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Microeconomics

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PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

ANSWER KEY

PART 1. WILLINGNESS TO PAY AND TO ACCEPT

Two goods. Good 2 is a consumption good that the consumer owns in the amount ω, whereas good 1 is an environmental good that the consumer enjoys. The consumer's preferences on (x_1, x_2) combinations of the two goods are represented by a utility function $u : \Re^2_{++} \to \Re$.

In what follows we consider two given levels of the environmental good, namely $\bar{x}$ and $\bar{\bar{x}}$, with $\bar{\bar{x}} > \bar{x} > 0$.

Define the consumer's *WTP* (for "willingness to pay for a move from $\bar{x}$ to $\bar{\bar{x}}$") as the maximal amount of good 2 that the consumer would be willing to part with in exchange for an <u>increase</u> in the amount of good 1 from $\bar{x}$ to $\bar{\bar{x}}$.

Define the consumer's *WTA* (for "willingness to accept for a move from $\bar{\bar{x}}$ to $\bar{x}$") as the minimal amount of good 2 that the consumer would be willing to receive (and add to ω) in exchange for a <u>decrease</u> in the amount of good 1 from $\bar{\bar{x}}$ to $\bar{x}$.

Question 1(a). Write the equations that implicitly define *WTP* and *WTA.* Graphically illustrate these two notions in a graph with axes x_1 and x_2, labeled Figure 1. Comment on the graph.

ANSWER.

WTP: $u(\bar{x}, \omega) = u(\bar{\bar{x}}, \omega - WTP)$.

WTA: $u(\bar{x}, \omega + WTA) = u(\bar{\bar{x}}, \omega)$.

See Figure 1. Both *WTP* and *WTA* measure the vertical distance between the indifference curves through $(\bar{x}, \omega)$ and through $(\bar{\bar{x}}, \omega)$. But *WTP* measures this distance for the abscissa value $\bar{\bar{x}}$, whereas *WTA* takes the abscissa value $\bar{x}$.

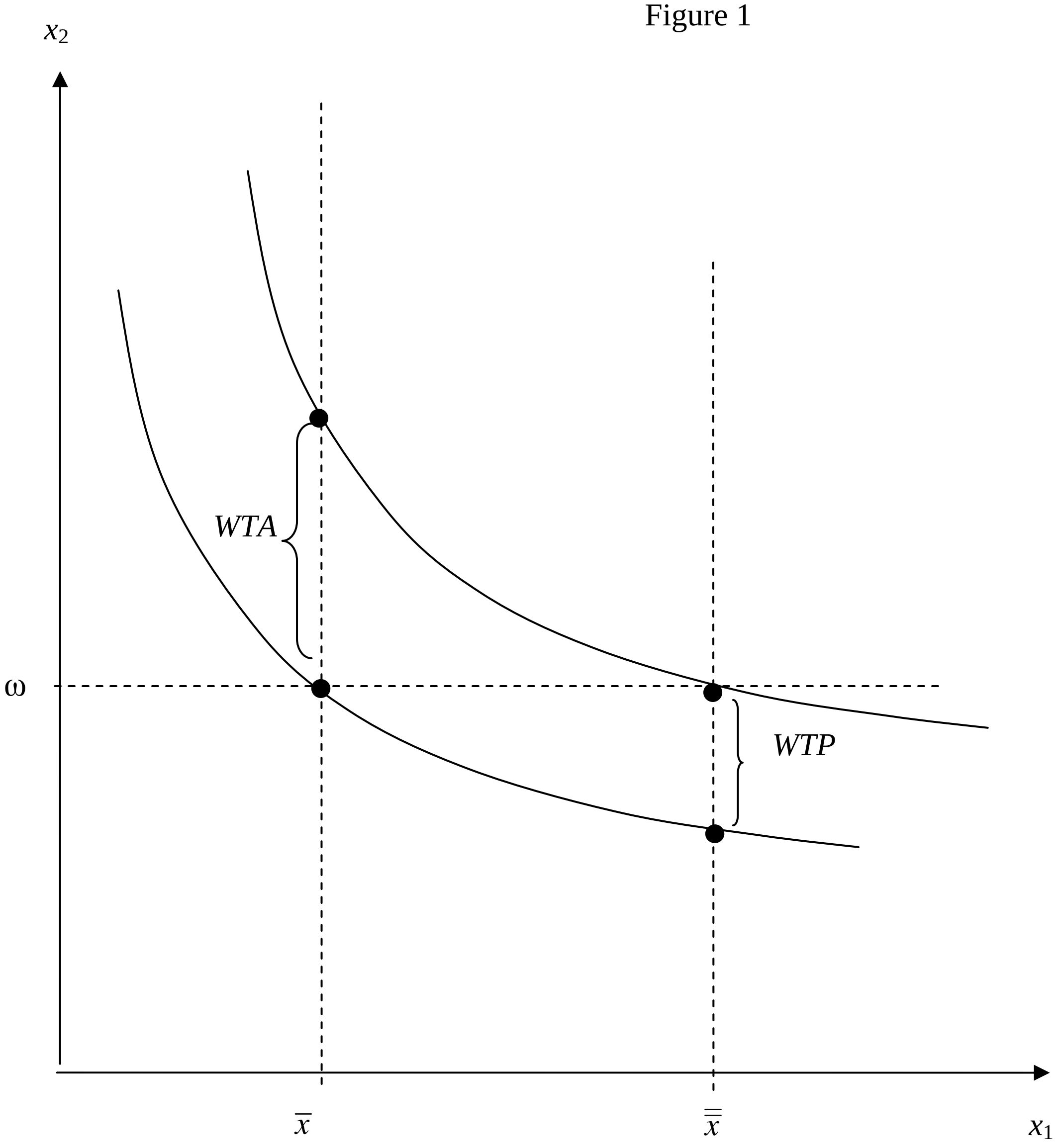
Question 1(b). How do *WTP* and *WTA* compare when *u* is quasilinear (with good 2 as numeraire)?

ANSWER. Let *u* be quasilinear with good 2 as numeraire, i. e., $u(x_1, x_2) = \varphi(x_1) + x_2$.

Then *WTP* satisfies $\varphi(\bar{x}) + \omega = \varphi(\bar{\bar{x}}) + \omega - WTP$, i. e., $WTP = \varphi(\bar{\bar{x}}) - \varphi(\bar{x})$. On the other hand, *WTA*: $\varphi(\bar{x}) + \omega + WTA = \varphi(\bar{\bar{x}}) + \omega$, i. e., $WTA = \varphi(\bar{\bar{x}}) - \varphi(\bar{x})$. Therefore, $WTA = WTP$ in the quasilinear case.

Graphically, the indifference curves are vertically parallel in this case, and therefore the vertical distance between them does not vary with the abscissa.

Figure 1



Question 1(c). Can you establish a conceptual parallel with the traditional (Mas-Colell *et al*., 1995, Ch. 3) notions of Equivalent and Compensation Variations for a consumer with wealth w who buys L goods and takes as given the L-dimensional price vector?

ANSWER. Let v be the indirect utility function, and consider two price vectors, p^0 and p^1, such that p^1 is better than p^0 for the consumer, i. e., $v(p^1, w) > v(p^0, w)$, which implies that both the EV and the CV of a move from p^0 to p^1 are positive. Recall that the CV is implicitly defined by:

$$v(p^1, w - CV) = v(p^0, w),$$

whereas EV is implicitly defined by

$$v(p^0, w + EV) = v(p^1, w).$$

Comparing with 1(a) above, we can establish the following parallelisms:

$$u \leftrightarrow v$$

$$\omega \leftrightarrow w$$

$$\bar{x} \leftrightarrow p^0$$

$$\bar{\bar{x}} \leftrightarrow p^1,$$

and accordingly

$$WTP \leftrightarrow CV$$

$$WTA \leftrightarrow EV.$$

The parallelism carries over to the observation that, under quasilinearity, when the price of the numeraire good does not change, $EV = CV$ (compare with 1(b)).

For 1(d)-1(k) below, we specialize to the CES utility function

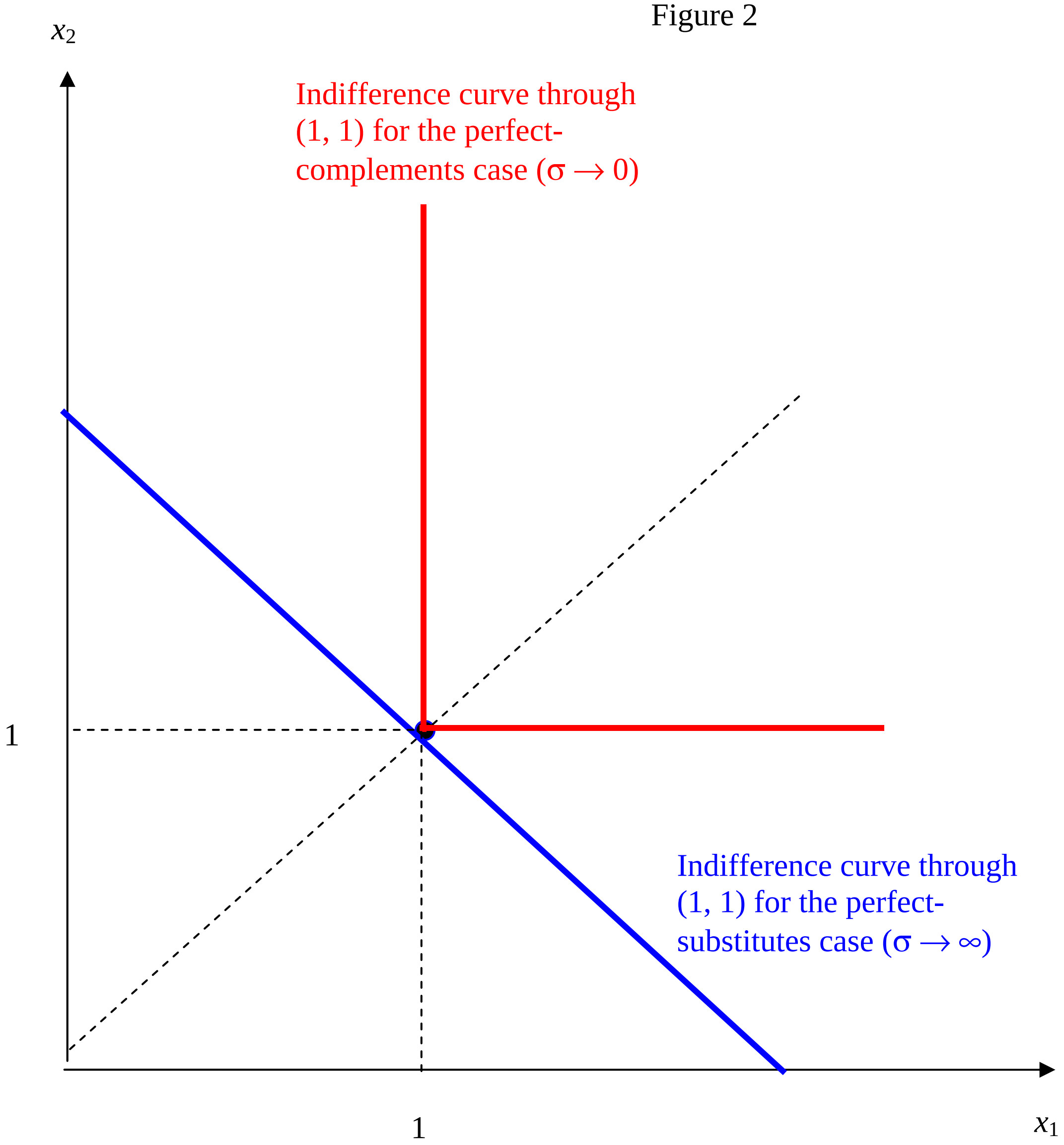
$$u(x_1, x_2) = \left[\left(x_1\right)^{\frac{\sigma-1}{\sigma}} + \left(x_2\right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}, \qquad (1)$$

where σ is a positive parameter, $\sigma \in (0,1) \cup (1,\infty)$.

Question 1(d). Interpret the parameter σ. What is the limit case of this CES utility function as $\sigma \to \infty$? What is the limit case of this CES utility function as $\sigma \to 0$? Give verbal and graphical (Figure 2) interpretations of these limit cases .

ANSWER. Of course, σ is the elasticity of substitution. As $\sigma \to \infty$, the limit is the perfect-substitutes utility function $x_1 + x_2$. As $\sigma \to 0$, the limit is the perfect-complements utility function $\min \{x_1, x_2\}$. See Figure 2.

Figure 2


x_2
Indifference curve through (1, 1) for the perfect-complements case ($\sigma \to 0$)
1
Indifference curve through (1, 1) for the perfect-substitutes case ($\sigma \to \infty$)
1
x_1

Question 1(e). Explicitly compute *WTP* and *WTA* for (1).

ANSWER.

WTP defined by $u(\bar{x},\omega)=u(\bar{\bar{x}},\omega-WTP)$, i. e.,

$$\left[\bar{x}^{\frac{\sigma-1}{\sigma}}+\omega^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}=\left[\bar{\bar{x}}^{\frac{\sigma-1}{\sigma}}+\left(\omega-WTP\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},$$

i. e., $\bar{x}^{\frac{\sigma-1}{\sigma}}+\omega^{\frac{\sigma-1}{\sigma}}=\bar{\bar{x}}^{\frac{\sigma-1}{\sigma}}+\left(\omega-WTP\right)^{\frac{\sigma-1}{\sigma}}$,

i. e., $\left(\omega-WTP\right)^{\frac{\sigma-1}{\sigma}}=\bar{x}^{\frac{\sigma-1}{\sigma}}+\omega^{\frac{\sigma-1}{\sigma}}-\bar{\bar{x}}^{\frac{\sigma-1}{\sigma}}$,

i. e., $\omega-WTP=\left[\bar{x}^{\frac{\sigma-1}{\sigma}}+\omega^{\frac{\sigma-1}{\sigma}}-\bar{\bar{x}}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$,

i. e., $WTP=\omega-\left[\bar{x}^{\frac{\sigma-1}{\sigma}}+\omega^{\frac{\sigma-1}{\sigma}}-\bar{\bar{x}}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$.

WTA defined by $u(\bar{x},\omega+WTA)=u(\bar{\bar{x}},\omega)$, i. e.,

$$\left[\bar{x}^{\frac{\sigma-1}{\sigma}}+\left(\omega+WTA\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}=\left[\bar{\bar{x}}^{\frac{\sigma-1}{\sigma}}+\omega^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},$$

i. e., $\bar{x}^{\frac{\sigma-1}{\sigma}}+\left(\omega+WTA\right)^{\frac{\sigma-1}{\sigma}}=\bar{\bar{x}}^{\frac{\sigma-1}{\sigma}}+\omega^{\frac{\sigma-1}{\sigma}}$,

i. e., $\left(\omega+WTA\right)^{\frac{\sigma-1}{\sigma}}=\bar{\bar{x}}^{\frac{\sigma-1}{\sigma}}+\omega^{\frac{\sigma-1}{\sigma}}-\bar{x}^{\frac{\sigma-1}{\sigma}}$,

i. e., $\omega+WTA=\left[\bar{\bar{x}}^{\frac{\sigma-1}{\sigma}}+\omega^{\frac{\sigma-1}{\sigma}}-\bar{x}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$,

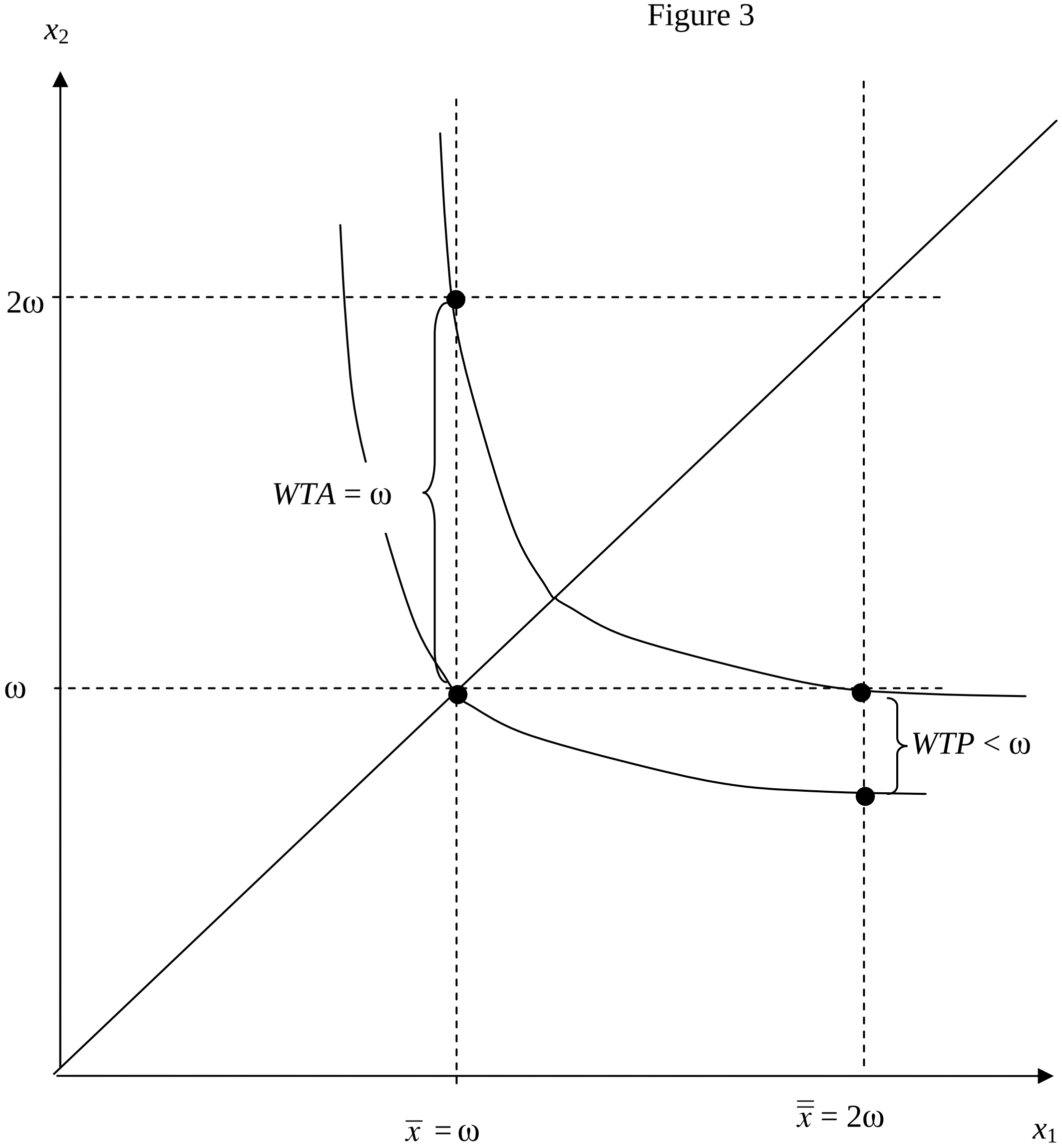
$$WTA=-\omega+\left[\bar{\bar{x}}^{\frac{\sigma-1}{\sigma}}+\omega^{\frac{\sigma-1}{\sigma}}-\bar{x}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}.$$

For 1(f)-1(k) below, consider the simple case where $\bar{x}=\omega$ and $\bar{\bar{x}}=2\omega$ (in addition to the CES assumption).

Question 1(f). Draw a graph, labeled Figure 3, that specializes Figure 1 to this case.

ANSWER.

Figure 3



Question 1(g). Specialize to this case the expressions obtained in 1(e) above. Illustrate the *WPA* in Figure 3.

ANSWER.

$$WTP=\omega-\left[\omega^{\frac{\sigma-1}{\sigma}}+\omega^{\frac{\sigma-1}{\sigma}}-(2\omega)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}==\omega-\omega\left[2-2^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}. \qquad (2)$$

$$WTA=-\omega+\left[(2\omega)^{\frac{\sigma-1}{\sigma}}+\omega^{\frac{\sigma-1}{\sigma}}-\omega^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}=-\omega+2\omega=\omega.$$

Question 1(h). What can you say about the magnitude of the ratio *WTA* /*WTP*? Argue in detail. Going back to 1(c) above, can you find a parallelism with the traditional Equivalent Variation and Compensating Variation measures?

ANSWER. We can see in Figure 3 that *WTP* < *WTA*. This can be algebraically proved from (2) by checking that $\left[2-2^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}>0$. Indeed, because $\sigma>0$, $2^{\frac{1}{\sigma}}>1$, and hence

$$2-2^{\frac{\sigma-1}{\sigma}}=2\left[1-\frac{1}{2^{\frac{1}{\sigma}}}\right]>0.$$

It follows that $\frac{WTA}{WTP}>1$, in accordance with Figure 1.

Question 1(i). What can you say about the limit of *WTA* as $\sigma\to\infty$? What can you say about the limit of *WTP* as $\sigma\to\infty$? What does this imply for the limit of the ratio *WTA* /*WTP* as $\sigma\to\infty$? Comment.

ANSWER. We see that *WTA* = ω, independently of σ. On the other hand

$$\lim_{\sigma\to\infty}WTP=\omega-\omega\lim_{\sigma\to\infty}\left[2-2^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}=\omega-0=\omega$$

(because $\lim_{\sigma\to\infty}\frac{\sigma-1}{\sigma}=\lim_{\sigma\to\infty}\frac{\sigma}{\sigma-1}=1$). Hence $\lim_{\sigma\to\infty}\frac{WTA}{WTP}=1$.

Intuitively, as $\sigma \to \infty$, the two goods become perfect substitutes, and a change in good 1 is perfectly equivalent to a change in good 2 of the same magnitude. Note also that the utility function then tends to linearity, which is a special case of quasilinearity, in which case, as seen in 1(b) above, *WTA* and *WTP* coincide.

Question 1(j). What can you say about the limit of *WTA* as $\sigma \to 0$? What can you say about the limit of *WTP* as $\sigma \to 0$? What does this imply for the limit of the ratio *WTA* /*WTP* as $\sigma \to 0$? Comment.

ANSWER. Again, $WTA = \omega$, independently of σ. On the other hand

$$\lim_{\sigma\to 0} WTP = \omega - \omega \lim_{\sigma\to 0}\left[2 - 2^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} = \omega - \omega = 0$$

(because $\lim_{\sigma\to 0}\dfrac{\sigma}{1-\sigma} = 0$, and $\dfrac{\sigma-1}{\sigma} \to -\infty$ as $\sigma \to 0$). Hence $\dfrac{WTA}{WTP} \to \infty$.

Intuitively, as $\sigma \to 0$, the two goods become perfect complements, and any move from $(\bar{x}, \omega) = (\omega, \omega)$ that reduces the amount of good 2 makes the consumer worse off. Thus, the consumer is not willing to sacrifice any positive amount of good 2 in exchange for an increase in good 1.

Question 1(k). Going back to 1(c) above once more, comment on the extent of the parallelism with the traditional Equivalent Variation and Compensating Variation measures.

ANSWER. We can summarize the parallelism as follows:

(A) All these measures are positive for an improvement in the economic environment, which is a change from $\bar{x}$ to $\bar{\bar{x}}$ in our problem here, and from p^0 to p^1 in the traditional measures.

(B) Under quasilinearity, $EV = CV$ and $WTA = WTP$.

(C) Similarly, under normality, $EV > CV$, and in our case $WTA > WTP$ as long as $\sigma < \infty$ (CES implies normality).

(D) But the parallelism is not perfect. As seen in 1(j), the ratio *WTA* /*WTP* grows without bounds as the two goods become complementary, whereas in the traditional case the ratio *WTA* /*WTP* is bounded, even when the goods are perfect complements.

QUESTION 2. RISK, UTILITY AND PRUDENCE ANSWER KEY

Let g and h be real-valued functions defined on an open interval of the positive real line, with positive first order derivative, and negative second order derivative. We will be using the following lemma (which you do not have to prove: it is an adaptation of Proposition 6.C.2, definitions (i) and (ii), in Mas-Colell *et al.*, 1995).

Lemma. *The following two conditions are equivalent*:

(i) $-\frac{g''(x)}{g'(x)} \geq -\frac{h''(x)}{h'(x)}, \forall x$.

(ii) If $\int h(x)dF(x) = h(\bar{x})$, then $\int g(x)dF(x) \leq g(\bar{x})$, for all lotteries F, $\forall \bar{x}$.

If either of these conditions is satisfied, then we say that g is (weakly) *more concave* than h.

Question 2(a). Interpret condition (i) in the above Lemma when g and h are understood as von Neumann-Morgenstern-Bernoulli (vNMB) utility functions.

ANSWER. Condition (i) means that the coefficients of absolute risk aversion (ARA) satisfy $r_A(x, g) \geq r_A(x, h)$, i. e., that g is more risk averse than h.

In what follows we view x as the sum of ω and z, where ω is interpreted as the (nonrandom) wealth of the consumer, and z is distributed according to a cumulative distribution function F with mean zero. We consider a consumer with vNMB utility function u, defined on an open interval of the positive real line, thrice differentiable, with $u'(x) > 0$ and $u''(x) < 0$.

Definition. Given F and ω we define the *risk premium* ρ implicitly by the equation

$$\int u(\omega + z)dF(z) = u(\omega - \rho).$$

Definition. Given F and ω we define the *utility premium* ψ as $\psi = u(\omega) - \int u(\omega + z)dF(z)$.

Definition. The vNMB utility function u displays (weak) *prudence* if $u'''(x) \geq 0, \forall x$, and *strict prudence* if $u'''(x) > 0, \forall x$.

Question 2(b). Interpret the risk premium and the utility premium in words.

ANSWER. Risk premium. We interpret ρ as the amount that the consumer with wealth ω would be willing to pay to avoid facing the risk represented by $F(z)$, or as the money loss suffered by the consumer with wealth ω when facing the risk represented by $F(z)$.

Note that ω - ρ is the certainty equivalent of the lottery, in terms of final wealth, induced by *F*.

Utility premium. We interpret ψ as the loss in utility suffered by the consumer with wealth ω when facing the risk represented by *F*(*z*).

Question 2(c). We are interested in analyzing how the risk premium ρ varies with wealth ω. Show that $\frac{d\rho}{d\omega} \leq 0$ *if and only if u displays* (weakly) *decreasing absolute risk aversion.* [Hint: use the Lemma above, and consider the increasing functions – *u*' and *u*].

ANSWER. Proof. *u* displays (weakly) decreasing absolute risk aversion iff

$$\frac{d\left(-\frac{u''(x)}{u'(x)}\right)}{dx} \leq 0\,, \qquad (1)$$

i. e., iff $$-\frac{u'''(x).u'(x) - u''(x).u''(x)}{[u'(x)]^2} \leq 0\,,$$

i. e., iff $$-u'''(x).u'(x) - [-[u''(x)]^2 \leq 0\,,$$

or, dividing through by the negative expression $u'(x).u''(x)$,

$$-\frac{u'''(x)}{u'(x)} - \left[-\frac{u''(x)}{u''(x)}\right] \geq 0\,,$$

i. e., by the Lemma, (1) holds iff the increasing function $-u'$ is more concave than u.

By implicitly differentiating the equation

$$\int u(\omega + z)dF(z) - u(\omega - \rho) = 0\,, \qquad (2)$$

which defines ρ, we compute

$$\frac{d\rho}{d\omega} = -\frac{\int u'(\omega + z)dF(z) - u'(\omega - \rho)}{u'(\omega - \rho)} = \frac{\int -u'(\omega + z)dF(z) - [-u'(\omega - \rho)]}{u'(\omega - \rho)}\,. \qquad (3)$$

Hence, $\frac{d\rho}{d\omega} \leq 0$ iff the numerator of (3) is nonpositive whenever (2) holds. But this is guaranteed by the Lemma (condition (ii)) whenever the function $-u'$ is more concave than u. Therefore (1) holds iff $\frac{d\rho}{d\omega} \leq 0$. ■

Question 2(d). Does decreasing absolute risk aversion imply prudence? Argue your answer.

ANSWER. YES. As just seen

$$\frac{d\left(-\frac{u''(x)}{u'(x)}\right)}{dx} \leq 0 \Leftrightarrow -u'''(x).u'(x) - \{-[u''(x)]^2\} \leq 0,$$

which implies that $-u'''(x) \leq 0$, i. e., $u'''(x) \geq 0$, the definition of prudence. ■

Note that the converse is not necessarily true, because the positive term $[u''(x)]^2$ may more than compensate for a negative $-u'''(x).u'(x)$.

Question 2(e). Again, we are interested in analyzing how the utility premium ψ varies with wealth ω. How is the sign of $\frac{d\psi}{d\omega}$ related to prudence?

ANSWER. We compute

$$\frac{d\psi}{d\omega} = u'(\omega) - \int u'(\omega + z)dF(z) \ .$$

Because $\omega = \int (\omega + z)dF(z)$,

$-u'(\omega) - \int -u'(\omega + z)dF(z) \ \geq 0 \Leftrightarrow$ - *u'* is a concave function [by Jensen's inequality],

i. e., $\frac{d\psi}{d\omega} = u'(\omega) - \int u'(\omega + z)dF(z) \ \underline{\leq} \ 0 \Leftrightarrow$ - *u'* is a concave function

$$\Leftrightarrow \frac{d^2(-u'(x))}{dx^2} \leq 0$$

$$\Leftrightarrow -u'''(x) \leq 0 \ ,$$

which again is the definition of prudence. We conclude:

$$\frac{d\psi}{d\omega} \leq 0 \text{ iff } u \text{ displays prudence.}$$

Question 2(f). Summarize, according to your answers to 2(c), 2(d) and 2(e), the relations among the following four properties:

$\frac{d\rho}{d\omega} \leq 0$,

Decreasing absolute risk aversion,

Prudence,

$$\frac{d\psi}{d\omega} \leq 0 .$$

Illustrate these relations for the vNMB utility functions: $-e^{-rx} (r > 0)$ and - $(a - x)^2$ (with domain $x < a$), and comment.

ANSWER. $\frac{d\rho}{d\omega} \leq 0$ $\Leftrightarrow$ decreasing ARA [2(c)]

decreasing ARA $\Rightarrow$ prudence [2(d)]

prudence $\Leftrightarrow \frac{d\psi}{d\omega} \leq 0$. [2(e)]

In particular, $\frac{d\rho}{d\omega} \leq 0 \Rightarrow \frac{d\psi}{d\omega} \leq 0$, but the converse is not necessarily true.

* vNMB function $-e^{-rx}$. This is the case of CARA. The argument in 2(c) shows that, then, $\frac{d\rho}{d\omega} = 0$. Yet the third-order derivative of $-e^{-rx}$ is $r^3 e^{-rx} > 0$. Hence, this function displays strict prudence, and, by the argument in 2(e), $\frac{d\psi}{d\omega} < 0$.

* Quadratic vNMB function - $(a - x)^2$. Its third-order derivative is zero: hence, it does not display strict prudence: only weak prudence, and accordingly $\frac{d\psi}{d\omega} = 0$. It actually displays increasing ARA, and by the argument in 2(c), $\frac{d\rho}{d\omega} > 0$. This in particular shows that the converse of the implication "$\frac{d\rho}{d\omega} \leq 0 \Rightarrow \frac{d\psi}{d\omega} \leq 0$" is false. The property that $\frac{d\rho}{d\omega} > 0$, i. e., that the risk premium is increasing with wealth, is counterintuitive, and suggests that the quadratic vNMB function is unrealistic.

These examples show that risk and utility premia may behave quite differently, despite their conceptual similarity.

Answer Key -

(a) Pareto optimal allocation

$$\max \; u(m, c, \ell, e) \quad \text{subject to}$$

$$\left.\begin{array}{ll} m + na \leq m_0 & \rho_m \\ c \leq n(1-p(a))\, f(L) & \rho_c \\ \ell + nL \leq L_0 & \rho_\ell \\ e \geq n\, p(a)\, \tilde{e}(L,a) & \eta \end{array}\right\} \text{multipliers}$$

(b) $\frac{\partial u}{\partial m} = \rho_m \quad \frac{\partial u}{\partial c} = \rho_c \quad \frac{\partial u}{\partial \ell} = \rho_\ell \quad \frac{\partial u}{\partial e} = -\eta$

$$/a \qquad -\cancel{n}\rho_m - \rho_c \cancel{n}\, p'(a) f(L) - \cancel{n}\eta\, p'(a)\, \tilde{e}(L,a) - \overset{\eta}{\cancel{n}}\, p(a)\frac{\partial \tilde{e}}{\partial a} = 0$$

$$/L \qquad \rho_c \cancel{n}(1-p(a))\, f'(L) - \rho_\ell \cancel{n} - \eta \cancel{n}\, p(a)\frac{\partial \tilde{e}}{\partial L} = 0.$$

Eliminating the multipliers and re-arranging

$$\frac{\partial u}{\partial m} = -p'(a)\left[\frac{\partial u}{\partial c} f(L) - \frac{\partial u}{\partial e}\tilde{e}(L,a)\right] + \frac{\partial u}{\partial e}\, p(a)\frac{\partial \tilde{e}}{\partial a} \qquad ①$$

$$\frac{\partial u}{\partial c}(1-p(a))\, f'(L) = \frac{\partial u}{\partial \ell} - \frac{\partial u}{\partial e}\, p(a)\frac{\partial \tilde{e}}{\partial L} \qquad ②$$

On the left side of ① : marginal cost (per firm) of one unit of money transferred from consumer to firms' investment.

On the right side of ① : marginal benefit which is the sum of three terms

- marginal benefit of $p'(a)\, f(L)$ additional units of consumptn
- marginal benefit of $p'(a)\ \tilde{e}(L,a)\ (< 0)$ decrease in pollution due to the decrease in number of firms which have an accident
- marginal benefit of the reduction of pollution $\frac{\partial \tilde{e}}{\partial a}$ from the firms which do have an accident.

On the left side of ② : marginal benefit of one more unit of labor given to the firm which creates $(1-p(a)\, f'(L)$ additional consumption.

On the right side of 2 : marginal cost which is the sum of the cost of the loss of leisure and of the additional pollution by the firms which have an accident

(c) competitive equilibrium with fines for the firms which pollute

- markets : money price p_m
  energy price p_e
  labor price w

- representative firm maximizes expected profit:

$$\max_{a \geq 0, L \geq 0} (1-p(a))\, p_c\, f(L) + p(a)\left(-k\, \tilde{e}(L,a)\right) - wL - p_m a$$

where k is the cost of one unit of pollution.

FOCs :
$$-p'(a)\left(p_c f(L) + k\, \tilde{e}(L,a)\right) - p(a)\, k \frac{\partial \tilde{e}}{\partial a} - p_m = 0$$

$$(1-p(a))\, p_c\, f'(L) - k\, p(a) \frac{\partial \tilde{e}}{\partial L} - w = 0$$

- representative agent maximizes $u(m, c, l, e)$, taking e as given, under the budget constraint

$$p_m m + p_c c + w l = p_m m_0 + w L_0 + n\pi + T$$

where π is the profit of the representative firm and T the amount of fines paid by the firms : $T = n\, p(a)\, k\, \tilde{e}(L,a)$

FOCs : $\frac{\partial u}{\partial m} = \lambda p_m \quad \frac{\partial u}{\partial c} = \lambda p_c \quad \frac{\partial u}{\partial l} = \lambda w$ where λ is the marginal utility of income

- markets clear :

$$m + na = m_0$$
$$c = (1-p(a))\, n\, f(L)$$
$$nL + l = L_0 .$$

(4)

Replacing the prices by their expression as a function of the marginal utilities of the consumer gives

$$-p'(a)\left(\frac{\partial u}{\partial e}\cdot\frac{1}{\lambda}f(L)+k\tilde{e}(L,a)\right)-p(a)k\frac{\partial\tilde{e}}{\partial a}-\frac{\partial u}{\partial m}\cdot\frac{1}{\lambda}=0,$$

$$(1-p(a))\frac{\partial u}{\partial c}\cdot\frac{1}{\lambda}f'(L)-kp(a)\frac{\partial\tilde{e}}{\partial L}-\frac{\partial u}{\partial\ell}\cdot\frac{1}{\lambda}=0.$$

If $k=-\frac{\partial u}{\partial e}.\frac{1}{\lambda}$ (calculated at the equilibrium values), then the FOC's for Pareto optimality are satisfied in equilibrium. Not surprisingly, the money cost of one unit of pollution for a firm which has an accident must be equal to the marginal disutility of production divided by the marginal utility of income to transform utils into dollars. If all programs of maximization are convex, this proves that an equilibrium with the appropriate fines is Pareto optimal.

Answer keys for Questions 4 and 5 Micro Prelim June 2010

4. **(a) and (b.1)**:

$G(b_1, b_2)$:

		Player 2	
		B	*H*
Player 1	*B*	–20 –20	20 0
	H	0 20	0 0

Nash equilibria: (H,B) and (B,H)

$G(b_1, \text{not-}b_2)$:

		Player 2	
		B	*H*
Player 1	*B*	–20 –40	20 20
	H	0 0	0 20

Nash equilibrium: (B,H)

$G(\text{not-}b_1, b_2)$:

		Player 2	
		B	*H*
Player 1	*B*	–40 –20	0 0
	H	20 20	20 0

Nash equilibrium: (H,B)

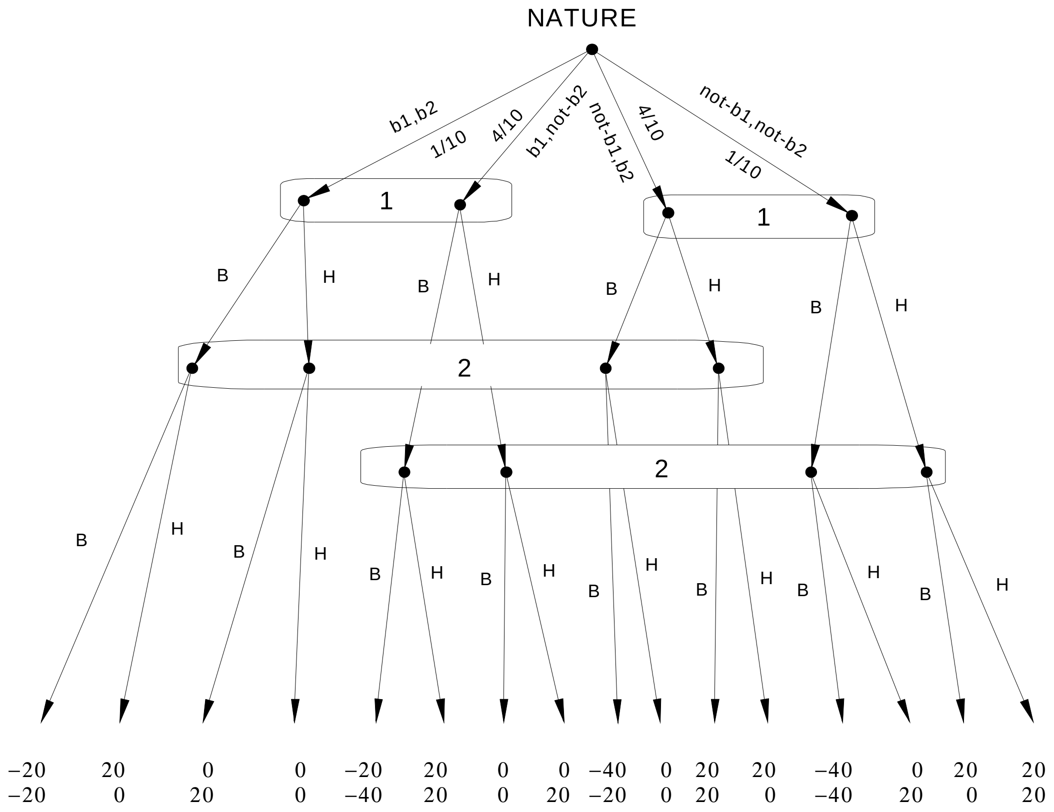
$G(\text{not-}b_1, \text{not-}b_2)$:

		Player 2	
		B	*H*
Player 1	*B*	–40 –40	0 20
	H	20 0	20 20

Nash equilibrium: (H,H)

(b.2) Let p be the probability that Player 1 plays B and q the probability that Player 2 plays B. Then the mixed-strategy equilibrium is given by the solution to $-20q + 20(1-q) = 0$ and $-20p + 20(1-p) = 0$, which is $p = q = \frac{1}{2}$.

(c) First of all note that the common prior is $\begin{pmatrix} b_1, b_2 & b_1, \text{not-}b_2 & \text{not-}b_1, b_2 & \text{not-}b_1, \text{not-}b_2 \\ \frac{1}{10} & \frac{4}{10} & \frac{4}{10} & \frac{1}{10} \end{pmatrix}$. Thus the extensive form is as follows:



(d) One possible strategy for player 1 is (B,H) which means "if I prefer going to the bar then I go to the bar and if I prefer going home then I go home".

(e) Each player has four strategies: (B,B), (B,H), (H,B) and (H,H).

(f) You were asked to fill in one row of the following strategic form (payoffs are expected payoffs):

Player 2

		B,B	B,H	H,B	H,H
	B,B	–30, –30	–10 , 0	–10 , –20	10, 10
Player	B,H	0, –10	16, 16	4 , –16	20, 10
1	H,B	–20 , –10	–16 , 4	–4 , –4	0, 10
	H,H	10 , 10	10 , 20	10 , 0	10 , 10

(g) There is a unique pure-strategy Nash equilibrium, namely $\big((B,H),(B,H)\big)$ where each player goes to the bar if she prefers going to the bar and goes home if she prefers going home. This can be found either the long way, by filling in all the payoffs in the above matrix, or by reasoning as follows. For each player, going home is strictly better than going to the bar if the player prefers going home, no matter what she anticipates the other player doing (in other words, H strictly dominates B at the information set where not-b_i holds). Thus the question is what to do if you prefer going to the bar. You know that the other player is going home if she prefers going home, thus you only need to consider her choice if she prefers going to the bar; if her plan is to go home also in that case, then B gives you 20 for sure and H gives you 0 for sure, hence B is better; if her plan is to go to the bar, then H gives you 0 for sure while B gives you the lottery $\begin{pmatrix} -20 & 20 \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$, that is, an expected payoff of 12; hence B is better in that case too.

(h) In the true state, both players prefer going to the bar, thus **(h.1)** they both end up going to the bar and **(h.2)** get a payoff of -20. (h.3) (B,B) is not a Nash equilibrium of the game that they are actually playing (game $G(b_1,b_2)$).

(i) If the true game is $G(b_1,b_2)$ they end up playing (B,B) which is not a Nash equilibrium of that game, if the true game is $G(b_1,\text{not-}b_2)$ they end up playing (B,H) which is a Nash equilibrium of that game, if the true game is $G(\text{not-}b_1,b_2)$ they end up playing (H,B) which is a Nash equilibrium of that game and if the true game is $G(\text{not-}b_1,\text{not-}b_2)$ they end up playing (H,H) which is a Nash equilibrium of that game. Since the probability of $G(b_1,b_2)$ is $\frac{1}{10}$, the probability that they end-up playing a Nash equilibrium of the actual game is $\frac{9}{10}$.

5. In what follows let $p_i = \frac{n_i}{n_H + n_M + n_L}$ ($i \in \{H,M,L\}$) be the proportion of type i in the population.

(a) OPTION 1. In order to get any applicants at all, the firm needs to offer a wage $w \geq w_0$. In this case every type would apply and the expected profit per worker is $\left(p_H x_H + p_M x_M + p_L x_L\right) R - w$. The maximum profit the firm can make with Option 1 is obtained by setting $w = w_0$ in which case the expected profit is

$$E\pi_1 = n\left[\left(p_H x_H + p_M x_M + p_L x_L\right) R - w_0\right].$$

OPTION 2**.** In order to attract the L types, it must be that $bx_L \geq w_0$, that is, $b \geq \frac{w_0}{x_L}$. Since $x_H > x_M > x_L$, $\frac{w_0}{x_L} > \frac{w_0}{x_M} > \frac{w_0}{x_H}$ and therefore every type would apply. The expected profit per worker in this case is $\left(p_H x_H + p_M x_M + p_L x_L\right)\left(R - b\right)$. The maximum expected profit is obtained by setting $b = \frac{w_0}{x_L}$, yielding a total expected profit of

$$E\pi_2 = n\left[\left(p_H x_H + p_M x_M + p_L x_L\right) R - \frac{\left(p_H x_H + p_M x_M + p_L x_L\right)}{x_L} w_0\right].$$

Since $\left(p_H x_H + p_M x_M + p_L x_L\right) > x_L$, $\frac{\left(p_H x_H + p_M x_M + p_L x_L\right)}{x_L} > 1$ and thus $\frac{\left(p_H x_H + p_M x_M + p_L x_L\right)}{x_L} w_0 > w_0$. Hence $E\pi_2 < E\pi_1$.

(b) In order to discourage the *L* types from applying, it must be that $bx_L < w_0$, that is, $b < \frac{w_0}{x_L}$. In order to encourage the *M* types to apply it must be $bx_M \geq w_0$, that is, $b \geq \frac{w_0}{x_M}$. Since $x_H > x_M$, $\frac{w_0}{x_M} > \frac{w_0}{x_H}$ and therefore type *H* would also apply. The expected profit per worker in this case (using Bayes' rule) is $\left(\frac{p_H}{p_H + p_M} x_H + \frac{p_M}{p_H + p_M} x_M\right)(R-b)$. The maximum value is obtained by setting $b = \frac{w_0}{x_M}$ yielding an expected profit of

$$E\pi_3 = n\left[\left(\frac{p_H}{p_H + p_M} x_H + \frac{p_M}{p_H + p_M} x_M\right) R - \left(\frac{p_H}{p_H + p_M} x_H + \frac{p_M}{p_H + p_M} x_M\right)\frac{w_0}{x_M}\right]$$

(recall the assumption that $n_H + n_M \geq n$ and thus the firm is able to fill all the *n* vacancies with Option 3).

First of all, note that $\left(\frac{p_H}{p_H + p_M} x_H + \frac{p_M}{p_H + p_M} x_M\right) > (p_H x_H + p_M x_M + p_L x_L)$.

Proof. Since $x_H > x_L$, $p_H x_H > p_H x_L$. Similarly, since $x_M > x_L$, $p_M x_M > p_M x_L$. Thus $p_H x_H + p_M x_M > (p_H + p_M) x_L$ so that (multiplying both sides by p_L) $p_L p_H x_H + p_L p_M x_M > (p_H + p_M) p_L x_L$. Hence

$$p_H \frac{p_L}{p_H + p_M} x_H + p_M \frac{p_L}{p_H + p_M} x_M > p_L x_L. \qquad (*)$$

Since $p_L = 1 - p_H - p_M$, (*) is the same as

$$p_H \frac{1 - p_H - p_M}{p_H + p_M} x_H + p_M \frac{1 - p_H - p_M}{p_H + p_M} x_M > p_L x_L. \qquad (**)$$

Adding $p_H x_H + p_M x_M$ to both sides of (**) we get

$$p_H x_H \left(\frac{1 - p_H - p_M}{p_H + p_M} + 1\right) + p_M x_M \left(\frac{1 - p_H - p_M}{p_H + p_M} + 1\right) > p_H x_H + p_M x_M + p_L x_L. \qquad (***)$$

Now, the LHS of (***) is equal to $\left(\frac{p_H}{p_H + p_M} x_H + \frac{p_M}{p_H + p_M} x_M\right)$. Q.E.D.

Thus we have that on the revenue side Option 3 is better than Option 1:

$$\left(\frac{p_H}{p_H + p_M} x_H + \frac{p_M}{p_H + p_M} x_M\right) R > (p_H x_H + p_M x_M + p_L x_L) R.$$

Now consider costs. Under Option 1, the cost per worker is w_0, while under Option 3 it is (when choosing the lowest admissible value for *b*, namely $b = \frac{w_0}{x_M}$)

$\left(\frac{p_H}{p_H + p_M} x_H + \frac{p_M}{p_H + p_M} x_M\right)\frac{w_0}{x_M}$. Now, this is greater than w_0 if and only if

$\dfrac{\dfrac{p_H}{p_H+p_M}x_H+\dfrac{p_M}{p_H+p_M}x_M}{x_M}>1$, that is, iff $\dfrac{p_H}{p_H+p_M}x_H+\dfrac{p_M}{p_H+p_M}x_M>x_M$ which is the case since $x_H>x_M$. Thus under Option 3 total cost is greater than under Option 1.

(c) Let $n_H=30, n_M=10, n_L=10$ (and n < 40). Then $p_H=\frac{3}{5}, p_M=\frac{1}{5}$,

$E\pi_1=2{,}650n$ and $E\pi_3=2{,}585n$.

(d) Let $n_H=10, n_M=30, n_L=10$ (and n < 40). Then $p_H=\frac{1}{5}, p_M=\frac{3}{5}$,

$E\pi_1=2{,}350n$ and $E\pi_3=2{,}528n$.

(e) In order to discourage the *M* types from applying, it must be that $bx_M<w_0$, that is, $b<\dfrac{w_0}{x_M}$.

Since $x_M>x_L$, $\dfrac{w_0}{x_M}<\dfrac{w_0}{x_L}$ and therefore type *L* would also choose not to apply. In order to encourage the *H* types to apply, we need $bx_H\geq w_0$, that is, $b\geq\dfrac{w_0}{x_H}$. The expected profit per worker in this case is $x_H\left(R-b\right)$. The largest possible value is obtained by setting $b=\dfrac{w_0}{x_H}$, yielding a an expected profit of

$$E\pi_4=\left(x_HR-w_0\right)\min\{n_H,n\}$$

Since $x_H>x_M$, if $\min\{n_H,n\}=n$, that is, $n_H\geq n$, this is greater than

$$E\pi_3=n\left[\left(\frac{p_H}{p_H+p_M}x_H+\frac{p_M}{p_H+p_M}x_M\right)R-\left(\frac{p_H}{p_H+p_M}x_H+\frac{p_M}{p_H+p_M}x_M\right)\frac{w_0}{x_M}\right]$$

In fact, $x_H>\left(\dfrac{p_H}{p_H+p_M}x_H+\dfrac{p_M}{p_H+p_M}x_M\right)>x_M$ since the since the expression in brackets is a convex combination between x_M and x_H with $x_M<x_H$. Thus revenue is higher under Option 4 and cost is lower since $\dfrac{\left(\dfrac{p_H}{p_H+p_M}x_H+\dfrac{p_M}{p_H+p_M}x_M\right)}{x_M}>1.$

Thus the maximum profit under Option 4 is greater than the maximum profit under Option 3 if $\min\{n_H,n\}=n$.

(f) If $\min\{n_H,n\}=n_H$, that is, there is a shortage of type *H* workers, then $E\pi_4=\left(x_HR-\dfrac{w_0}{x_H}\right)n_H$ and this could be less than $E\pi_3$. With the given parameter values, if n_H = 1 and n_M = 3, $E\pi_3=2{,}528n$ and $E\pi_4=3{,}250n_H=3250.$ Thus if n = 2, then $E\pi_3=5{,}056>E\pi_4=3250.$