# PRELIM

## MACROECONOMICS, 200D

Winter quarter, 2008/2009.

*Feel free to impose additional structure on the problems below, but please state your assumptions clearly.*

**Question 1** Consider the following (real valued) sequence $\{x_t\}_{t=0}^{\infty}$ defined by: $x_0 = 1$, $x_1 = 1 + a$, $x_2 = 1 + a + a^2$, $x_n = 1 + a + a^2 + \ldots + a^n$ and so on. Prove that if $|a| < 1$, then $\{x_t\}_{t=0}^{\infty}$ has a unique limit point. (Hint: Define $x_{n+1}$ recursively. You are then allowed to evoke the contraction mapping theorem.)

**Answer** Note that $x_{n+1} = 1 + ax_n = Tx_n$. Thus $|x_{n+1} - x_n| = |1 + ax_n - 1 - ax_{n-1}|$ which in turn equals $|a||x_n - x_{n-1}|$. Thus if $|a| < 1$, $Tx_n$ is a contraction mapping which, by the contraction mapping theorem has a unique fixed point.

**Question 2** A large share of macroeconomic models build on the Ramsey growth model augmented with a labor-leisure choice

$$V(k_0) = \max_{\{c_t,k_{t+1},\ell_t\}_{t=0}^{\infty}} \sum \beta^t u(c_t, \ell_t) \tag{1}$$

$$\text{s.t} \qquad c_t + k_{t+1} = f(k_t, \ell_t) \tag{2}$$

In class, however, we only considered problems of the type

$$V(x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum \beta^t F(x_t, x_{t+1}) \tag{3}$$

$$\text{s.t} \qquad x_{t+1} \in \Gamma(x_t) \tag{4}$$

Define $x_t$, $F(,\cdot,\cdot)$ and $\Gamma(\cdot)$ such that these two problems coincide exactly.

**Answer** $x_{t+1} = (c_t, k_{t+1}, \ell_t)$, $F(x_t, x_{t+1}) = u(c_t, \ell_t)$ and $\Gamma(x_t) = \{(c_t, k_{t+1}, \ell_t) : c_t + k_{t+1} = f(k_t, \ell_t)\}$.

**Question 5** Consider an infinitely lived individual born in time zero, endowed with a cake of size $x_0$. The cake is storable (without depreciation) and infinitely divisible. The agent derives contemporary utility from (cake) consumption through $u(c_t)$, and has as her ultimate objective to maximize her infinitely discounted sum of utility streams. Her discount rate is, as usual, given by $\beta$.

*(i)* Formulate the mathematical problem corresponding to the above description (i.e formulate the Sequence Problem).

*(ii)* Provide a recursive formulation and show that its solution also solves *SP* in *(i)* (i.e Theorem 2).

*(iii)* Let $u(c) = \ln(c)$. Derive both the value function, $v(x)$, and the policy function, $x' = g(x)$.

**Answer**

*(i)* The sequence problem is given by

$$\tilde{v}(x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(x_t - x_{t+1}) \tag{5}$$

*(ii)* Define

$$v(x_0) = \max_{x_1}\{u(x_0 - x_1) + \beta v(x_1)\} \tag{6}$$

and assume that $\lim \beta^n v(x_n) \to 0$ for all $\{x_{t+1}\}_{t=0}^{\infty}$. Then for any $\{x_{t+1}\}_{t=0}^{\infty}$ we have

$$\begin{aligned}
v(x_0) &\geq u(x_0 - x_1) + \beta v(x_1) && (7)\\
&\geq u(x_0 - x_1) + \beta u(x_1 - x_2) + \beta^2 v(x_2) && (8)\\
&\geq u(x_0 - x_1) + \beta u(x_1 - x_2) + \ldots + \beta^n u(x_n - x_{n+1}) + \beta^{n+1} v(x_{n+1}) && (9)\\
&\geq \sum_{t=0}^{\infty} \beta^t u(x_t - x_{t+1}) && (10)\\
& && (11)
\end{aligned}$$

And for some $x_1^*, x_2^*, \ldots$ we have

$$\begin{aligned}
v(x_0) &= u(x_0 - x_1^*) + \beta v(x_1^*) && (12)\\
&= u(x_0 - x_1^*) + \beta u(x_1^* - x_2^*) + \ldots + \beta^n u(x_n^* - x_{n+1}^*) + \beta^{n+1} v(x_{n+1}^*) && (13)\\
&= \sum_{t=0}^{\infty} \beta^t u(x_t^* - x_{t+1}^*) && (14)\\
& && (15)
\end{aligned}$$

Thus, by definition

$$v(x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(x_t - x_{t+1}) = \tilde{v}(x_0) \tag{16}$$

*(iii)* Using guess and verify one arrives at

$$v(x) = \frac{1}{1-\beta}\left(\ln(1-\beta) + \frac{\beta}{1-\beta}\ln\beta + \ln x\right), \quad g(x) = \beta x \tag{17}$$

**Question 6** Consider the following sequence problem

$$V(a_0, w_0) = \max_{\{c_t\}_{t=0}^{\infty}} \sum \beta^t u(c_t) \tag{18}$$

$$\text{s.t} \qquad a_0 + \sum_{t=0}^{\infty} q^t w_t = \sum_{t=0}^{\infty} q^t c_t \tag{19}$$

$$w_{t+1} = (1-\delta)\bar{w} + \delta w_t \tag{20}$$

*(i)* For which value of $\delta$ does (20) converge to a unique fixed point? What is the value of this fixed point?

*(ii)* Define

$$a_t = \sum_{s=0}^{\infty} q^s (c_{t+s} - w_{t+s}) \tag{21}$$

Use this definition to show that the problem above has a recursive representation. No formal proof is necessary, but explain the logic behind your result clearly. What interest rate do the "bonds" $a_t$ pay?

*(iii)* In a deterministic economy, such as the one above, it appears as if a one period bond is sufficient to allocate resources efficiently. Is this also true in a stochastic economy? Why or why not?

**Answer**

*(i)* Following question 1, equation (20) is a contraction map if $|\delta| < 1$. The fixed point that solves $w = (1-\delta)\bar{w} + \delta w$ equals $w = \bar{w}$.

*(ii)* The recursive formulation is given by

$$v(a, w) = \max_{a'} \{u(c) + \beta v(a', w')\} \tag{22}$$

$$\text{s.t} \qquad a + w = c + qa' \tag{23}$$

$$w' = (1-\delta)\bar{w} + \delta w \tag{24}$$

The reason is that given an optimal plan of consumption, we can define

$$a_1^* = \sum_{s=0}^{\infty} q^s (c_{1+s}^* - w_{1+s}) \tag{25}$$

and by construction

$$a_0 + w_0 = c_0^* + qa_1^* \tag{26}$$

Note that $v(a_1^*, w_1) \geq V_1^*$ (where $V_1^*$ is the time 1 continuation value of the optimal plan), since the optimal plan is feasible under the "continuation constraint". If the inequality was strict, however, there would exist a continuation plan, call it $\{c'\}$, such that

$$a_1^* = \sum_{s=0}^{\infty} q^s (c'_{1+s} - w_{1+s}) \tag{27}$$

yielding utility

$$u(c_0^*) + \beta v(a_1^*, w_1) > v(a_0, w_0) \tag{28}$$

Since any plan with

$$a_1^* = \sum_{s=0}^{\infty} q^s (c_{1+s} - w_{1+s}) \tag{29}$$

satisfies the present value budget constraint, $\{c^*\}$ can not have been optimal. This is a contradiction and thus $v(a_1^*, w_1) = V_1^*$.

Since $q$ is the price of bonds, the implied interest rate is $r = 1/q - 1$.

*(iii)* This is not true in a stochastic economy. In a deterministic economy, one period bonds are sufficient for market completeness. There exist one market (one bond) for each good (time period). In the stochastic case there exist one market for each time period, but not for each node in an event tree, and markets are thus incomplete.