

Microeconomics Prelim June 2005

Answer Keys

1.

a. Proof of the property. Let $x^0 \in B^1 \subset B^0$, and let x^0 be chosen at (B^0, s) .

First, if $x^0 = s$, then, by (1), $u(s) \geq u(x^*(B^0)) - c(s)$. Because $B^1 \subset B^0$, $u(x^*(B^0)) \geq u(x^*(B^1))$, and hence $u(s) \geq u(x^*(B^1)) - c(s)$ and, again by (1), s is chosen at (B^1, s) .

Second, let $x^*(B^0) \neq s$ and let $x^0 = x^*(B^0)$ be chosen at (B^0, s) . Then, by (2), $u(s) \leq u(x^*(B^0)) - c(s)$. Because $x^0 = x^*(B^0) \in B^1$ and because $u(x^*(B^0)) \geq u(x^*(B^1))$, the uniqueness of the maximizers implies that $x^*(B^1) = x^*(B^0) = x^0$. Thus, x^0 is also chosen at (B^1, s) .

Revealed preference interpretation. If x^0 is chosen at (B^0, s) , then x^0 is revealed (weakly) preferred to any point in B^0 when the *status quo* is s . The consistency of choices would then dictate that x^0 be also revealed (weakly) preferred to any point in a subset B^1 of B^0 , as long as x^0 belongs to it and as long as the *status quo* is still s . This is precisely what the property says.

b. Consider the function

$f: X \times X \rightarrow \mathfrak{R}: f(x, s) = u(x) - g(x, s)$, where $g(s, s) = 0$ and, for $x \neq s$, $g(x, s) = c(s) > 0$.

When $X = \mathfrak{R}_+^N$, f is clearly discontinuous at any point in its domain of the form (s, s) , even when both u and c are continuous functions.

We now show that the maximization of f on B is equivalent to the choice rules (1) and (2).

First, let x^0 be chosen at (B, s) . If $x^0 = s$, then, by (1), $f(s, s) \equiv u(s) \geq u(x^*(B)) - c(s) \geq u(x) - c(s)$, for all $x \in B$, with $f(x, s) \equiv u(x) - c(s)$, for all $x \in B$, $x \neq s$, i. e., $x^0 = s$ maximizes f on B . (Trivially, $f(s, s) \geq f(s, s)$.) If, on the contrary, $x^0 \neq s$, then, by (2), $x^0 = x^*(B)$ and $f(s, s) \equiv u(s) \leq u(x^*(B)) - c(s) \equiv f(x^*(B), s)$. Moreover, if $x \in B$, then $u(x^*(B)) \geq u(x)$, by the definition of $x^*(B)$, and hence $f(x^*(B), s) \equiv u(x^*(B)) - c(s) \geq u(x) - c(s) \equiv f(x, s)$, for all $x \in B$, $x \neq s$. Summarizing, if x^0 is chosen at (B, s) , then x^0 maximizes $f(x, s)$ on B .

Conversely, let x^0 solve $\max_x f(x, s)$ subject to $x \in B$. If $x^0 = s = x^*(B)$, then (1) is trivially satisfied. If $x^0 = s \neq x^*(B)$, then, because of maximization, $f(s, s) \geq f(x, s)$ for all $x \in B$, $x \neq s$, i. e., $u(s) \geq u(x) - c(s)$ for all $x \in B$, $x \neq s$, or $u(s) + c(s) \geq u(x)$ for all $x \in B$, $x \neq s$. This together with the fact that $s \neq x^*(B)$ implies that $u(s) + c(s) \geq u(x^*(B))$ (otherwise there would be an x in B , $x \neq s$, with $u(s) + c(s) < u(x)$, a contradiction). Hence, (1) is satisfied. Last, let $x^0 \neq s$. Then $f(x^0, s) \equiv u(x^0) - c(s) \geq f(x, s) \equiv u(x) - g(x, s)$ for all $x \in B$, i. e., $u(x^0) \geq u(x) + [c(s) - g(x, s)] \geq u(x)$, $x \in B$, i. e., $x^0 = x^*(B)$. Moreover, because $x \in B$, $u(x^*(B)) - c(s) \geq f(s, s) = u(s)$, which is (2). Thus, the maximization of f on B yields the *status-quo*-biased choice of (1) and (2).

Summarizing, the choice rules (1) and (2) are equivalent to the maximization of a discontinuous function that subtracts any moving costs from the utility of consumption.

c.1. Standard Cobb-Douglas maximization subject to the budget constraint given by the initial endowments $(\omega_1, \omega_2) = (2, 2)$. Hence:

$$(x_1^*(p), x_2^*(p)) = (1 + p, 1 + (1/p)); v(p) = (1 + p)^{0.5} (1 + (1/p))^{0.5} = (1 + p) p^{-0.5},$$

$$x_2^*(1) = 2 = v(1).$$

Graphs: $x_2^*(p) = 1 + (1/p)$, always decreasing, see Figure 1.

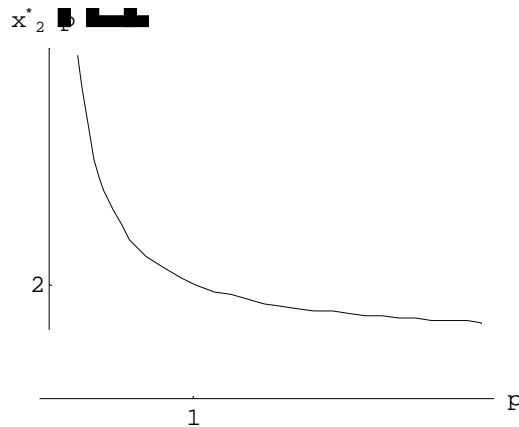


Figure 1

One can compute $v'(p) = 0.5 p^{-0.5}(1 - (1/p))$. Hence, $v(p)$ is decreasing for $p < 1$ and increasing for $p > 1$. See Figure 2.

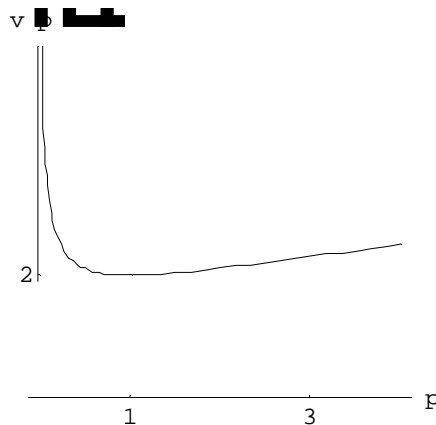


Figure 2

c.2. Following 2, the consumer maximizes $u(x) - g(x, s)$. If the consumer stays at the *status quo* $s \equiv (2, 2)$, her utility is $2^{0.5}2^{0.5} = 2$. This is obviously her choice if $p = 1$, in which case

$(x_1^*(p), x_2^*(p)) = s$. If $p \neq 1$, then choosing $(x_1^*(p), x_2^*(p))$ entails a moving cost of $c(s) = 0.2$: the consumer then chooses $(x_1^*(p), x_2^*(p))$ if $v(p) - 0.2 \leq 2$, and chooses s if $v(p) - 0.2 \geq 2$. See Figure 3.

To find the prices at which $v(p) - 0.2 = 2$, we solve the equation

$(1 + p) p^{-0.5} - 0.2 = 2$, i. e., $(1 + p) p^{-0.5} = 2.2$, i.e., $(1 + p)^2 = 4.84 p$, i.e., $1 + 2p + p^2 = 4.84 p$, i.e., $p^2 - 2.84 p + 1 = 0$, with solutions $p = 0.412$ and $p = 2.428$, as indicated in Figure 3.

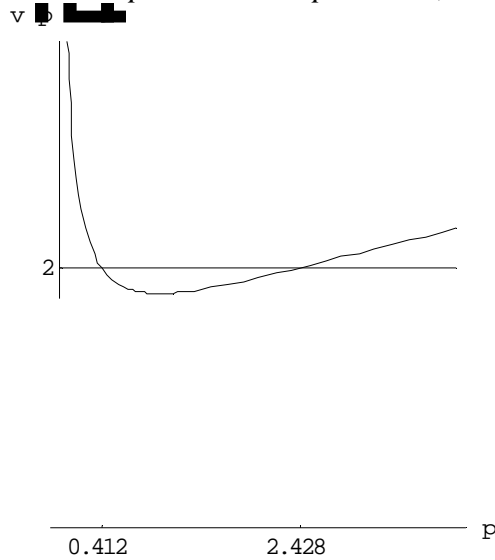


Figure 3

Thus, maximizing $u(x) - g(x, s)$ subject to the budget constraint given by p requires

* setting $(x_1, x_2) = (2, 2) \equiv s$, if $p \in [0.412, 2.428]$

* setting $(x_1, x_2) = (x_1^*(p), x_2^*(p))$, if either $p \leq 0.412$ or $p \geq 2.428$.

Intuitively, at price 1 the consumer has no incentive to move from $s = (2, 2)$ even if it entailed zero moving costs. For prices close to 1, the gains from moving are too small to compensate for the moving costs, and the consumer stays at $s = (2, 2)$. But if p is very small or very large, the consumer can afford points far away from $s = (2, 2)$ that yield a high utility, more than compensating the moving cost. See Figure 4.

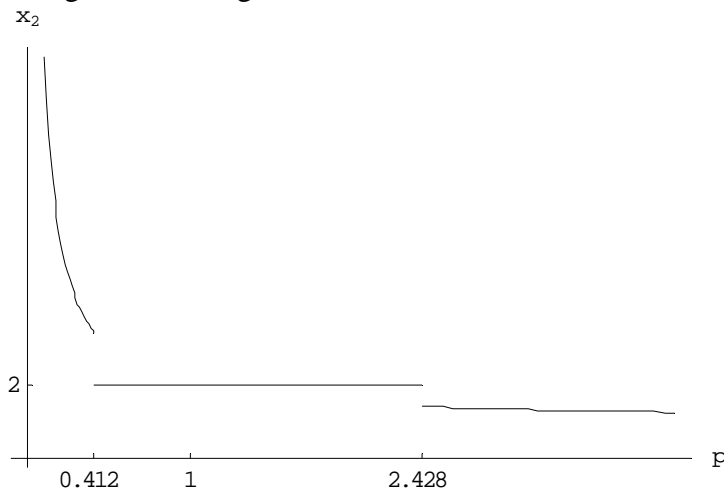
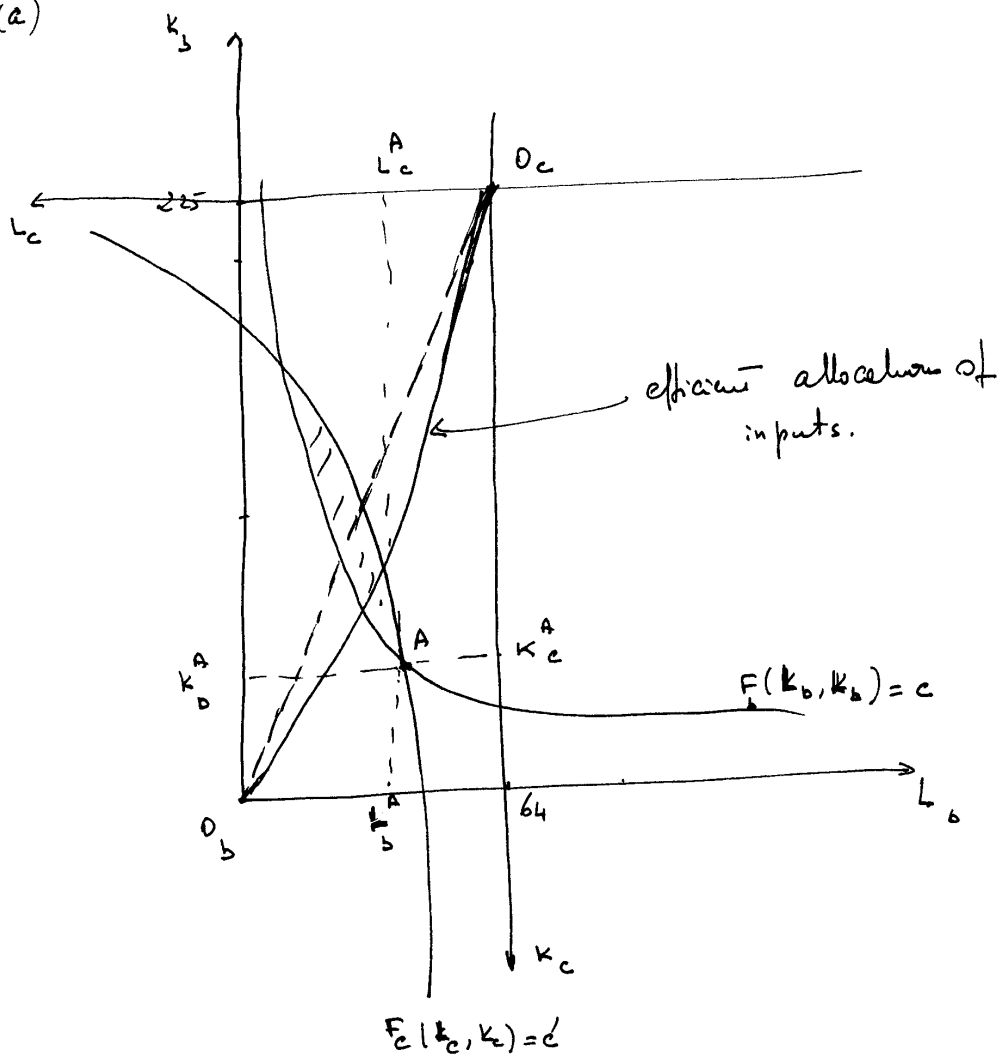


Figure 4

2.

2 (a)



By construction, a point ^A inside the Edgeworth box is such that

$$K_b^A + K_c^A = 225$$

$$L_b^A + L_c^A = 64$$

i.e. it represents a feasible allocation of

the available inputs among the 2 sectors. If the isoquants

$$F_b(L_b, K_b) = F_b(L_b^A, K_b^A) \quad \text{and} \quad F_c(L_c, K_c) = F_c(L_c^A, K_c^A)$$

cut at A, then all allocations in the shaded region give more of $\textcircled{2}$ the 2 outputs. At an efficient allocation the isoquants must be tangent

$$(b) \text{ tangency } \Leftrightarrow \frac{\frac{\partial F_b}{\partial L_b} = \frac{\partial F_c}{\partial L_c}}{\frac{\partial F_b}{\partial K_b} = \frac{\partial F_c}{\partial K_c}} \Leftrightarrow \frac{\frac{2}{3} L_b^{-1/3} K_b^{1/3}}{\frac{1}{3} L_b^{2/3} K_b^{-2/3}} = \frac{\frac{1}{3} L_c^{-2/3} K_c^{2/3}}{\frac{2}{3} L_c^{1/3} K_c^{-1/2}}$$

$$\Leftrightarrow 2 \frac{K_b}{L_b} = \frac{1}{2} \frac{K_c}{L_c}$$

$$\text{feasibility } \Leftrightarrow \begin{aligned} K_b + L_c &= 64 \\ K_b + K_c &= 225 \end{aligned}$$

$$\text{efficient allocation } \Leftrightarrow 2 \frac{K_b}{L_b} = \frac{1}{2} \frac{225 - K_b}{64 - L_b}$$

$$\Leftrightarrow K_b = \frac{225 L_b}{256 - 3L_b}$$

$$\text{Study the function } g(x) = \frac{225x}{256 - 3x} \quad g'(x) = \frac{225 \times 256}{(256 - 3x)^2} > 0$$

$$g''(x) > 0 \quad (g' \text{ is increasing in } x).$$

$g(64) = 225$. Thus the graph of g goes through $0_b, 0_c$ and, since g is convex, is below the diagonal of the Edgeworth box

interpretation: Sector b (bread) is more labor intensive ⁽³⁾
 and sector c (cars) is more capital intensive. Thus,
 in an efficient allocation of inputs, $\frac{K_b}{L_b}$ must be
 less than the average ratio $\frac{225}{64}$ while $\frac{K_c}{L_c}$ must be
 larger than the average ratio $\frac{225}{64}$, which is the slope
 of the diagonal.

(c) Equilibrium.

Given that in both firms the marginal product of labor and
 capital tend to ∞ when the corresponding input tends to
 zero, we do not need to worry about non-negativity constraints,
 when studying the max. problem of firms.

Firm b:
$$\max p_b L_b^{2/3} K_b^{1/3} - wL_b - rK_b$$

$$\begin{cases} \frac{2}{3} p_b L_b^{-1/3} K_b^{1/3} = w \\ \frac{1}{3} p_b L_b^{2/3} K_b^{-2/3} = r \end{cases}$$

$$\Rightarrow \begin{cases} 2 \frac{K_b^{1/3}}{L_b^{1/3}} = \frac{w}{r} \Rightarrow \boxed{K_b = \left(\frac{w}{2r}\right) L_b} \\ \frac{2}{3} p_b L_b^{-1/3} K_b^{1/3} = w \end{cases}$$

Substituting for K_b in the second relation gives the ⁽⁴⁾
 condition on prices which ensures zero profit:

$$\frac{2}{3} P_b L_b^{-1/3} \left(\frac{w}{2r} L_b \right)^{1/3} = w$$

$$P_b = \frac{3}{2} w \left(\frac{2r}{w} \right)^{1/3} = \frac{3}{2} w^{-2/3} (2r)^{1/3} = P_b$$

$$y_b = L_b^{2/3} K_b^{1/3} \Rightarrow y_b = \left(\frac{w}{2r} \right)^{1/3} L_b$$

Firm c: same calculation gives

$$\frac{1}{2} \frac{K_c}{L_c} = \frac{w}{r} \Leftrightarrow K_c = \left(\frac{2w}{r} \right) L_c$$

$$P_c = 3 w^{1/3} \left(\frac{r}{2} \right)^{2/3}$$

$$y_c = \left(\frac{2w}{r} \right)^{1/3} L_c$$

Consumer:

$$\max \alpha x_b^{1-\alpha} x_c^\alpha$$

$$P_b x_b + P_c x_c = 64w + 225r$$

$$\alpha x_b = \alpha \frac{64w + 225r}{P_b} = \alpha \frac{64w + 225r}{3/2 w^{2/3} (2r)^{1/3}}$$

$$\alpha x_c = (1-\alpha) \frac{64w + 225r}{P_c} = (1-\alpha) \frac{64w + 225r}{3 w^{1/3} \left(\frac{r}{2} \right)^{2/3}}$$

market clearing: normalize $w=1$. The equilibrium ^{price of capital} ~~interest rate~~ must be found by clearing either the market for labor or the market for capital, knowing that the firms will produce the demand of the consumer (constant returns).

$$L_b = y_b \left(\frac{2r}{3} \right)^{1/3} = \alpha \frac{64 + 225r}{3 \left(\frac{2r}{3} \right)^{1/3}} \cdot \left(\frac{2r}{3} \right)^{1/3}$$

$$= \frac{2}{3} \alpha (64 + 225r)$$

$$L_c = y_c \left(\frac{r}{2} \right)^{2/3} = (1-\alpha) \frac{64 + 225r}{3 \left(\frac{r}{2} \right)^{2/3}} \cdot \left(\frac{r}{2} \right)^{2/3}$$

$$= \frac{1-\alpha}{3} (64 + 225r)$$

$$L_b + L_c = 64 \quad (\Rightarrow) \quad \frac{2}{3} \alpha (64 + 225r) + \frac{1-\alpha}{3} (64 + 225r) = 64$$

$$(\Rightarrow) (64 + 225r) \left(\frac{1+\alpha}{3} \right) = 64$$

$$(\Rightarrow) \frac{1+\alpha}{3} \cdot 225r = 64 \left(\frac{2-\alpha}{3} \right)$$

$$(\Rightarrow) \boxed{r = \frac{64}{225} \frac{2-\alpha}{1+\alpha}}$$

The function $\alpha \rightarrow \frac{2-\alpha}{1+\alpha}$ is decreasing in α , thus ⑥
the price of capital decreases when α increases. This
was expected since, when α increases, the demand
for bread — the labor intensive good — increases and the
relative price of labor increases — or the relative price of
capital falls.

3.

3. (a) Pareto optimum allocations

(7)

$$\max \alpha \left(\frac{1}{2} \ln x_1 + \frac{1}{2} \ln y \right) + (1-\alpha) \left(\frac{1}{3} \ln x_2 + \frac{2}{3} \ln y \right)$$

$$x_1 + x_2 + y \leq 6$$

(at a P.O. both consumers consume the produced quantity of the public good).

$$FOC_3 \quad \frac{\alpha}{2x_1} = \lambda \quad \frac{1-\alpha}{3x_2} = \lambda \quad \frac{1}{2} \alpha \frac{1}{y} + \frac{2}{3} (1-\alpha) \frac{1}{y} = \lambda$$

$$x_1 = \frac{\alpha}{2\lambda} \quad x_2 = \frac{1-\alpha}{3\lambda} \quad y = \frac{\frac{1}{2} \alpha + \frac{2}{3} (1-\alpha)}{\lambda}$$

Using the feasibility constraint:

$$\frac{\alpha}{2\lambda} + \frac{1-\alpha}{3\lambda} + \frac{\frac{1}{2} \alpha + \frac{2}{3} (1-\alpha)}{\lambda} = 6 \quad \Rightarrow \quad \lambda = \frac{1}{6}$$

$$\Rightarrow \boxed{x_1 = 3\alpha \quad x_2 = 2(1-\alpha) \quad y = 6 \left(\frac{2}{3} - \frac{1}{6} \alpha \right)}$$

$$\frac{2}{3} - \frac{1}{6} \alpha \geq \frac{2}{3} - \frac{1}{6} = \frac{1}{2} \quad \Rightarrow \quad y \geq \frac{6}{2} = 3.$$

The production of public good decrease with α , since agent 2 likes the public good relatively more than agent 1. Decreasing the weight on agent 2 decreases the P.O. level of public good. But the minimum of public good produced is 3 units.

(b) Voluntary contributions.

(8)

$$\text{agent 1: } \max_{z_1 \geq 0} \frac{1}{2} \ln(4 - z_1) + \frac{1}{2} \ln(z_1 + z_2)$$

$$\text{FOC: } -\frac{1}{2(4 - z_1)} + \frac{1}{2} \frac{1}{z_1 + z_2} \leq 0 \quad = 0 \quad \text{if } z_2 > 0$$

assume equality $z_1 + z_2 = 4 - z_1$

$$2z_1 = 4 - z_2 \quad z_1 = \frac{4 - z_2}{2} \quad \text{if } z_2 \leq 4.$$

Since agent 2 cannot contribute more than 2 units, agent 1 will always contribute.

$$\text{agent 2: } \max_{z_2 \geq 0} \frac{1}{3} \ln(2 - z_2) + \frac{2}{3} \ln(z_1 + z_2)$$

$$\text{FOC: } -\frac{1}{3(2 - z_2)} + \frac{2}{3} \frac{1}{z_1 + z_2} \leq 0 \quad = 0 \quad \text{if } z_2 > 0$$

$$\text{assume equality } \frac{z_1 + z_2}{2} = 2 - z_2 \Leftrightarrow \frac{3}{2} z_2 = 2 - z_1$$

$$z_2 = \frac{4 - z_1}{3} \quad \text{if } z_1 \leq 4. \quad \text{Since agent 1 cannot contribute}$$

more than 4 units, agent 2 will always contribute.

$$\text{equilibrium: } \begin{cases} z_1 = \frac{4 - z_2}{2} \\ z_2 = \frac{4 - z_1}{3} \end{cases} \Leftrightarrow z_1 = \frac{6}{5}, \quad z_2 = \frac{8}{5} \quad y = \frac{14}{5} < 3.$$

Since $y < 3$, the allocation cannot be Pareto optimal. ③
 There is underproduction of the public good, because agents equalize marginal cost to private marginal benefit instead of social marginal benefit.

(e) (i) constrained demands

agent 1: $\max \frac{1}{2} \ln x_1 + \frac{1}{2} \ln y$

$$x_1 + py \leq 4 \quad \lambda_1$$

$$y \leq \bar{y} \quad \mu_1$$

FOCs: $\frac{1}{2x_1} = \lambda_1 \quad \frac{1}{2y} = \lambda_1 p + \mu_1 \quad \mu_1 (y - \bar{y}) = 0.$

assume $\mu_1 = 0 \quad x_1 = \frac{1}{2\lambda_1} \quad y = \frac{1}{2\lambda_1 p} \quad \frac{1}{2\lambda_1} + \frac{1}{2\lambda_1} = 4 \quad \lambda_1 = \frac{1}{4}$

$x_1 = 2 \quad y = \frac{2}{p}$. This is the solution if $\frac{2}{p} \leq \bar{y}$, i.e. if

$p \geq \frac{2}{\bar{y}}$. If $p < \frac{2}{\bar{y}}$ the agent is rationed at \bar{y} .

$$y_i(p, \bar{y}) = \begin{cases} \bar{y} & \text{if } p \leq \frac{2}{\bar{y}} \\ \frac{2}{p} & \text{if } p \geq \frac{2}{\bar{y}} \end{cases}$$

agent 2: The same type of calculation gives

(10)

$$y_2(p, \bar{y}) = \begin{cases} \bar{y} & \text{if } p \leq \frac{4}{3\bar{y}} \\ \frac{4}{3p} & \text{if } p \geq \frac{4}{3\bar{y}} \end{cases}$$

(ii) In order that both agents buy \bar{y} , the price must be such that $p \leq \frac{2}{\bar{y}}$ and $p \leq \frac{4}{3\bar{y}} < \frac{2}{\bar{y}}$

Thus the max. \bar{y} for which both agents buy \bar{y} at price $\frac{1}{2}$ is $\frac{1}{2} = \frac{4}{3\bar{y}} \Rightarrow \bar{y} = \frac{8}{3}$

$$x_1 = 4 - \frac{1}{2} \times \frac{8}{3} = 4 - \frac{4}{3} = \frac{8}{3}$$

$$y = \frac{8}{3}$$

$$x_2 = 2 - \frac{1}{2} \times \frac{8}{3} = 2 - \frac{4}{3} = \frac{2}{3}$$

(iii) In order that no agent is entrained it must be that $p \geq \frac{2}{\bar{y}}$. Budget balance of the public firm implies

$$\frac{4}{3p} \times p + \frac{2}{p} \times p = \bar{y} \Rightarrow \bar{y} = \frac{10}{3} \quad p = \frac{2}{10/3} = 6/10$$

Agent 1 buys $\frac{2}{6/10} = \frac{10}{3} = \bar{y}$ while agent 2

$$\text{buys } \frac{4}{3 \times 6/10} = \frac{20}{9} < \frac{10}{3}$$

The equilibrium allocation is

$$\alpha_1 = 4 - 2 = 2 \quad y_1 = \bar{y} = \frac{10}{3}$$

$$\alpha_2 = \frac{2}{3} \quad y_2 = \frac{20}{9} < \bar{y}$$

Compared to the equilibrium in (i) agent 2 is worse off since she consumes the same quantity of private good and less public good $\left(\frac{20}{9} < \frac{8}{3} = \frac{24}{9}\right)$

Agent 1 has utility $\frac{1}{2} \ln \frac{20}{3}$ while in (ii) his utility was $\frac{1}{2} \ln \frac{8 \times 8}{9}$. Since $\frac{64}{9} \approx 7.1 > \frac{20}{3} \approx 6.7$ agent 1 has more utility in (ii). There is clearly waste in the allocation of (iii) since agent 2 is artificially prevented from consuming the whole available public good.

(iv) Comparison between (ii) and V.C. equilibrium

$$V.C.: \alpha_1 = 4 - \frac{6}{5} = \frac{14}{5} \quad \alpha_2 = 2 - \frac{8}{5} = \frac{2}{5} \quad y = \frac{14}{5}$$

$$u_1^{V.C.} = \frac{1}{2} \left(\ln \frac{14}{5} + \ln \frac{14}{5} \right) = \ln \frac{14}{5}$$

$$u_2^{V.C.} = \frac{1}{3} \ln \frac{2}{5} + \frac{2}{3} \ln \frac{14}{5} \\ = \frac{1}{3} \ln \frac{2 \times 14^2}{5^3}$$

$$\text{in (ii)} \quad \bar{u}_1 = \ln \frac{8}{3} \quad \bar{u}_2 = \frac{1}{3} \ln \frac{2}{3} \left(\frac{8}{3}\right)^2$$

(12)

Since $\frac{14}{5} > \frac{8}{3}$ and $\frac{14^2}{5^3} < \frac{8^2}{3^3}$
 $u_1^{vc} > \bar{u}_1$ and $u_2^{vc} < \bar{u}_2$

so that the allocations are not comparable. In the voluntary contribution equilibrium agent 1 contributes up to the point where his marginal cost = private marginal benefit, while in (ii) where all agents face the same price, he is retained and marginal cost < private marginal benefit. Thus in (ii) Agent 1 "free rides" on agent 2. ~~As a consequence~~ As a consequence the provision of public good is smaller in (ii) than in the voluntary contribution equilibrium, and this hurts agent 2 who has a lot of utility weight on the public good. So using a common price for the public good does not seem to be a panacea, even when it can work.

4. (a) The strategy sets are $S_1 = S_2 = [0, \infty)$. The payoff functions are as follows:

$$\pi_1(x, y) = \begin{cases} X - y & \text{if } x \geq y \\ 0 & \text{if } y > x \end{cases} \quad \text{and} \quad \pi_2(x, y) = \begin{cases} A - Y + y & \text{if } x \geq y \\ A & \text{if } y > x \end{cases}$$

(b) **Suppose that $X > Y$.** If (x, y) is a Nash equilibrium **with $x \geq y$** then it must be that $y \leq X$ (otherwise player 1 could increase its payoff by reducing x below y) and $y \geq Y$ (otherwise player 2 would be better off by increasing y above x). Thus it must be $Y \leq y \leq X$, which is possible, given our assumption. However, it cannot be that $x > y$, because player 2 would be getting a higher payoff by increasing y to x . Thus it must be $x \leq y$, which implies that $x = y$. Thus the following are Nash equilibria:

all the pairs (x, y) with $Y \leq y \leq X$ and $x = y$.

Now consider pairs (x, y) with $x < y$. Then it cannot be that $y < X$, because player 1 could increase its payoff by increasing x to y . Thus it must be $y \geq X$ (hence – by our supposition – $y > Y$). Furthermore, it must be that $x \leq Y$ (otherwise player 2 could increase its profits by reducing y to (or below) x). Thus

(x, y) with $x < y$ is a Nash equilibrium if and only if $x \leq Y$ and $y \geq X$.

(c) **Suppose that $X < Y$.** For the same reasons given above, an equilibrium with $x \geq y$ requires $Y \leq y \leq X$. However, this is not possible given that $X < Y$. Hence,

there is no Nash equilibrium (x, y) with $x \geq y$.

Thus we must restrict attention to pairs (x, y) with $x < y$. As explained before, it must be that $y \geq X$ and $x \leq Y$. Thus,

(x, y) with $x < y$ is a Nash equilibrium if and only if $X \leq y$ and $x \leq Y$.

(d) Pareto efficiency requires that the chemical plant be shut down if $X < Y$ that it remain operational if $X > Y$. Now, when $X < Y$ all the equilibria have $x < y$ which leads to shut-down, hence a Pareto efficient outcome. When $X > Y$, there are two types of equilibria: one where $x = y$ and the plant remains operational (a Pareto efficient outcome) and the other where $x < y$ in which case the plant shuts down, yielding a Pareto inefficient outcome.

(e) Yes, for player 1 choosing $x = X$ is a dominant strategy. Proof. Fix an arbitrary y . We must show that $x = X$ gives at least as high a payoff against y as any other x . Three cases are possible. **Case 1: $y < X$.** In this case $x = X$ or any other x s.t. $x \geq y$ yields $\pi_1 = X - y > 0$, while $x < y$ yields $\pi_1 = 0$. **Case 2: $y = X$.** In this case 1's payoff is zero no matter what x he chooses. **Case 3: $y > X$.** In this case $x = X$ or any other x s.t. $x < y$ yields $\pi_1 = 0$, while $x \geq y$ yields $\pi_1 = X - y < 0$.

(f) No, choosing $y = Y$ is not a dominant strategy for player 2. For example, if $x > Y$ then choosing $y = Y$ yields $\pi_2 = A$ while choosing a y s.t. $Y < y \leq x$ yields $\pi_2 = A - Y + y > A$.

5. Inverse demand is given by $P_H(Q) = \frac{12-Q}{b_H}$ and $P_L(Q) = \frac{12-Q}{b_L}$.

(a) Let $W_H(Q)$ be the willingness to pay of the H consumer for Q units of output. Then

$$W_H(Q) = \frac{1}{b_H} \left(12Q - \frac{Q^2}{2} \right). \text{ For Option 1 the monopolist would choose within the set}$$

$\{(Q, V) : V = W_H(Q)\}$. The corresponding profits are $\pi_1(Q) = \frac{1}{b_H} \left(12Q - \frac{Q^2}{2} \right) - Q$. This

function is strictly concave. Solving $\frac{d\pi_1}{dQ} = 0$ we get $Q_1^* = 12 - b_H$, $V_1^* = \frac{144 - b_H^2}{2b_H}$ with

corresponding profits $\pi_1^* = \frac{(12 - b_H)^2}{2b_H}$.

(b) Let $W_L(Q)$ be the willingness to pay of the L consumer for Q units of output. Then

$$W_L(Q) = \frac{1}{b_L} \left(12Q - \frac{Q^2}{2} \right). \text{ For Option 2 the monopolist would choose within the set}$$

$\{(Q, V) : V = W_L(Q)\}$. The corresponding profits are $\pi_2(Q) = 2 \left[\frac{1}{b_L} \left(12Q - \frac{Q^2}{2} \right) - Q \right]$. This

function is strictly concave. Solving $\frac{d\pi_2}{dQ} = 0$ we get $Q_2^* = 12 - b_L$, $V_2^* = \frac{144 - b_L^2}{2b_L}$ with

corresponding profits $\pi_2^* = \frac{(12 - b_L)^2}{b_L}$.

(c) Let (Q_H, V_H) and (Q_L, V_L) be the two packages. Then the incentive compatibility constraints are:

- (1) $V_L \leq W_L(Q_L)$: the L -consumer is willing to buy “her” package
- (2) $W_L(Q_L) - V_L \geq W_L(Q_H) - V_H$: incentive compatibility constraint for the L -consumer (she does not prefer the H package to the L package)
- (3) $V_H \leq W_H(Q_H)$: the H -consumer is willing to buy “her” package
- (4) $W_H(Q_H) - V_H \geq W_H(Q_L) - V_L$ incentive compatibility constraint for the H -consumer (she does not prefer the L package to the H package);

Since $W_H(Q) > W_L(Q)$, (3) follows, as a strict inequality, from (1) and (4). Let S be the set of pairs $((Q_H, V_H), (Q_L, V_L))$ that satisfy (1) – (4) and let $((Q_H^*, V_H^*), (Q_L^*, V_L^*)) \in S$ be a pair that

maximizes the firm's profits within S . Then (1) and (4) must be satisfied as equalities, that is,

$$V_L^* = W(Q_L^*) = \frac{1}{b_L} \left(12Q_L^* - \frac{(Q_L^*)^2}{2} \right) \quad \text{and}$$

$$V_H^* = W_H(Q_H^*) - W_H(Q_L^*) + V_L^* = \frac{1}{b_H} \left(12Q_H^* - \frac{(Q_H^*)^2}{2} - 12Q_L^* + \frac{(Q_L^*)^2}{2} \right) + \frac{1}{b_L} \left(12Q_L^* - \frac{(Q_L^*)^2}{2} \right)$$

(d) Let S^* be the set of pairs $((Q_H, V_H), (Q_L, V_L))$ that satisfy the two boxed equalities. Then if $((Q_H, V_H), (Q_L, V_L))$ maximizes the firm's profits in S it must belong to S^* . Within S^* the profit of the firm is given by

$$\pi = (V_H - Q_H) + (V_L - Q_L) = \frac{1}{b_H} \left(12Q_H - \frac{(Q_H)^2}{2} - 12Q_L + \frac{(Q_L)^2}{2} \right) + \frac{2}{b_L} \left(12Q_L - \frac{(Q_L)^2}{2} \right) - Q_H - Q_L$$

The Hessian of π is

$$H = \begin{pmatrix} \frac{\partial^2 \pi}{\partial (Q_H)^2} = -\frac{1}{b_H} & \frac{\partial^2 \pi}{\partial Q_L \partial Q_H} = 0 \\ \frac{\partial^2 \pi}{\partial Q_L \partial Q_H} = 0 & \frac{\partial^2 \pi}{\partial (Q_L)^2} = \frac{b_L - 2b_H}{b_L b_H} \end{pmatrix}$$

Which is negative definite since $b_L < 2b_H$. Thus the maximum of π is found by solving

$$\frac{\partial \pi}{\partial Q_H} = \frac{12 - b_H - Q_H}{b_H} = 0 \quad \text{and} \quad \frac{\partial \pi}{\partial Q_L} = \frac{24b_H - 12b_L - b_L b_H + b_L Q_L - 2b_H Q_L}{b_H b_L} = 0$$

The solution is

$$Q_H^* = 12 - b_H \quad \text{and} \quad Q_L^* = \frac{24b_H - 12b_L - b_L b_H}{2b_H - b_L}$$

(e) When $b_H = 2$ and $b_L = 3$ then $\pi_1^* = 25$, $\pi_2^* = 27$ and $\pi_3^* = 28$. Thus Option 3 is the best. The two packages are $(Q_H = 10, V_H = 26)$ and $(Q_L = 6, V_L = 18)$ with implied per-unit prices of $P_H = 2.6$ and $P_L = 3$. The L -consumer gets zero surplus, while the H -consumer gets a surplus of $W_H(10) - 26 = 35 - 26 = 9$. Thus total surplus is $\pi_3^* + 9 = 28 + 9 = 37$.

(f) With first-degree price discrimination (in the form of a two-part tariff) the monopolist would set $P = MC = 1$ and sell $Q_H = 12 - b_H = 10$ units to the H -consumer and $Q_L = 12 - b_L = 9$ units to the L -consumer and extract all the surplus by charging tariffs of $T_H = W_H(10) - 10 = 35 - 10 = 25$ and $T_L = W_L(9) - 9 = 13.5$ with corresponding profits of $\pi = 25 + 13.5 + 19 - 19 = 38.5$.