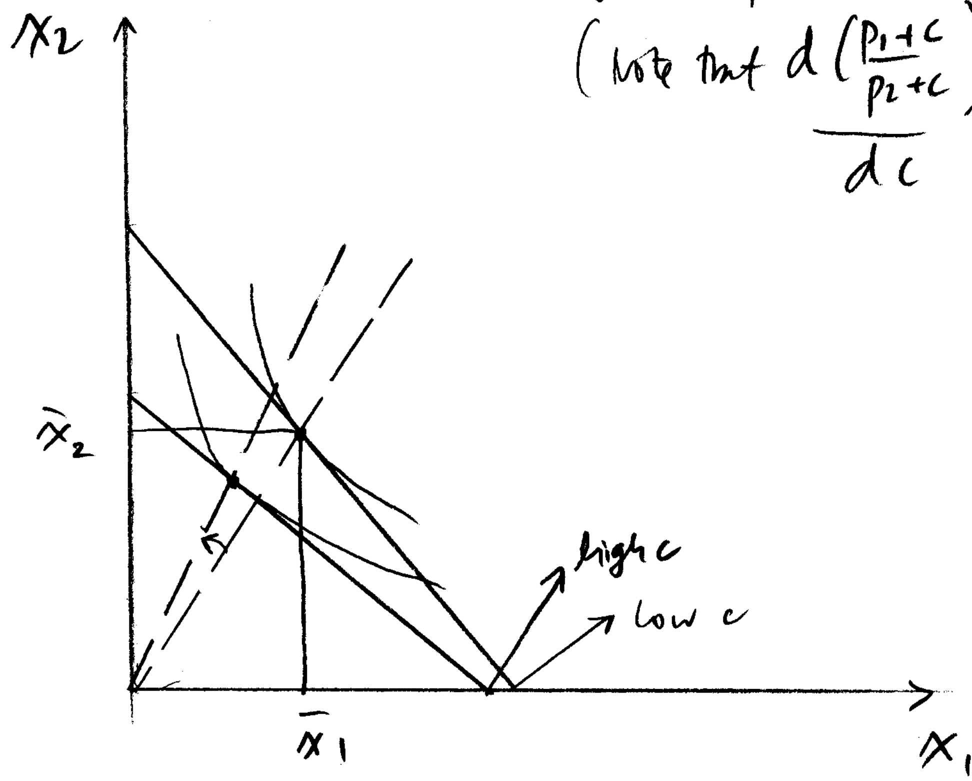
Question 1.

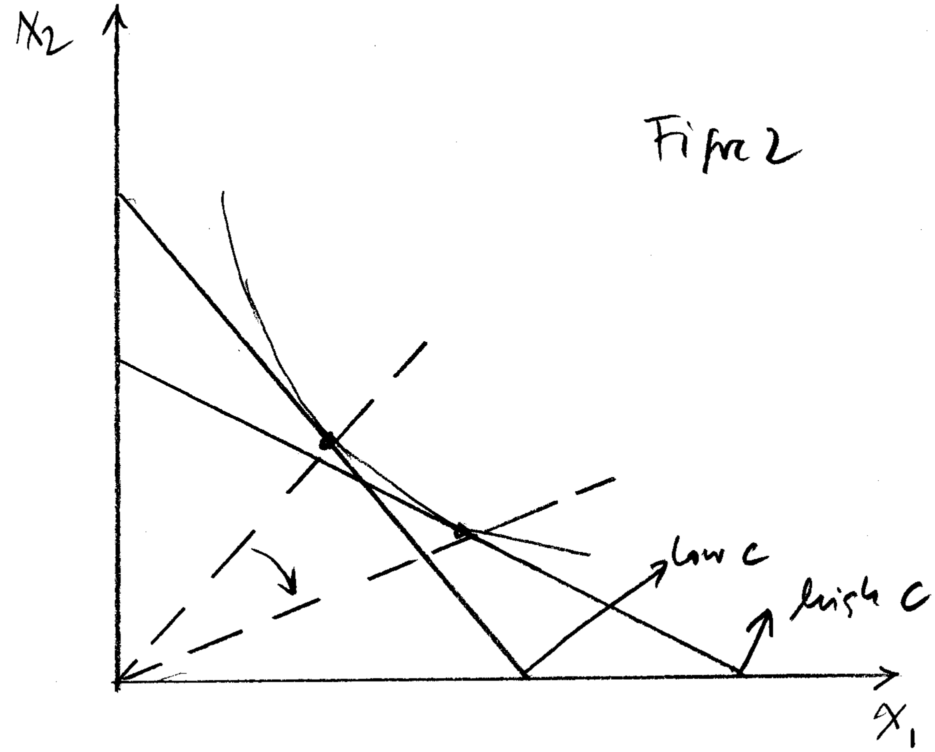
1.1. Clearly, as c increases the budget set shrinks, whereas the budget line becomes flatter see figure 1, i.e., good 1 becomes relatively cheaper.

(Note that $\dfrac{d\left(\frac{p_1+c}{p_2+c}\right)}{dc} < 0$ for $p_1 > p_2$)



It is graphically clear that, after the increase in c, $\frac{x_1}{x_2}$ may well decrease, as illustrated in Figure 1.

1.2: Now the claim is correct. Let c increase. Graphically, we see in Figure 2 that, by



the negativity of the own-price substitution effect, the compensated demand for good 1 must increase, which in the case of two goods means that the compensated demand for the other good must decrease (the goods must be net substitutes).

Hence, $\frac{h_1}{h_2}$ must increase with an increase of c.

Analytically, we compute

$$\frac{d\left(\dfrac{h_1(p_1+c,\,p_2+c,\,u)}{h_2(p_1+c,\,p_2+c,\,u)}\right)}{dc} = \frac{\left(\dfrac{\partial h_1}{\partial p_1}+\dfrac{\partial h_1}{\partial p_2}\right)h_2-\left(\dfrac{\partial h_2}{\partial p_1}+\dfrac{\partial h_2}{\partial p_2}\right)h_1}{h_2^2}. \quad (1)$$

By the 0-homogeneity of Hicksian demand in prices, we have that:

$$\left.\begin{aligned} \frac{\partial h_1}{\partial p_1}p_1+\frac{\partial h_1}{\partial p_2}p_2 &= 0 \\ \frac{\partial h_2}{\partial p_1}p_1+\frac{\partial h_2}{\partial p_2}p_2 &= 0 \end{aligned}\right\} \quad (2)$$

Hence, the numerator of (1) becomes

$$\left[\frac{\partial h_1}{\partial p_1}-\frac{\partial h_1}{\partial p_1}\frac{p_1}{p_2}\right]h_2-\left[-\frac{\partial h_2}{\partial p_2}\frac{p_2}{p_1}+\frac{\partial h_2}{\partial p_2}\right]h_1$$

$$=\underbrace{\frac{\partial h_1}{\partial p_1}}_{-}\underbrace{\left[1-\frac{p_1}{p_2}\right]}_{-}\underbrace{h_2}_{+}+\underbrace{\frac{\partial h_2}{\partial p_2}}_{-}\underbrace{\left[\frac{p_2}{p_1}-1\right]}_{-}\underbrace{h_1}_{+} > 0,$$

because $\dfrac{\partial h_j}{\partial p_j}<0$ and $p_1>p_2$.

Comment. The claim, even when restricted to the compensated demand functions, has no general validity, although it may hold in some special cases. There is a striking diffrence between the 2-good and the 3-good case. In the two-good case, (2) implies that $\frac{\partial h_i}{\partial p_j} > 0$ for $i \neq j$, i.e., the two goods must be net substitutes. But with 3 goods, (2) becomes

$$\frac{\partial h_1}{\partial p_1} p_1 + \frac{\partial h_1}{\partial p_2} p_2 + \frac{\partial h_1}{\partial p_3} p_3 = 0$$

$$\frac{\partial h_2}{\partial p_1} p_1 + \frac{\partial h_2}{\partial p_2} p_2 + \frac{\partial h_2}{\partial p_3} p_3 = 0,$$

allowing for the possibility that $\frac{\partial h_1}{\partial p_2} = \frac{\partial h_2}{\partial p_1} < 0$ and $\frac{\partial h_1}{\partial p_3} > 0$, $\frac{\partial h_2}{\partial p_3} > 0$; i.e., goods 1 and 2 may be net complements, with the sign of (1) indeterminate.

2.1(A)

$$\max_{z_{j1},\dots,z_{jM}} \left(\sum_{m=1}^{M} h_m z_{jm}\right)^{\beta} \left(\frac{\sum_{m=1}^{M} h_m z_{jm}}{\sum_{m=1}^{M} z_{jm}}\right)^{\theta} A - \sum_{m=1}^{M} w_m z_{jm}$$

$$\text{s.t.} \quad z_{jm} \geqq 0, \quad j = 1, \dots, M$$

NOT ASKED

KT conditions: $\dfrac{\partial f}{\partial z_{jn}} - w_n \leqq 0$,

with equality if $z_{jn} > 0$,

$n = 1, \dots, M$,

where
$$\frac{\partial f}{\partial z_{jn}} = \beta\left(\sum h_m z_{jm}\right)^{\beta-1} h_n \left(\frac{\sum h_m z_{jm}}{\sum z_{jm}}\right)^{\theta} A$$

$$+ A\left(\sum h_m z_{jm}\right)^{\beta} \theta \left(\frac{\sum h_m z_{jm}}{\sum z_{jm}}\right)^{\theta-1} \left(\frac{h_n \sum z_{jm} - \sum h_m z_{jm}}{\left(\sum z_{jm}\right)^2}\right)$$

2.1(c)

$$f(z_{j1}, z_{j2}) = (h_1 z_{j1} + h_2 z_{j2})^{\beta} \left(\frac{h_1 z_{j1} + h_2 z_{j2}}{z_{j1} + z_{j2}}\right)^{\theta} A.$$

with $$\frac{\partial f}{\partial z_{j2}} = \beta (h_1 z_{j1} + h_2 z_{j2})^{\beta - 1} h_2 \left(\frac{h_1 z_{j1} + h_2 z_{j2}}{z_{j1} + z_{j2}}\right)^{\theta} A$$

$$+ (h_1 z_{j1} + h_2 z_{j2})^{\beta} \theta \left(\frac{h_1 z_{j1} + h_2 z_{j2}}{z_{j1} + z_{j2}}\right)^{\theta - 1} \left(\frac{h_2 z_{j1} + \cancel{h_2 z_{j2}} - h_1 z_{j1} - \cancel{h_2 z_{j2}}}{(z_{j1} + z_{j2})^2}\right)$$

$$= A (h_1 z_{j1} + h_2 z_{j2})^{\beta + \theta - 1} (z_{j1} + z_{j2})^{-\theta}$$

$$\left[\beta h_2 + \theta \frac{z_{j1} + z_{j2}}{(z_{j1} + z_{j2})^2} z_{j1} (h_2 - h_1)\right]$$

The sign of $\frac{\partial f}{\partial z_{j2}}$ is that of the term in square brackets,

i.e., $$\beta h_2 + \frac{\theta z_{j1} (h_2 - h_1)}{z_{j1} + z_{j2}},$$

i.e., that of $$\beta h_2 z_{j1} + \beta h_2 z_{j2} + \theta z_{j1} (h_2 - h_1)$$

Answer key - Prelim June 06 -

Question 3

(a) $\max\ u^1(x_1^1, x_2^1)$ subject to

$u^i(x_1^i, x_2^i) \geq v_i, \quad i = 2, \ldots I \qquad \alpha^i$

$\sum x_1^i \leq y_1 \qquad p_1$

$\sum x_2^i \leq y_2 \qquad p_2$

$y_1 \leq f(l_1, k_1) \qquad \mu_1$

$y_2 \leq g(l_2, k_2) \qquad \mu_2$

$l_1 + l_2 \leq \sum w_l^i \qquad p_l$

$k_1 + k_2 \leq \sum w_k^i \qquad p_k$

FOCs:

$$\alpha^i \frac{\partial u^i}{\partial x_1^i} = p_1, \quad i = 1 \ldots I \quad (\text{with } \alpha^1 = 1)$$

$$\alpha^i \frac{\partial u^i}{\partial x_2^i} = p_2 \qquad ''$$

$$p_1 = \mu_1, \qquad p_2 = \mu_2$$

$$\mu_1 \frac{\partial f_1}{\partial l_1} = p_l \qquad \mu_2 \frac{\partial g}{\partial l_2} = p_l$$

$$\mu_1 \frac{\partial f}{\partial k_1} = p_k \qquad \mu_2 \frac{\partial g}{\partial k_2} = p_k$$

all variables and multipliers being positive.

(b) FOCs at m.c. pricing equilibrium

firm 1 : $\min \bar{w} l_1 + \bar{r} k_1$

$f(l_1, k_1) \geq \bar{y}_1 \qquad \nu_1$

where (-) means that the functions are taken at the equilibrium value of the appropriate variables

FOCs. $\boxed{\bar{w} = \nu_1 \frac{\partial f}{\partial l_1}(-) \qquad \bar{r} = \nu_1 \frac{\partial f}{\partial k_1}(-)}$

$\bar{p}_1$ = marginal cost $\Leftrightarrow \bar{p}_1 = \frac{\partial C}{\partial y_1}$ where C is the solution of the problem above. Envelope thm $\Rightarrow$ $\boxed{\bar{p}_1 = \nu_1}$

firm 2 $\quad \max \bar{p}_2 g(l_2, k_2) - \bar{w} l_2 - \bar{r} k_2$

FOCs : $\boxed{\bar{p}_2 \frac{\partial g}{\partial l_2}(-) = \bar{w} \qquad \bar{p}_2 \frac{\partial g}{\partial k_2}(-) = \bar{r}}$

consumer i $\quad \max u^i(x^i_1, x^i_2)$

$\bar{p}_1 x^i_1 + \bar{p}_2 x^i_2 = \bar{w} w^i_l + \bar{r} w^i_k \qquad - t_i \qquad \lambda^i$

Note : both firms 1 and 2 make zero profit so there is no profit to distribute.

$\boxed{\frac{\partial u^i}{\partial x^i_1}(-) = \lambda^i \bar{p}_1 \qquad \frac{\partial u^i}{\partial x^i_2} = \lambda^i \bar{p}_2}$

Matching FOCs: let $p_1 = \bar{p}_1$ $\quad p_2 = \bar{p}_2$ $\quad \alpha_i = \frac{1}{\lambda_i}$

$\bar{w} = p_\ell \quad \bar{r} = p_\ell$

$\mu_1 = \nu_1$, $\mu_2 = p_2$

Because of the relation $\bar{p} = \nu_1$, the FOCs for Pareto optimally are satisfied.

(e) the production function of firm 1 is not differentiable but the cost minimizing condition, which must hold both at a P.O. (not use more inputs than needed) and at an equilibr. imply that

$\boxed{k_1 = \ell_1}$ will always hold

Once this holds everything is differentiable

(i) P.O. $\quad \max \sqrt{x_1 x_2}$

$x_1 = y_1 = \ell_1^2$

$x_2 = y_2 = \sqrt{k_2 \ell_2}$

$\ell_1 = k_1 \quad \ell_1 + \ell_2 = 1$

$k_1 + k_2 = 1$

$\ell_1 = k_1$ and the resource constraints on the factors imply $\ell_2 = k_2$ so that the problem can be simplified to

$\max \sqrt{\ell_1^2 \, \ell_2}$

$\ell_1 + \ell_2 = 1$

or $\max_{l_1 \geq 0} \sqrt{l_1^2(1-l_1)}$ or $\max_{l_1 \geq 0} l_1^2(1-l_1)$

FOC : $2l_1(1-l_1) - l_1^2 = 0 \Rightarrow 2(1-l_1) - l_1 = 0 \Rightarrow$

$\boxed{l_1^* = \frac{2}{3}}$ $\boxed{l_2^* = \frac{1}{3}}$ $\boxed{y_1^* = \left(\frac{2}{3}\right)^2 = \frac{4}{9} \quad y_2^* = \frac{1}{3}}$

(ii) Marginal Cost Pricing Equilibrium.

firm 1 : $c(y_1; w, r) = (w+r)\sqrt{y_1}$

since it must be that $\boxed{k_1 = l_1}$ and $y_1 = l_1^2 \Leftrightarrow \boxed{l_1 = \sqrt{y_1}}$

$\Rightarrow$ $\boxed{p_1 = \frac{w+r}{2\sqrt{y_1}}}$ (price charged by firm 1 if it produces y_1)

taxes needed : $p_1 y_1 - (w+r)\sqrt{y_1} = \frac{w+r}{2\sqrt{y_1}} \cdot y_1 - (w+r)\sqrt{y_1}$

$= -\frac{(w+r)\sqrt{y_1}}{2}$

thus $\boxed{t = -\frac{(w+r)\sqrt{y_1}}{2}}$

firm 2: $\max \; p_2 \sqrt{l_2 k_2} - w l_2 - r k_2$

$$\frac{p_2 \sqrt{k_2}}{2\sqrt{l_2}} = w \qquad \frac{p_2 \sqrt{l_2}}{2\sqrt{k_2}} = r$$

$$\Rightarrow \sqrt{\frac{k_2}{l_2}} = \frac{2w}{p_2} \qquad \sqrt{\frac{k_2}{l_2}} = \frac{p_2}{2r}$$

$$\Rightarrow \frac{2w}{p_2} = \frac{p_2}{2r} \iff p_2^2 = 4wr$$

$$\boxed{p_2 = 2\sqrt{wr}}$$

(Condition for zero profit with constant returns)

and $\boxed{\frac{k_2}{l_2} = \frac{w}{r}}$ optimal choice of inputs

Consumer $\max \; \sqrt{x_1 x_2}$

$$p_1 x_1 + p_2 x_2 = w + r - t$$

$$\boxed{x_1 = \frac{w+r-t}{2p_1} \qquad x_2 = \frac{w+r-t}{2p_2}}$$

equilibrium: $\left.\begin{matrix} l_1 + l_2 = 1 \\ k + k_2 = 1 \end{matrix}\right\}$ since $l_1 = k_1 \Rightarrow l_2 = k_2$

which implies $w = r$ for the FOCs of firm 2 to be satisfied. Let's normalize $w = r = 1$

then $x_1 = y_1 = \frac{2 - t}{2 p_1}$ $\Rightarrow$ since $p_1 = \frac{2}{2\sqrt{y_1}}$ and $t = +\frac{2}{2}\sqrt{y_1}$

$$\Rightarrow y_1 = \frac{2 - \sqrt{y_1}}{2\ {}^1/\sqrt{y_1}} = \frac{\sqrt{y_1}\ (2 - \sqrt{y_1})}{2} = \frac{2\sqrt{y_1} - y_1}{2}$$

$$\Rightarrow \frac{3}{2} y_1 = \sqrt{y_1} \Rightarrow \frac{3}{\cancel{2}} \sqrt{y_1} = 1 = y_1 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}.$$

which is the optimal level.

Thus $l_1 = \frac{2}{3} = k_1$, $l_2 = \frac{1}{3} = k_2$

let us check that the market for good 2 clears. $p_2 = 2\sqrt{1} = 2$

$$y_2 = \sqrt{l_2 k_2} = \frac{1}{3} \qquad x_2 = \frac{2 - t}{2 \times 2} = \frac{2 - 2/3}{4} = \frac{4}{3} \cdot \frac{1}{4} = \frac{1}{3}$$

OK

When there is only one allocation which satisfies the FOC for P.O, since the marginal cost pricing equilibrium satisfies these FOCs, then it must be Pareto optimal. When there are many allocations which satisfy the FOCs for P.O., some optimal, the other not, then it may be that some marginal cost pricing equilibria are not Pareto optimal.

Answer Key - Prelim June 06 - Question 4

(a) suppose all other students choose $\tilde{h}$ and look at the optimal response of student i

$$\max \quad \Phi\left(g_i(h^i, \tilde{h})\right) + (\bar{L} - h_i)$$

$$g_i(h^i, \tilde{h}) = \frac{h^i}{\frac{1}{I}\left((I-1)\tilde{h} + h^i\right)} = \frac{I h_i}{h_i + (I-1)\tilde{h}}$$

$$g_i'(h_i, \tilde{h}) = \frac{I\left[h_i + (I-1)\tilde{h}\right] - I h_i}{\left(h_i + (I-1)\tilde{h}\right)^2} = \frac{I(I-1)\tilde{h}}{\left(h_i + (I-1)\tilde{h}\right)^2}$$

FOC for optimal h_i

$$\frac{I(I-1)\tilde{h}}{\left(h_i + (I-1)\tilde{h}\right)^2}\,\varphi'\left(g_i(h_i, \tilde{h})\right) - 1 = 0$$

(since $\varphi'(g) \to +\infty$ if $g \to 0$, $h_i = 0$ cannot be a solution.

Symmetric Nash Equilibrium : $h_i = \tilde{h}$

$$\frac{I(I-1)\tilde{h}}{(I\tilde{h})^2}\,\varphi'(1) - 1 = 0.$$

$$\Rightarrow \quad \tilde{h} = \frac{I-1}{I}\,\varphi'(1) \quad \text{or} \quad 1 = \frac{I-1}{I\tilde{h}}\,\varphi'(1)$$

(b) The N.E solution $\tilde{h}$ increases with I. When $I = 1$ $\tilde{h} = 0$

and when $I \to \infty$ $\tilde{h} = \varphi'(1) \iff 1 = \frac{\varphi'(1)}{\tilde{h}}$.

If I is very large, a change in effort from the part of one student does not affect the average performance of the class.

The marginal cost of an increase Δh_i of working time is Δh_i

the marginal change in grade is $\Delta g_i = \frac{\Delta h_i}{\tilde{h}}$, so that

$$MC_i = MB_i \implies 1 = \frac{\varphi'(1)}{\tilde{h}}$$

When there are few students, a change Δh_i in effort changes both the performance of student i and the average performance so that Δg_i is smaller. The exact calculation is given by

$$g_i'(\tilde{h}, \tilde{h}) = \frac{I-1}{I\tilde{h}}$$

so that the marginal benefit of an additional Δh_i is $\varphi'(1) \frac{I-1}{I\tilde{h}} \Delta h_i$

and $MB_i = MC_i \implies 1 = \varphi'(1) \frac{I-1}{I\tilde{h}}$.

(c) Pareto optimal solution: $h_i = h_j \implies g_i = 1$ for all i

$$\max_{h \geq 0} \varphi(1) + L - h$$ is obtained for $h^* = 0$

(with $\lim \frac{h}{h} = 1$ when $h \to 0$)

Since there is no social utility from the work of a student and the "sorting" given by the grade does not change with the level of effort, the most efficient solution is $h_i = 0$ for all i.

(d)(i) The result of (c) is unintuitive. So it is worthwhile to introduce the value of graduate work in the term $\psi(h_i)$ which captures the useful information or ability that student acquires working. As usual, it is likely that the benefit will exhibit decreasing returns to effort.

(ii) The Nash equilibrium is defined by the equation

$$\psi'(\tilde{h}) + \frac{I-1}{I\tilde{h}}\psi'(1) = 1$$

and the symmetric Pareto optimal solution is defined by

$$\psi'(h^*) = 1$$

Thus $\psi'(\tilde{h}) < \psi'(h^*) \Rightarrow \tilde{h} > h^*$ since ψ is concave.

The relative performance grading still involves a "rat" race captured by the term $\frac{I-1}{I\tilde{h}}\psi'(1)$.

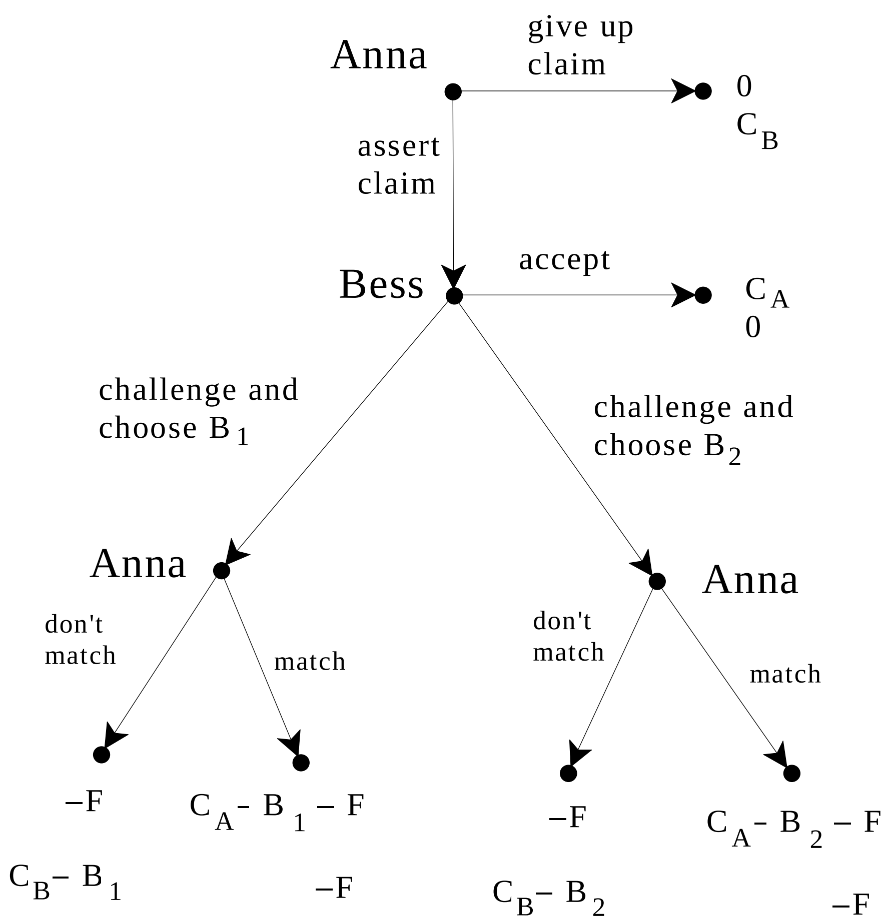
(d) In ~~this~~ model, grading by relative performance has no useful role. The standard justification is that there is randomness in the relation between work and performance and a relative performance evaluation "factors" out" the common shocks — like a difficult exam.

Microeconomics Prelim 2006

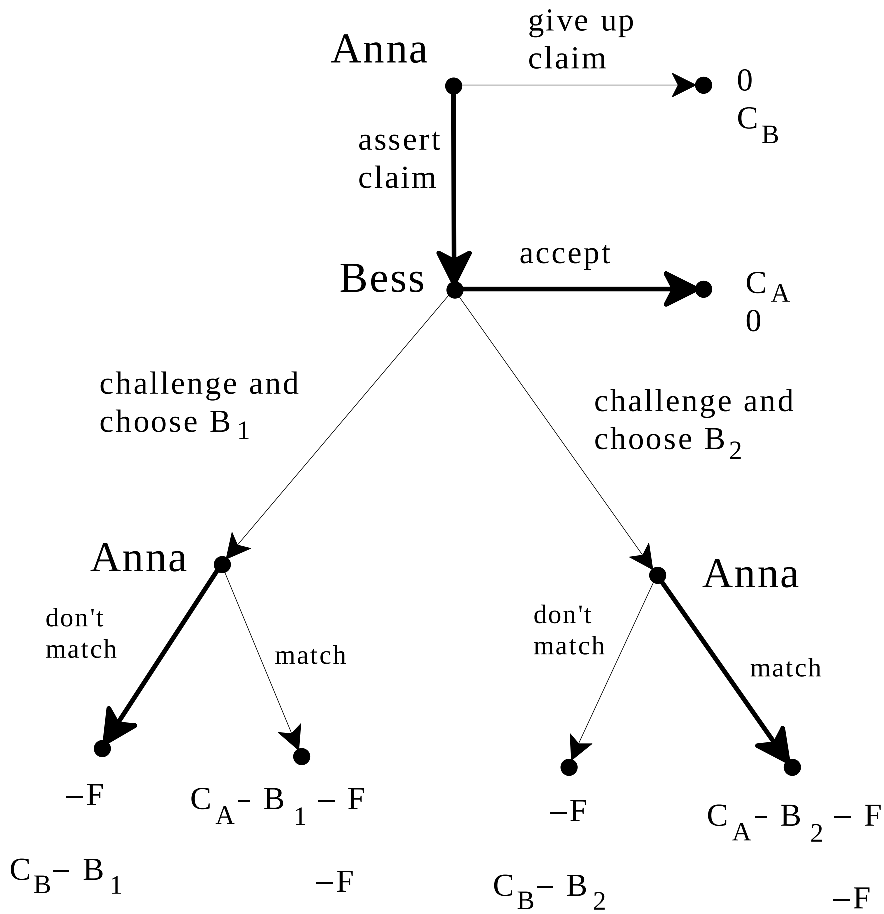
Answer Keys

5.

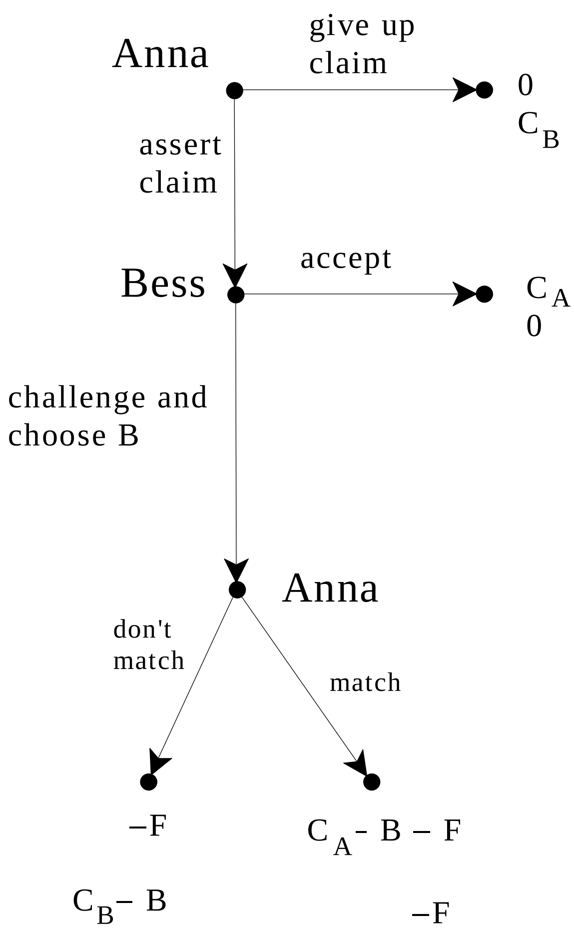
(a) The game is as follows:



(b) The backward-induction solution is marked by thick arrows below:



(c) The structure of the game is as follows:



Suppose that Anna is the true mother and she thus values the child more than Bess does: $C_A > C_B$. At the last node Anna will choose "match" if $C_A > B$ and "don't match" if $B \geq C_A$. In the first case Bess's payoff will be $-F$, while in the second case it will be $C_B - B$ which is negative since $B \geq C_A$ and $C_A > C_B$. Thus in either case Bess's payoff would be negative. Hence at her decision node Bess will choose "accept" (Bess can get the child at this stage only if she bids more than the child is worth to her). Anticipating this, Anna will assert her claim at the first decision node. Thus at the backward-induction solution the child goes to Anna, the true mother. The payoffs are C_A for Anna and 0 for Bess. **Note that no money changes hands**.

(d) Suppose that Bess is the true mother and she thus values the child more than Anna does: $C_B > C_A$. At the last node Anna will choose "match" if $C_A > B$ and "don't match" if $B \geq C_A$. In the first case Bess's payoff will be $-F$, while in the second case it will be $C_B - B$, which will be positive as long as $C_B > B$. Hence at her decision node Bess will choose to challenge and bid any amount B such that $C_B > B > C_A$. Anticipating this, at her first decision node Anna will give up (and get a payoff of 0), because if she asserted her claim then her final payoff would be $-F$. Thus at the backward-induction solution the child goes to Bess, the true mother. The payoffs are 0 for Anna and C_B for Bess. **Note that no money changes hands**.

(e) As pointed out above, in both cases no money changes hands at the backward-induction solution. Thus Solomon collects no money at all and both Ann and Bess pay nothing.