ANSWER KEY

Micro Prelim, Fall 2010

QUESTION 1. WHAT KIND OF COMPENSATION?

An economy is comprised of I persons, numbered 1 to I. Person i is endowed with a preference relation $\succsim^i$ defined on an abstract set A of *feasible states*. We assume that $\succsim^i$ can be represented by a utility function $\beta^i : A \to \Re$.

1(a). What do we mean when we say that a feasible state $\overline{\overline{s}}$ is *better than another feasible state* $\overline{s}$ *according to the Pareto criterion*? Graphically illustrate in utility space for the case of two people.

ANSWER. A feasible state $\overline{\overline{s}}$ is *better than another feasible state* $\overline{s}$ *according to the Pareto criterion* if $\overline{\overline{s}} \succsim^i \overline{s}$ (or $\beta^i(\overline{\overline{s}}) \geq \beta^i(\overline{s})$) for all *i*, with strict preference (or utility inequality) for at least one *i*. See Figure 1.

A *feasible redistribution correspondence* is a correspondence Φ: $A \to\to A$ interpreted as follows: $\Phi(s)$ is the *set of states that can be reached from state s by feasible redistributions.*

1(b). Let Φ be given. What do we mean when we say that a feasible state $\overline{\overline{s}}$ is *better than another feasible state* $\overline{s}$ *according to the weak potential compensation criterion for the redistribution correspondence* Φ? Graphically illustrate in utility space for the case of two people.

ANSWER. A feasible state $\overline{\overline{s}}$ is *better than another feasible state* $\overline{s}$ *according to the weak potential compensation criterion for the redistribution correspondence* Φ if there is a state $s' \in \Phi(\overline{\overline{s}})$ that all persons prefer to $\overline{s}$. See Figure 2.

Now we endow the set A with some structure. Assume that $A \subset \Re^{IN+M}$, i. e., a state s is of the form $s = (s_1^1,\ldots,s_N^1; s_1^2,\ldots,s_N^2;\ldots; s_1^i,\ldots,s_N^i;\ldots; s_1^I,\ldots,s_N^I; s_1,\ldots,s_M)$, and that, for $i = 1,\ldots, I$, the utility function β^i is increasing in s_j^i and constant with respect to s_j^h, $h \neq i$, $j = 1,\ldots, N$. The interpretation is that there are N goods, or attributes, that are desirable and do not generate externalities among persons, and that the amount s_j^i of desirable good or attribute $j \in \{1,\ldots,N\}$ is allocated to Person i. In addition, there may be some other M components of the state, e. g., public goods or bads, prices, non-desirable private goods, or private goods that generate externalities in consumption. It follows that Person *i* may care only about the components $(s_1^i,...,s_N^i;s_1,...,s_M)$ of the state. The formulation is general enough to cover the type of states studied in Walrasian demand theory (see 1(d) below) as well as the example in 1(f)-(i) below.

Given $j \in \{1,...,N\}$, we define the correspondence Φ_j: $A \rightarrow\rightarrow A$ as follows:

$\Phi_j(s) = \{s' \in \Re^{IN+M} : \sum_i s_j^i{}' = \sum_i s_j^i, s_k^i{}' = s_k^i, for\ k \neq j, i = 1,...,I, s_m{}' = s_m, for\ m = 1,...,M\}$.

In words, according to the correspondence Φ_j, state *s'* can be reached from *s* by redistributing good or attribute *j* among the *I* people, leaving unchanged all other components of the vector *s*.

We adopt the notation $s = (s_{-ij}; s_j^i)$, where the vector s_{-ij} comprises all the components of the vector *s* except s_j^i.

Given $j \in \{1,...,N\}$, $i \in \{1,...,I\}$ and two feasible states $\bar{s}$ and $\bar{\bar{s}}$, we define the *compensating modification* $CM_j^i(\bar{s},\bar{\bar{s}})$ (in good or attribute *j*, for Person *i*, and for a move from $\bar{s}$ and $\bar{\bar{s}}$) implicitly by the equation

$$\beta^i(\bar{\bar{s}}_{-ij}, \bar{\bar{s}}_j^i - CM_j^i(\bar{s},\bar{\bar{s}})) = \beta^i(\bar{s}). \quad (1)$$

1(c). Interpret $CM_j^i(\bar{s},\bar{\bar{s}})$ in words.

ANSWER. The compensating modification $CM_j^i(\bar{s},\bar{\bar{s}})$ is the change in the amount of desirable good or attribute *j* that makes *i* indifferent between $\bar{s}$, on the one hand, and $\bar{\bar{s}}$ together with the change, on the other. If $\bar{\bar{s}}$ is worse than $\bar{s}$ for *i*, then a positive amount of good or attribute *j* must be added to $\bar{\bar{s}}_j^i$ to bring *i*'s utility up to $\beta^i(\bar{s})$, i. e., $CM_j^i(\bar{s},\bar{\bar{s}}) < 0$. If $\bar{\bar{s}}$ is better than $\bar{s}$ for *i*, then a positive amount of good or attribute *j* must be subtracted from $\bar{\bar{s}}_j^i$ to bring *i*'s utility down to $\beta^i(\bar{s})$.

1(d). Argue that $CM_j^i(\bar{s},\bar{\bar{s}})$ generalizes the familiar notion of the compensating variation in the usual Walrasian context.

ANSWER. Consider states of the form $s = (w^1,...,w^I;p) \in \Re^{I+L}$, i. e., *N* = 1, good or attribute 1 is the wealth of the consumer, and *M* = *L*: the last *L* components of the state vector are the prices

$p = (p_1, \ldots, p_L)$ of the L goods. Person i's utility function is given by the indirect utility function $v^i(p, w^i)$.

We define the compensating variation of a change from the state defined by $\bar{p}$ to the state defined by $\bar{\bar{p}}$ implicitly by the equation $v^i(\bar{\bar{p}}, w^i - CV) = v^i(\bar{p}, w^i)$, which specializes (1) to the Walrasian context.

1(e). Prove the following fact. *If* $\sum_h CM_j^h(\bar{s}, \bar{\bar{s}}) > 0$, *then state* $\bar{\bar{s}}$ *is better than state* $\bar{s}$ *according to the potential compensation criterion for the redistribution correspondence* Φ_j.

ANSWER. Proof. We adapt the familiar argument of the Walrasian case. Let $\sum_h CM_j^h(\bar{s}, \bar{\bar{s}}) > 0$. Define $s_j^i{}' = \bar{\bar{s}}_j^i - CM_j^i + \frac{\sum_h CM_j^h}{I}$, and the vector s' by substituting $s_j^i{}'$ for $\bar{\bar{s}}_j^i$, $i = 1, \ldots, I$, and leaving all other components of $\bar{\bar{s}}$ unchanged.

First, the vector s' belongs to $\Phi_j(\bar{\bar{s}})$, because $\sum_i s_j^i{}' = \sum_i \bar{\bar{s}}_j^i - \sum_i CM_j^i + \sum_h CM_j^h = \sum_i \bar{\bar{s}}_j^i$, i. e., s' redistributes the amount of good or attribute j present in $\bar{\bar{s}}$, while leaving unchanged all other components of $\bar{\bar{s}}$ that may affect i's utility. Second, by the definition of CM_j^i, $\beta^i(\bar{\bar{s}}_{-ij}, \bar{\bar{s}}_j^i - CM_j^i(\bar{s}, \bar{\bar{s}})) = \beta^i(\bar{s})$, and hence $\beta^i(\bar{\bar{s}}_{-ij}, \bar{\bar{s}}_j^i - CM_j^i + \frac{\sum_h CM_j^h}{I}) > \beta^i(\bar{s})$, because $\frac{\sum_h CM_j^h}{I} > 0$, and good or attribute j is desirable, i. e., $\beta^i(\bar{\bar{s}}_{-ij}, s_j^i{}') = \beta^i(s') > \beta^i(\bar{s})$. ■

The rest of this question illustrates that choosing different notions of redistribution by choosing different j's has important implications.

To this end, consider an economy with $I = N = 2$, and $M = 0$, i. e., two goods, apples (good 1) and bananas (good 2), and two people, John (Person 1) and Mary (Person 2). A feasible state is now a consumption allocation, i. e., a vector in $\Re^4$ of the form $(s_1^1, s_2^1; s_1^2, s_2^2)$, where s_2^1 is John's consumption of bananas, s_1^2 is Mary's consumption of apples, etc.

John's preferences can be represented by the utility function

$$\beta^1(s) = 3s_1^1 + s_2^1,$$

while Mary's preferences can be represented by the utility function

$\beta^2(s) = s_1^2 + 3s_2^2$.

Remark. The linearity of these functions guarantees that, given Φ, the weak potential compensation criterion is automatically strong, so that if a state is better than another one according to the weak potential compensation criterion for Φ, it cannot be the case that the second one is better than the first one according to the same criterion for the same Φ.

Consider the following two states:

$\bar{s} = (\bar{s}_1^1, \bar{s}_2^1; \bar{s}_1^2, \bar{s}_2^2) = (100, 100, 100, 100)$,

$\bar{\bar{s}} = (\bar{\bar{s}}_1^1, \bar{\bar{s}}_2^1; \bar{\bar{s}}_1^2, \bar{\bar{s}}_2^2) = (50, 150, 50, 150)$.

1(f). Is $\bar{s}$ better than $\bar{\bar{s}}$ according to the Pareto criterion? Is $\bar{\bar{s}}$ better than $\bar{s}$ according to the Pareto criterion? Explain, and graphically illustrate in utility space.

ANSWER.

For John, $\beta^1(\bar{s}) = 300 + 100 = 400$ and $\beta^1(\bar{\bar{s}}) = 150 + 150 = 300$, i. e., John is better off at $\bar{s}$ than at $\bar{\bar{s}}$.

For Mary, $\beta^2(\bar{s}) = 100 + 300 = 400$ and $\beta^2(\bar{\bar{s}}) = 50 + 450 = 500$, i. e., Mary is better off at $\bar{\bar{s}}$ than at $\bar{s}$. Hence, $\bar{s}$ and $\bar{\bar{s}}$ are Pareto noncomparable in this society.

1(g). Consider the feasible redistribution correspondence Φ_1, i. e., redistribution is implemented in apples. Compute $CM_1^i(\bar{s}, \bar{\bar{s}})$, for *i* = 1, 2, and $CM_1^1(\bar{s}, \bar{\bar{s}}) + CM_1^2(\bar{s}, \bar{\bar{s}})$. Is $\bar{s}$ better than $\bar{\bar{s}}$ according to the potential compensation criterion for Φ_1? Is $\bar{\bar{s}}$ better than $\bar{s}$ according to the potential compensation criterion for Φ_1? Explain, specifying any relevant compensations, and graphically illustrate in utility space.

ANSWER.

For John: We solve $\beta^1(50 - CM_1^1, 150) = \beta^1(100, 100)$,

i. e., $3(50 - CM_1^1) + 150 = 400$,

i. e., $150 - 3CM_1^1 + 150 = 400$,

i. e., $CM_1^1(\bar{s}, \bar{\bar{s}}) = -100/3$.

For Mary: We solve $\beta^2(50 - CM_1^2, 150) = \beta^2(100, 100)$,

i. e., $50 - CM_1^2 + 450 = 400$,

i. e., $CM_1^2(\bar{s}, \bar{\bar{s}}) = 100$.

Hence, $CM_1^1(\bar{s}, \bar{\bar{s}}) + CM_1^2(\bar{s}, \bar{\bar{s}}) = -(100/3) + 100 > 0$. The sum of the compensating modifications is positive and the above fact implies that state $\bar{\bar{s}}$ is better than state $\bar{s}$ according to the potential compensation criterion for the redistribution correspondence Φ_1. Hence, there exists a state *s'* that can be reached from $\bar{\bar{s}}$ by redistributing apples and that both John and Mary prefer to $\bar{s}$. For instance, let $(s_1^1{}', s_2^1{}'; s_1^2{}', s_1^2{}') = (100,150;0,150)$, which yields

$\beta^1(s') = 300 + 150 = 450 > 400 = \beta^1(\bar{s})$,

$\beta^2(s') = 0 + 450 = 450 > 400 = \beta^2(\bar{s})$.

See Figure 3.

1(h). Consider now the feasible redistribution correspondence Φ_2, i. e., redistribution is implemented in bananas. Compute $CM_2^i(\bar{s}, \bar{\bar{s}})$, for *i* = 1, 2, and $CM_2^1(\bar{s}, \bar{\bar{s}}) + CM_2^2(\bar{s}, \bar{\bar{s}})$. Is $\bar{s}$ better than $\bar{\bar{s}}$ according to the potential compensation criterion? Is $\bar{\bar{s}}$ better than $\bar{s}$ according to the potential compensation criterion? Explain, specifying any relevant compensation, and graphically illustrate in utility space.

ANSWER.

For John: We solve $\beta^1(50, 150 - CM_2^1) = \beta^1(100, 100)$,

i. e., $150 + 150 - CM_1^1 = 400$,

i. e., $CM_2^1(\bar{s}, \bar{\bar{s}}) = -100$.

For Mary: We solve $\beta^2(50, 150 - CM_2^2) = \beta^2(100, 100)$,

i. e., $50 + 3(150 - CM_1^2) = 400$,

i. e., $50 + 450 - 3CM_1^2 = 400$,

i. e., $CM_2^2(\bar{s}, \bar{\bar{s}}) = 100/3$.

Now the sum $CM_2^1(\overline{s},\overline{\overline{s}})+CM_2^2(\overline{s},\overline{\overline{s}})$ is negative, while (as is easy to check) the sum $CM_2^1(\overline{\overline{s}},\overline{s})+CM_2^2(\overline{\overline{s}},\overline{s})$ is positive. Hence, state $\overline{s}$ is better than state $\overline{\overline{s}}$ according to the potential compensation criterion for the redistribution correspondence Φ_2.

Hence, there exists a state *s*’’ that can be reached from $\overline{s}$ by redistributing bananas and that both John and Mary prefer to $\overline{\overline{s}}$. For instance, let $(s_1^1", s_2^1"; s_1^2", s_1^2") = (100, 50; 100, 150)$, which yields

$$\beta^1(s") = 300 + 50 = 350 > 300 = \beta^1(\overline{\overline{s}}),$$

$$\beta^2(s") = 100 + 450 = 550 > 500 = \beta^2(\overline{\overline{s}}).$$

See Figure 3.

1(i). Compare your answers to 1(g) and 1(h), and comment.

Answer. We must be explicit about which type of redistributions are feasible when we refer to the potential compensation criterion. In the classical Walrasian case, we consider lumpsum redistributions of wealth, whereas in the present example we consider first redistributions of apples and then redistributions of bananas. Depending on the notion of feasible redistribution employed (as formally expressed by the correspondence Φ) the criterion may point in one direction or in the opposite one: $\overline{\overline{s}}$ is “better” than $\overline{s}$ when feasible redistributions entail apples, whereas $\overline{s}$ is “better: than $\overline{\overline{s}}$ when they entail bananas.

Note that this phenomenon is unrelated to the possible reversals of the weak potential compensation criterion in nonquasilinear economies, because the weak and strong criteria coincide in our example.

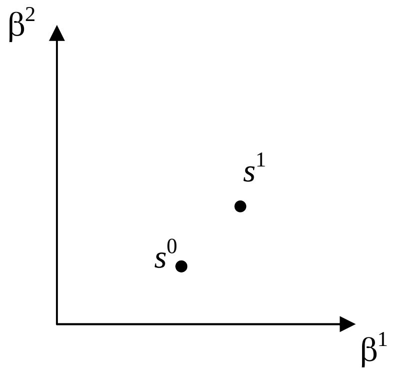


Figure 1
s^1 is better than s^0 according to the Pareto criterion

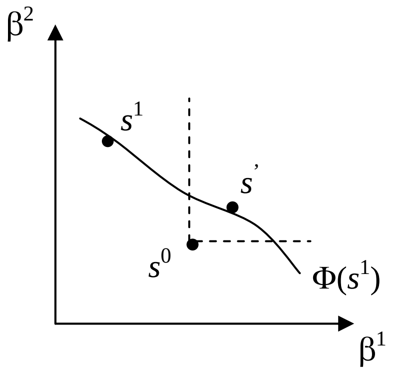


Figure 2
s^1 is better than s^0 according to the potential compensation criterion for Φ

Note.
In figures 1-3, the coordinates of a point labeled s are $(\beta^1(s), \beta^2(s))$

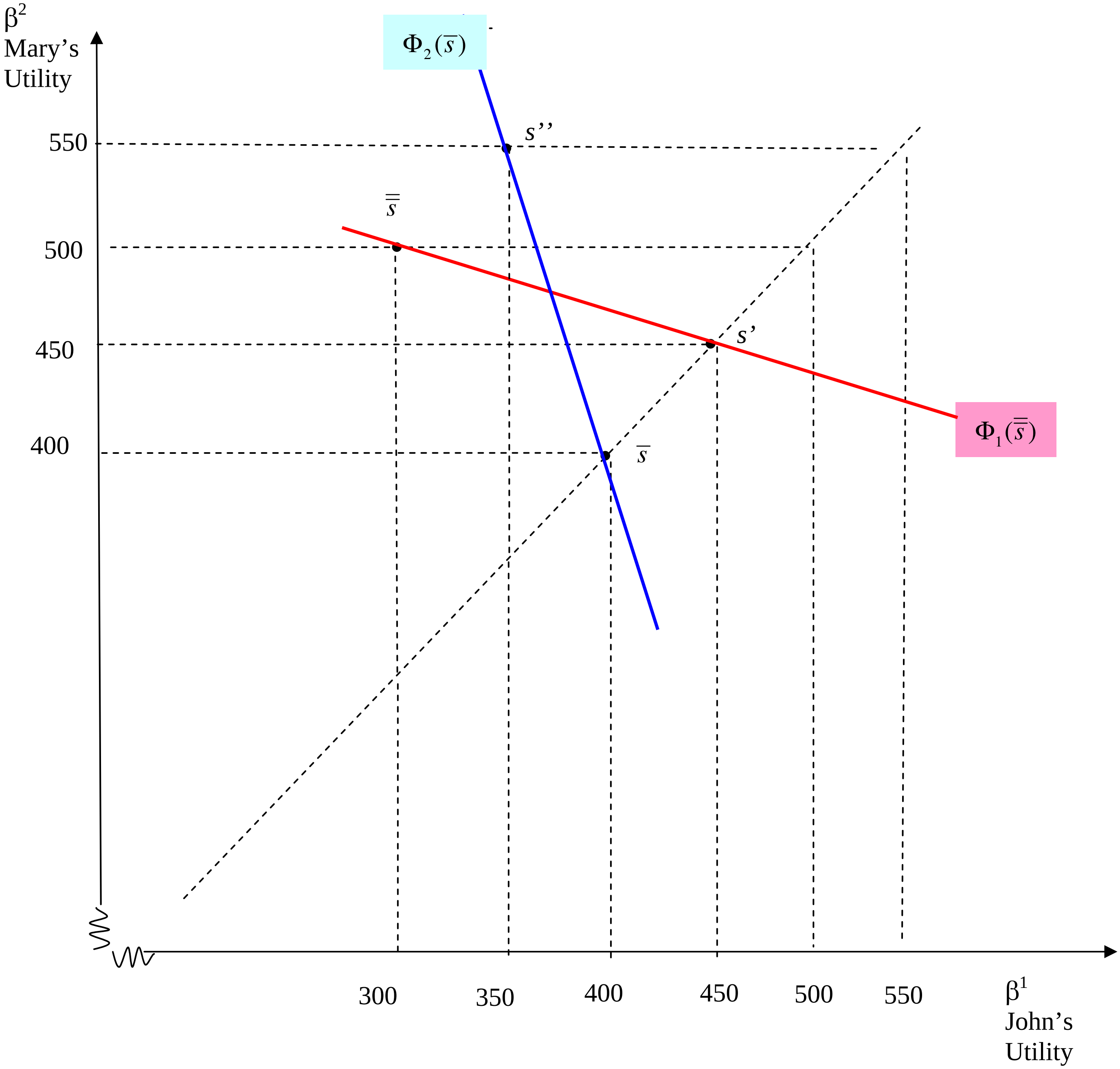


Figure 3
The potential compensation criterion depends
on the correspondence Φ

ANSWER KEY

Micro Prelim, Fall 2010

QUESTION 2. CONSTRAINED RISK AVERSION

A consumer plans to buy goods 1 and 2 at the market prices p_1 and p_2, but her wealth w is uncertain. *Ex ante*, she may have to choose among various lotteries that will affect her wealth. Her *ex ante* preferences satisfy the expected utility hypothesis, with the following von Neumann-Morgenstern-Bernoulli (vNMB) utility function defined on consumption vectors (x_1, x_2):

$$u(x_1, x_2) = \frac{[x_1{}^{\alpha} x_2{}^{1-\alpha}]^{1-\rho} - 1}{1-\rho}, \qquad (1)$$

where $\alpha \in (0,1)$ and $\rho \in (0,1) \cup (1,\infty)$.

2(a). Assume that, once w is known, she can always satisfy her (*ex post*) Walrasian demand for goods 1 and 2 at the (certain) prices p_1 and p_2. Call her in this case the *unconstrained consumer*.

2(a).1. What are her Walrasian demands $\tilde{x}_1(p_1, p_2, w)$ and $\tilde{x}_2(p_1, p_2, w)$? Graphically represent them in Figure 1 for given prices p_1 and p_2, and wealth w^0.

ANSWER. These are the familiar Walrasian demands for the Cobb-Douglas preferences, namely $\tilde{x}_1(p_1, p_2, w) = \alpha \dfrac{w}{p_1}$ and $\tilde{x}_2(p_1, p_2, w) = (1-\alpha)\dfrac{w}{p_2}$.

2(a).2. Because her only uncertainty concerns her wealth, we can write the vNMB utility of the unconstrained consumer as a function of w (with prices as parameters), to be denoted $\beta(w)$. What is this function?

Compute:

- Its first-order derivative $\beta'(w)$;
- Its second-order derivative $\beta''(w)$;
- Its coefficient of relative risk aversion evaluated at w, denoted $CRR[\beta, w]$.

ANSWER. The function β is her indirect utility function, which can be computed by substituting her Walrasian demands in (1), i. e.,

$$\beta(w)=\frac{\left[\left(\alpha\frac{w}{p_1}\right)^{\alpha}\left((1-\alpha)\frac{w}{p_2}\right)^{1-\alpha}\right]^{1-\rho}-1}{1-\rho}=\frac{\left[\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}w\right]^{1-\rho}-1}{1-\rho}=\frac{[\Psi w]^{1-\rho}-1}{1-\rho},$$

where $\Psi\equiv\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}$.

We compute

$$\beta'(w)=\frac{(1-\rho)[\Psi w]^{-\rho}\Psi}{1-\rho}=\Psi^{1-\rho}w^{-\rho}, \qquad (2)$$

$$\beta''(w)=-\rho\Psi^{1-\rho}w^{-\rho-1},$$

$$CRR[\beta,w]=-\frac{-\rho\Psi^{1-\rho}w^{-\rho-1}}{\Psi^{1-\rho}w^{-\rho}}w=\rho,$$

independent from *w*. This is the familiar case of *CRRA*, with coefficient ρ.

2(b). Consider a given wealth level w^0, and write $x_1^0\equiv\tilde{x}_1(p_1,p_2,w^0)$. Assume now that as her wealth changes, she is constrained to consume the exact amount $x_1^0\equiv\tilde{x}_1(p_1,p_2,w^0)$ of good 1 (perhaps because of previous commitments, or by institutional restrictions), spending what is left of her wealth in good 2. Refer to her in this case as the *constrained consumer*.

2(b). 1. Graphically represent her constrained choice in Figure 1.

2(b). 2. Now her vNMB utility is a different function of wealth, to be denoted $\beta_C(w)$. What is this function?

ANSWER. Given *w*, the consumption vector of the constrained consumer is $(x_1^0,\frac{w-p_1x_1^0}{p_2})$, which substituted into (1) gives:

$$\beta_C(w)=\frac{\left[\left(x_1^0\right)^{\alpha}\left(\frac{w-p_1x_1^0}{p_2}\right)^{1-\alpha}\right]^{1-\rho}-1}{1-\rho}.$$

2(b).3. Compute its first-order derivative $\beta_C'(w)$. How do $\beta_C'(w^0)$ and $\beta'(w^0)$ compare? Explain. Can you graphically illustrate the comparison between $\beta_C'(w^0)$ and $\beta'(w^0)$ in the (wealth, utility) plane?

ANSWER.

$$\beta_C\,'(w)=\frac{\left(x_1^0\right)^{\alpha(1-\rho)}(1-\alpha)(1-\rho)\left(\dfrac{w-p_1x_1^0}{p_2}\right)^{(1-\alpha)(1-\rho)-1}\dfrac{1}{p_2}}{1-\rho}=\frac{1-\alpha}{p_2}\left(x_1^0\right)^{\alpha(1-\rho)}\left(\frac{w-p_1x_1^0}{p_2}\right)^{-\rho+\rho\alpha-\alpha}\quad . \quad (3)$$

At w^0, x_1^0 is her Walrasian demand $\alpha\dfrac{w^0}{p_1}$ for good 1, and $\dfrac{w^0-p_1x_1^0}{p_2}$ equals her Walrasian demand for good 2 , $(1-\alpha)\dfrac{w^0}{p_2}$. Substituting these expressions into (3) gives

$$\beta_C\,'(w^0)=\frac{1-\alpha}{p_2}\left(\alpha\frac{w^0}{p_1}\right)^{\alpha(1-\rho)}\left((1-\alpha)\frac{w^0}{p_2}\right)^{-\rho+\rho\alpha-\alpha}=\left(\frac{\alpha}{p_1}\right)^{\alpha(1-\rho)}\left(\frac{1-\alpha}{p_2}\right)^{1-\rho+\rho\alpha-\alpha}\left(w^0\right)^{\alpha-\rho\alpha-\rho+\rho\alpha-\alpha}$$

$$=\left(\frac{\alpha}{p_1}\right)^{\alpha(1-\rho)}\left(\frac{1-\alpha}{p_2}\right)^{(1-\alpha)(1-\rho)}\left(w^0\right)^{-\rho}=\Psi^{1-\rho}\left(w^0\right)^{-\rho},$$

as in (2).

The fact that β and β_C have the same slope at w^0 is an implication of the Envelope Theorem. Note that $\beta(w)\geq\beta_C(w)$, with $\beta(w^0)=\beta_C(w^0)$, and $\beta_C(w)$ tangent to $\beta(w)$ at w^0, see Figure 2.

2(b).4. Compute the second-order derivative $\beta_C''(w)$ and evaluate it at w^0.

ANSWER. We compute from (3)

$$\beta_C\,''(w)=\frac{1-\alpha}{p_2}\left(x_1^0\right)^{\alpha(1-\rho)}(-\rho+\rho\alpha-\alpha)\left(\frac{w-p_1x_1^0}{p_2}\right)^{-\rho+\rho\alpha-\alpha-1}\frac{1}{p_2}.$$

In a manner parallel to the previous computation, we obtain

$$\beta_C\,''(w^0)=\frac{1-\alpha}{p_2}\left(\alpha\frac{w^0}{p_1}\right)^{\alpha(1-\rho)}(-\rho+\rho\alpha-\alpha)\left((1-\alpha)\frac{w^0}{p_2}\right)^{-\rho+\rho\alpha-\alpha-1}\frac{1}{p_2}$$

$$=\left(\frac{1-\alpha}{p_2}\right)^{-\rho+\rho\alpha-\alpha}\left(\frac{\alpha}{p_1}\right)^{\alpha(1-\rho)}(-\rho+\rho\alpha-\alpha)\frac{1}{p_2}\left(w^0\right)^{\alpha-\alpha\rho-\rho+\rho\alpha-\alpha-1}$$

$$=\left(\frac{1-\alpha}{p_2}\right)^{-\rho+\rho\alpha-\alpha}\left(\frac{\alpha}{p_1}\right)^{\alpha(1-\rho)}(-\rho+\rho\alpha-\alpha)\frac{1}{p_2}\left(w^0\right)^{-1-\rho}.$$

2(b).5. Compute the coefficient $CRR[\beta_C, w^0]$ of relative risk aversion of β_C evaluated at w^0.

ANSWER.

$$CRR[\beta_C, w^0] = -\frac{\beta_C''(w^0)}{\beta_C'(w^0)} w^0 = -\frac{\left(\frac{1-\alpha}{p_2}\right)^{-\rho+\rho\alpha-\alpha} \left(\frac{\alpha}{p_1}\right)^{\alpha(1-\rho)} (-\rho+\rho\alpha-\alpha)\frac{1}{p_2}\left(w^0\right)^{-1-\rho}}{\left(\frac{1-\alpha}{p_2}\right)^{1-\rho+\rho\alpha-\alpha} \left(\frac{\alpha}{p_1}\right)^{\alpha(1-\rho)} \left(w^0\right)^{-\rho}} w^0$$

$$= [\rho(1-\alpha)+\alpha]\left(\frac{1-\alpha}{p_2}\right)^{-\rho+\rho\alpha-\alpha-1+\rho-\rho\alpha+\alpha} \frac{1}{p_2} = [\rho(1-\alpha)+\alpha]\left(\frac{1-\alpha}{p_2}\right)^{-1} \frac{1}{p_2} = \rho + \frac{\alpha}{1-\alpha}.$$

2(c). Compare *CRR*[β, w^0] and *CRR*[β_C, w^0]. Who is more risk averse (at w^0), the constrained consumer, or the unconstrained consumer? Explain.

ANSWER. We see that *CRR*[β_C, w^0] - *CRR*[β, w^0] = $\frac{\alpha}{1-\alpha}$ > 0. Hence, the constrained consumer is more risk averse than the unconstrained one. Intuitively, wealth risk has more severe utility consequences for the constrained consumer, because her capacity to adapt is lower than that of the constrained consumer. Geometrically, Figure 2 indicates that the function β_C has more curvature than the function β at w^0.

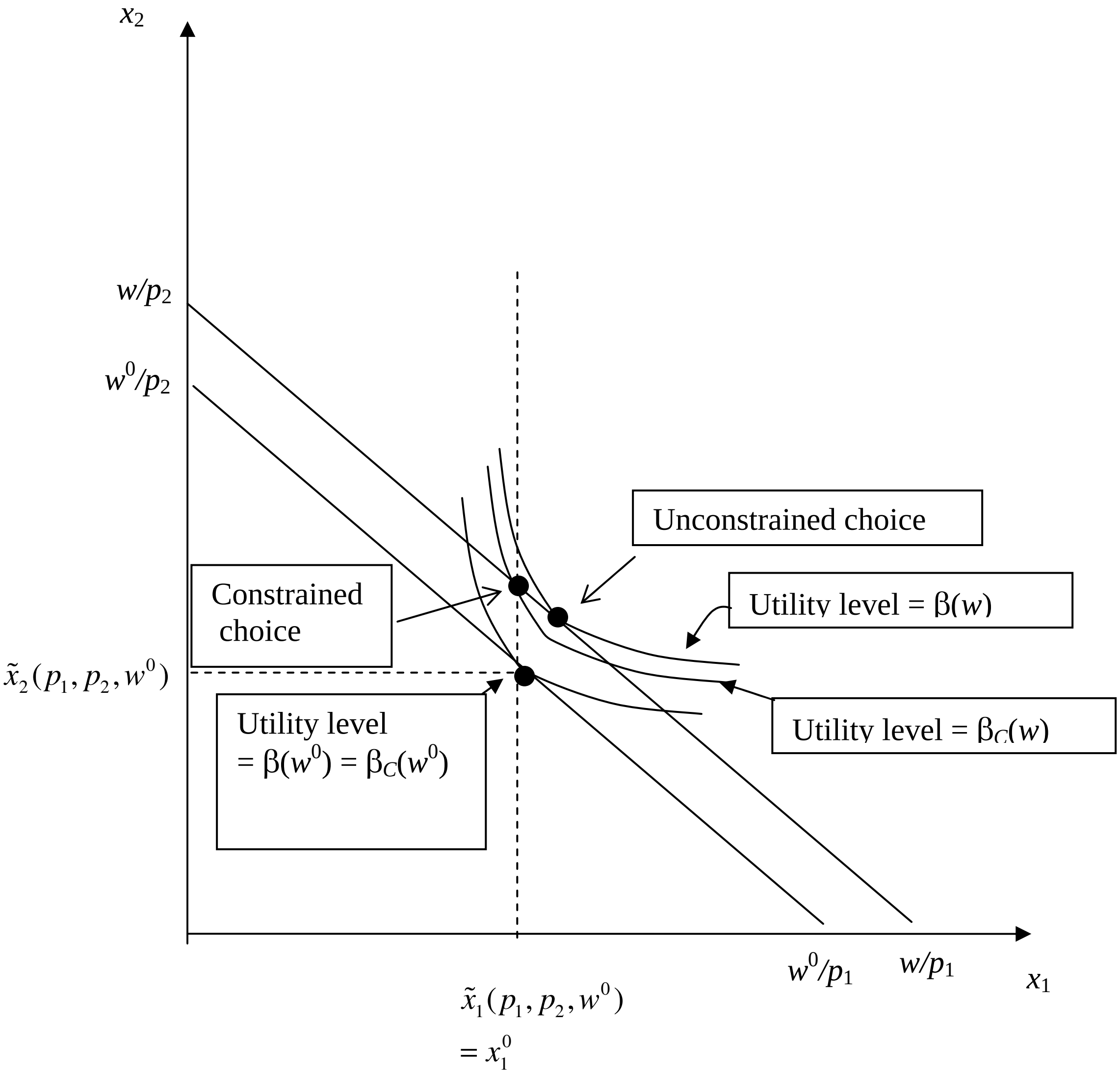


Figure 1.
Unconstrained vs. constrained consumer

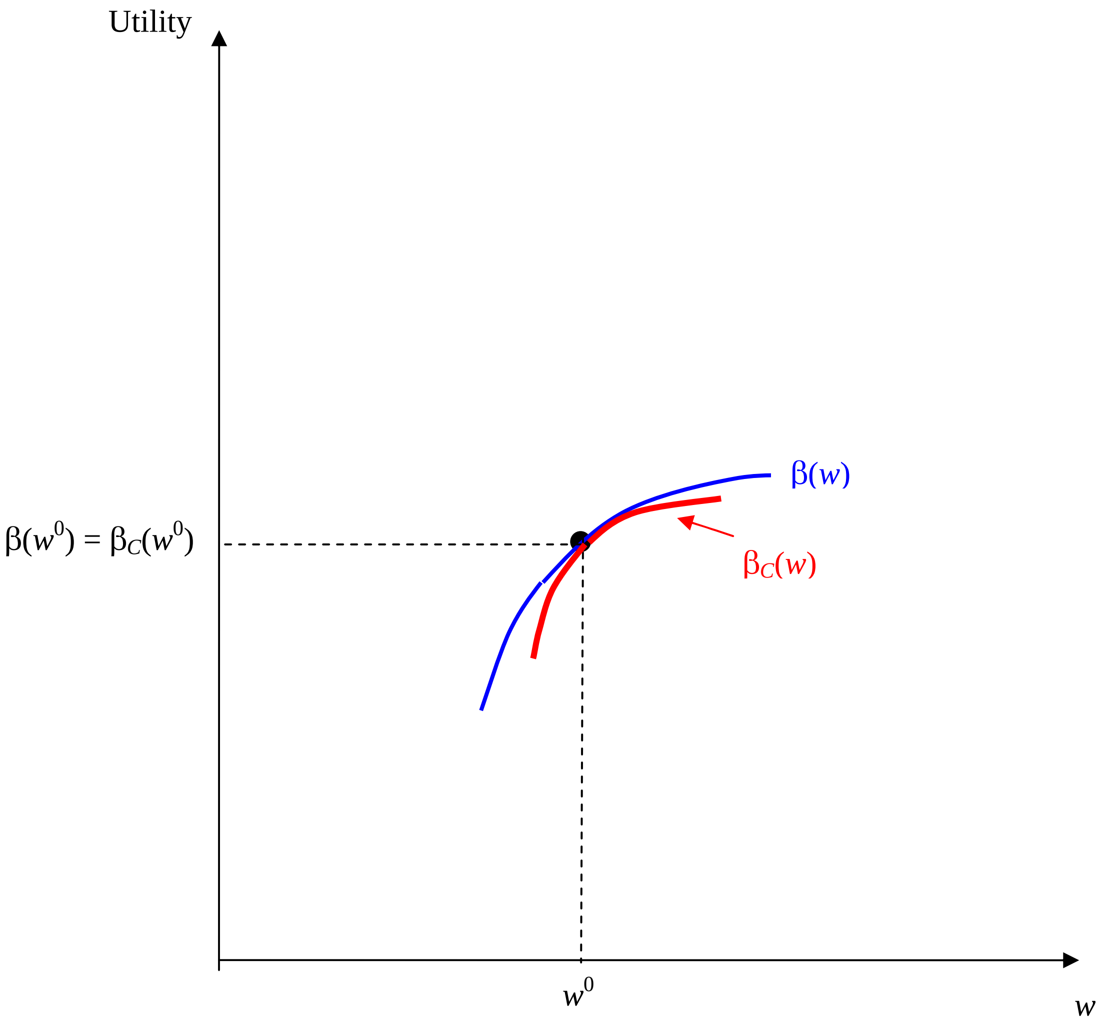


Figure 2.
Unconstrained vs. constrained vNMB function

Answer Key - Question 3
Fall 2010

(a) coeff. of risk aversion of agent i $\frac{1}{\alpha^i + x}$: risk aversion decreases with α^i for x fixed, and decreases with x^i (wealth) for α^i fixed.

(b) max $\sum P_s \ln(\alpha^i + x_s^i)$ s.t $\sum p_s x_s^i \le \sum p_s w_s^i$

$\Leftrightarrow$ max $\sum P_s \ln X_s^i$ s.t. $\sum p_s X_s^i \le \sum p_s w_s^i + \alpha^i \sum p_s$

or

$p.X^i \le p(w^i + \alpha^i \mathbb{1})$.

Thus $$\boxed{X_s^i = x_s^i + \alpha^i = P_s \frac{p.(w^i + \alpha^i \mathbb{1})}{p_s}} \quad (1)$$

(c) equilibrium $\sum x_s^i = w_s$. Summing (1) for all i

gives $$w_s + \alpha = P_s \frac{p(w + \alpha \mathbb{1})}{p_s}$$

$$p_s = P_s \frac{p(w + \alpha \mathbb{1})}{w_s + \alpha}$$

Substituting in (1) gives

Answer Key - Question 3
Fall 2010

(a) coeff. of risk aversion of agent i $\frac{1}{\alpha^i + x}$: risk aversion decreases with α^i for x fixed, and decreases with x^i (wealth) for α^i fixed.

(b) max $\sum P_s \ln(\alpha^i + x^i_s)$ s.t. $\sum p_s x^i_s \le \sum p_s w^i_s$

$\Leftrightarrow$ max $\sum P_s \ln X^i_s$ s.t. $\sum p_s X^i_s \le \sum p_s w^i_s + \alpha^i \sum p_s$

or

$$p \cdot X^i \le p(w^i + \alpha^i \mathbb{1}).$$

Thus $$\boxed{X^i_s = x^i_s + \alpha^i = P_s \frac{p \cdot (w^i + \alpha^i \mathbb{1})}{p_s}} \quad (1)$$

(c) equilibrium $\sum x^i_s = w_s$. Summing (1) for all i gives

$$w_s + \alpha = P_s \frac{p(w + \alpha \mathbb{1})}{p_s}$$

$$p_s = P_s \frac{p(w + \alpha \mathbb{1})}{w_s + \alpha}$$

Substituting in (1) gives

$$x^i_s + \alpha^i = \cancel{\rho_s} \frac{p(w^i + \alpha^i \mathbb{1})}{\cancel{\rho_s}\, p(W + \alpha \mathbb{1})} (W_s + \alpha)$$

or
$$x^i_s = \frac{p(w^i + \alpha^i \mathbb{1})}{p(W + \alpha \mathbb{1})} s + \frac{p(w^i + \alpha^i \mathbb{1})}{p(W + \alpha \mathbb{1})} \alpha - \alpha^i$$

$$\boxed{\begin{array}{l} x^i = b^i W + a^i \mathbb{1} \\ \text{with} \quad b^i = \dfrac{p(w^i + \alpha^i \mathbb{1})}{p(W + \alpha \mathbb{1})} \qquad a^i = b^i \alpha - \alpha^i \end{array}}$$

$\Sigma a^i = (\sum_i b^i)\alpha - \sum_i \alpha^i = 0$ since $\Sigma b^i = 1$.

(d) As seen in question (a) an agent's risk tolerance increases with α^i.

For a given wealth level, the coefficient b^i on the risky aggregate output increases with α^i, and the coeff on the sure consumption $\mathbb{1}$ decreases with α^i: at equality of wealth more risk tolerant agents have a more risky consumption in equilibrium than less risk tolerant agents.

(e) If $\alpha^1 \to +\infty$, then $\alpha \to +\infty$. For $i = 2, .., I$ $b^i \to 0$ and $a^i \to \dfrac{p.w^i}{p.\mathbb{1}}$ while $b^1 \to 1$: agent 1 becomes risk neutral and takes all the risks of the economy while the risk averse agents 2, .. I all have a sure consumption stream. Agent 1 provides insurance to all the other agents.

Answer Key, Question 4

Fall 2010

4. (a) optimal level of public good obtained by maximizing the utility of the representative agent

max $x + h(y)$ s. to $nx + y = nw$

FOC $1 = \lambda n$ $h'(y) = \lambda$ (neglecting non-negativity constraint)

Thus y^* is given by $\boxed{y_n^* = (h')^{-1}\left(\frac{1}{n}\right)}$

Since h is concave, h' and $(h')^{-1}$ are decreasing. When n increases, y_n^* increases.

(b) choice of the contribution z^i of the representative agent i

max $w - z^i + h(z^i + Z^{-i})$ s. to $z^i \geq 0$

where Z^{-i} is the total contribution of the other agents.

FOC: $h'(z^i + Z^{-i}) \leq 1$ $= 1$ if $z^i > 0$.

In a symmetric equilibrium $z^i = z$ for all i and, since $\lim h'(y) > 1$ when $y \to 0$, $z > 0$. Thus

$h'(nz) = h'(y) = 1$ so that the level of public good in the voluntary contribution equilibrium is

$$\boxed{y_n^{vc} = (h')^{-1}(1)}$$

At the equilibrium the private marginal benefit of additional contribution is fixed to 1 and this limits the possible level of public good. When n increases the contribution of each agent becomes negligible.

(c) choice of contribution of agent i when lottery is attached to contribution of public good :

$$\max \quad w - z^i + \frac{z^i}{z^i + Z^{-i}} R + h(z^i + Z^{-i} - R) \quad \text{subject to } z^i \geq 0 .$$

FOC : $$\frac{R}{z^i + Z^{-i}} - \frac{z^i}{(z^i + Z^{-i})^2} R + h'(z^i + Z^{-i} - R) \leq 1 \quad = 1 \text{ if } z^i > 0 .$$

At symmetric equilibrium $z^i = z > 0$ (if nobody contributes, agent i has interest to contribute trivially and win the prize for sure). The level of public good produced is thus given by the equation

$$\frac{R}{y+R} - \frac{(y+R)/n}{(y+R)^2} R + h''(y) = 1$$

or $$\boxed{\left(1 - \frac{1}{n}\right) \frac{R}{y+R} + h'(y) = 1} \qquad (1)$$

Let ~~us~~ $\hat{y}_n$ denote the solution to this equation
If $n > 1$, $1 - \frac{1}{n} > 0$ and $h'(\hat{y}_n) < 1$, So that $\hat{y}_n > \bar{y}_n^{vc}$.

To find $\frac{\partial \hat{y}_n}{\partial n}$ differentiate equation (1) to obtain

$$\frac{\partial \bar{y}}{\partial n} \underbrace{\left[h''(y) - \left(1 - \frac{1}{n}\right) \frac{R}{(y+R)^2} \right]}_{<0} = \underbrace{- \frac{1}{n^2} \frac{R}{y+R}}_{<0}$$

Thus $\frac{\partial \hat{y}_n}{\partial n} > 0$: the public good produced increases with the size of the economy. However $\hat{y}_n$ does not tend to ∞ when $n \to \infty$.
For suppose $\hat{y}_n \to \infty$. Then

$\left(1 - \frac{1}{n}\right) \frac{R}{\hat{y}_n + R} \to 0$ and $h'(\hat{y}_n) \to 0$, so that the

FOC (1) cannot be satisfied when n is large.

(d) The equation which gives the optimal contribution in a symmetric equilibrium is still (1), with $R = nr$ (the agents take R as given. Let $\tilde{y}_n$ denote the level of public good produced. To obtain $\partial \tilde{y}_n / \partial n$ differentiate the equation

$$\boxed{\left(1-\frac{1}{n}\right)\frac{nr}{y+nr}+h'(y)=1 \qquad (2)}$$

to obtain, after simplification,

$$\underbrace{\left[h''(y)-\left(1-\frac{1}{n}\right)\frac{nr}{(y+nr)^2}\right]}_{<0}\frac{\partial y}{\partial n}=\underbrace{-\frac{r(y+r)}{(y+nr)^2}}_{<0}$$

Thus $\dfrac{\partial \tilde{y}_n}{\partial n} > 0$.

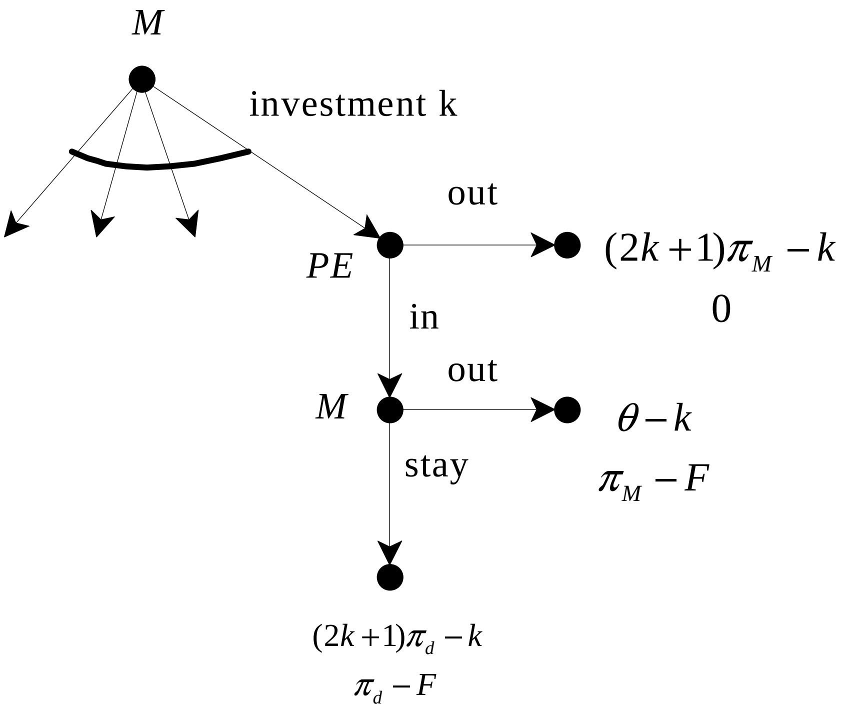
Suppose that $\tilde{y}_n \to \tilde{y}$ finite when $n \to \infty$. Then

$$\left(1-\frac{1}{n}\right)\frac{nr}{\tilde{y}_n+nr} \to 1 \quad \text{and} \quad h'(\tilde{y}_n) \to h'(\tilde{y}) > 0$$

so that equation (2) can not be satisfied when n is large. Thus $\tilde{y}_n \to \infty$ when $n \to \infty$. When the number of agents increase, the probability of winning the prize for each agent decreases and, to keep the "attractiveness" of winning the lottery sufficient to induce more than marginal contributions, the prize must increase with the size of the economy.

Answer keys for Question 5 Micro Prelim August 2010

(a) The extensive form is as follows:

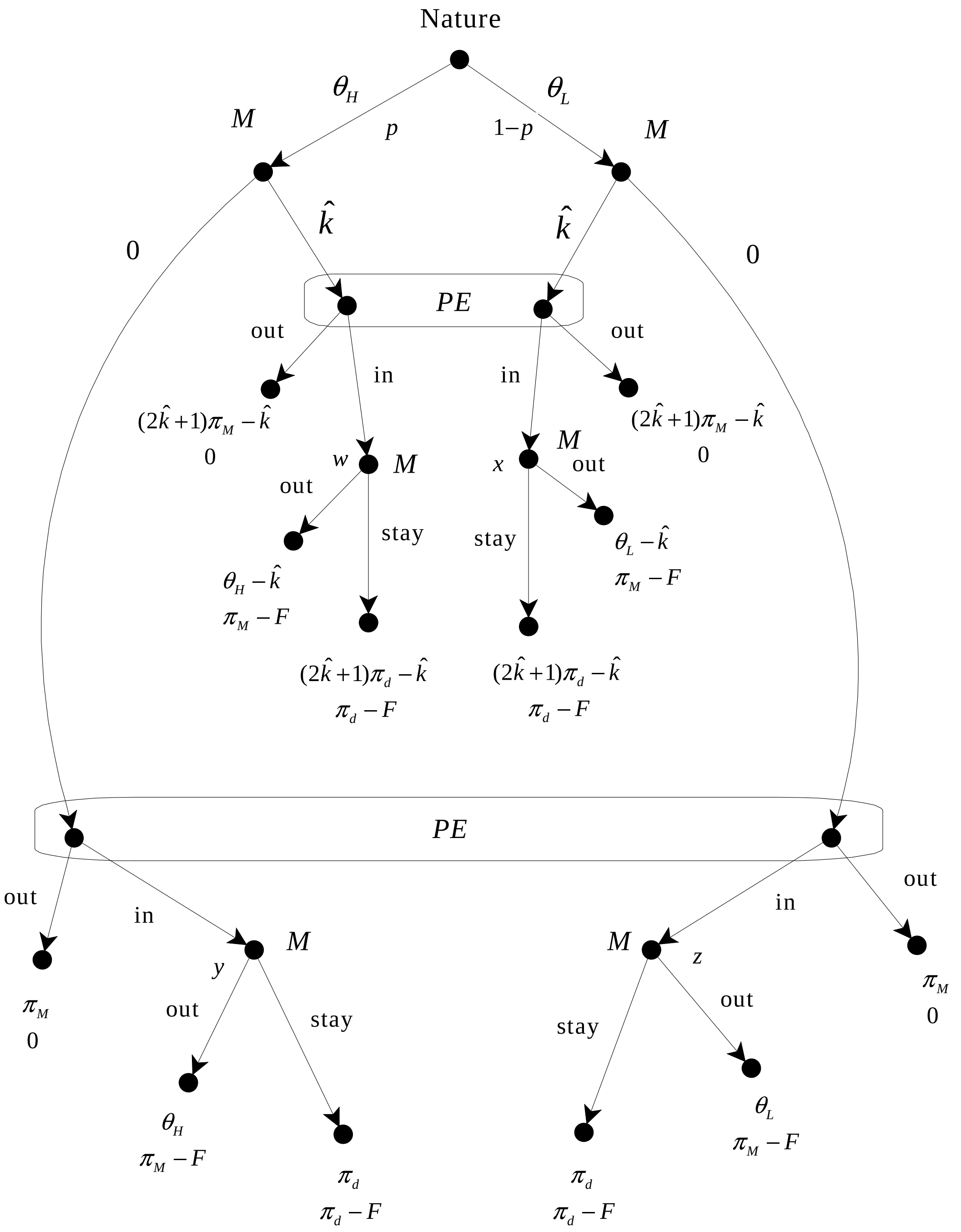


(b) If *PE* enters and *M* stays, then *PE* makes negative profits $\pi_d - F < 0$. Thus *PE* will enter only if it anticipates *M* to exit. Now, *M* will stay if and only if $(2k+1)\pi_d - k \geq \theta - k$, that is (since $\pi_d = \frac{1}{18}$), if and only if $\frac{2k+1}{18} \geq \theta$ or $k \geq 9\theta - \frac{1}{2}$. Thus, if *M* chooses $k \geq 9\theta - \frac{1}{2}$ then *PE* will stay out and *M*'s profits will be $(2k+1)\pi_M - k = \frac{(2k+1)}{8} - k$, which is maximized at $k = 9\theta - \frac{1}{2}$ and at this value *M*'s profits are $\Pi(\theta) = \frac{1}{2} - \frac{27}{4}\theta$. On the other hand, if *M* chooses $k < 9\theta - \frac{1}{2}$ then *PE* will enter and then *M* will exit, obtaining a payoff of $\theta - k$, which is maximized at *k* = 0. Thus we have the following.

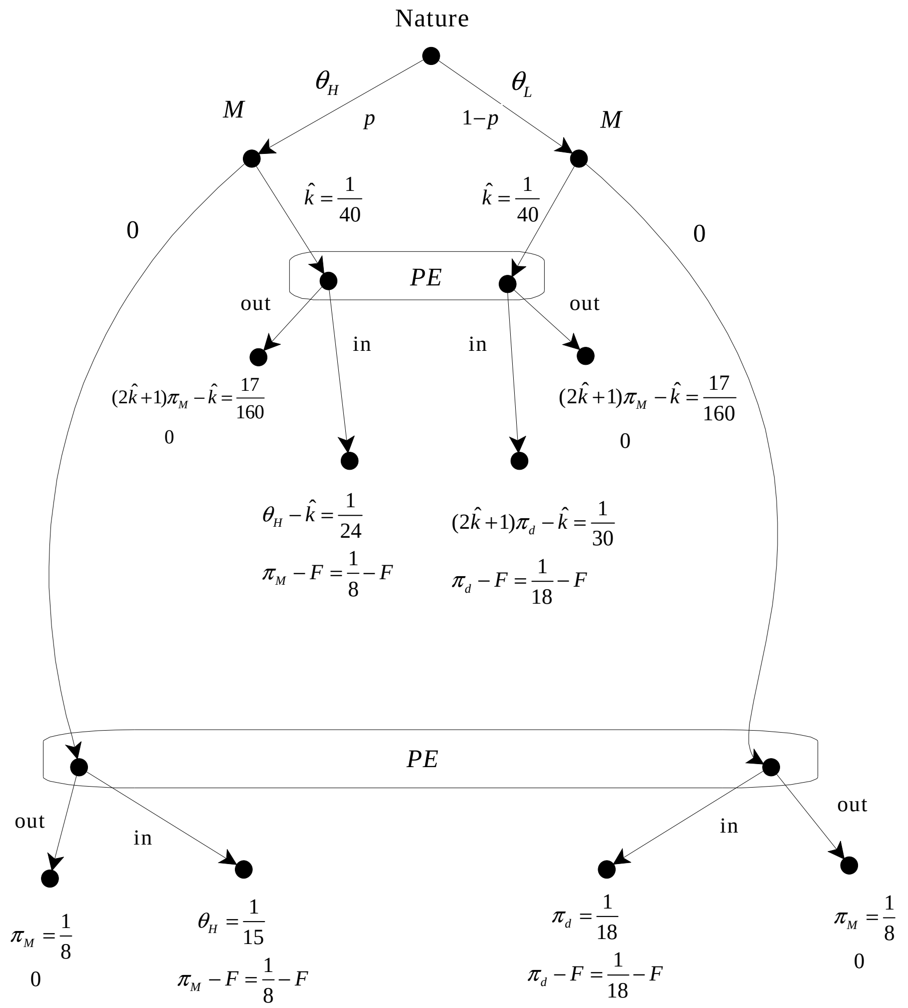
(b.1) Let $\theta = \frac{1}{16}$. Then $\Pi\left(\frac{1}{16}\right) = \frac{5}{64} > \frac{1}{16} = \theta$ and thus *M* will choose $k = 9\theta - \frac{1}{2} = \frac{9}{16} - \frac{1}{2} = \frac{1}{16}$ and, observing this, *PE* will stay out because if it entered then *M* would stay and *PE* would make negative profits equal to $\pi_d - F$. Thus at the subgame-perfect equilibrium *M*'s profits are $\frac{5}{64}$ and *PE*'s profits are 0.

(b.2) Let $\theta = \frac{3}{16}$. Then $\Pi\left(\frac{3}{16}\right) = -\frac{49}{64} < \frac{3}{16} = \theta$ and thus *M* will choose $k = 0$ and, observing this, *PE* will enter and then *M* will exit. Thus at the subgame-perfect equilibrium *M*'s profits are $\frac{3}{16}$ and *PE*'s profits are $\pi_M - F = \frac{1}{8} - F > 0$.

(c) The extensive form is as follows:



(d) Since $\pi_d = \frac{1}{18}$, when $\theta_L = \frac{1}{20}$ and $\theta_H = \frac{1}{15}$, for *M* at node *y* “out” is better than “stay” and at node *z* “stay is better than “out”. Furthermore, since $\hat{k} = \frac{1}{40}$, $(2\hat{k}+1)\pi_d = \frac{7}{120}$ thus for *M* at node *w* “out” is better than “stay” and at node *x* “stay” is better than “out”. Hence the game can be simplified as follows:



For a pure-strategy separating equilibrium there are only two possibilities:
(1) the θ_H type chooses $\hat{k}$ and the θ_L type chooses 0,
(2) the θ_H type chooses 0 and the θ_L type chooses $\hat{k}$.

In case (1) *PE* must assign probability 1 to the left node of its top information set and thus chooses "in" at that information set; furthermore, *PE* must assign probability 1 to the right node of its bottom information set and thus chooses "out" at that information set. But then the θ_H type of *M* gets a payoff of $\frac{1}{24}$ by choosing $\hat{k}$ and a payoff of $\frac{1}{8}=\frac{3}{24}$ by choosing 0. Hence it is not a Nash equilibrium.

In case (2) *PE* must assign probability 1 to the right node of its top information set and thus chooses "out" at that information set; furthermore, *PE* must assign probability 1 to the left node of its bottom information set and thus chooses "in" at that information set. But then the θ_H type of *M* gets a payoff of $\frac{1}{15}$ by choosing 0 and a payoff of $\frac{17}{160}>\frac{1}{15}$ by choosing $\hat{k}$. Hence it is not a Nash equilibrium.

(e) In order for the θ_H type to optimally choose $\hat{k}$, it must be that *PE*'s choice at its bottom information set is "in" and this is an optimal choice for *PE* if and only if *PE* attaches sufficiently high probability to the left node of its bottom information set. Since Bayes' rule does not apply to this information set, such beliefs are allowed by the notion of weak sequential equilibrium. Since both types of *M* choose $\hat{k}$, Bayes' rule requires *PE* to attach probability *p* to the left node of its top information set (and 1 − *p* to the right node). Thus "out" is optimal if and only if $0 \geq \frac{1}{8}p+\frac{1}{18}(1-p)-F$, that is, if and only if $p \leq \frac{72F-4}{5}$.