Answer Key to Prelim
Macroeconomics, 200d

Winter quarter, 2010.

Feel free to impose additional structure on the problems below, but please state your assumptions clearly.

Question 1

(a) If u is bounded, and if $\gamma > 0$, Blackwell's sufficient conditions apply and the operator $T(V)$ is a contraction mapping. As a consequence, there exist a unique V satisfying the Bellman equation.

(b)

$$V(k) = \max_c \{u(c)\Delta + e^{-\Delta\gamma} V(\Delta(f(k) - \delta k - c) + k)\} \tag{1}$$

(c) Rearranging and taking limits we find that

$$\gamma V(k) = \max_c \{u(c) + V'(k)(f(k) - \delta k - c)\} \tag{2}$$

Question 2

(a) A space is complete if every Cauchy sequence (defined in the metric of the space) converges.

(b) A norm is simply a measure of distance in a space. The sup-norm is a metric between two elements in a space, which reports the supremum (the least upper bound) of their absolute difference.

(c) The contraction mapping theorem states that a contraction mapping on a complete metric space has one unique fixed point, $v = T(v)$, and that the sequence $v_{n+1} = T(v_n)$ converges to this fixed point.

(d) The envelope theorem states under which conditions the value function is differentiable and provides an expression for its derivative.

In the course we focus on the space of bounded functions endowed with the sup-norm, which we know is complete. As a consequence, we use the contraction mapping theorem to show there exist one unique solution to the Bellman equation. We used the envelope theorem in order to derive the Euler equation, and consequently to characterize the solution to the problem closer.

Question 4

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Date: September 2, 2010
Time: 5 hours
Reading Time: 20 minutes

PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

Directions: Answer all questions. Feel free to impose additional structure on the problems below, but please state your assumptions clearly.

Short Answer Questions. *Keep your answers short and concise. (Each question is worth 7 points.)*

1. Consider the following optimization problem

$$V(k) = \max_c \{u(c) + e^{-\gamma} V(f(k) + (1-\delta)k - c)\}$$

   (a) What assumptions do we need on u, f and γ to guarantee that there exists a unique $V(\cdot)$ satisfying the above Bellman equation?

   (b) In the equation above, one time-period is equal to one unit of time. Reformulate the equation such that one time-period equals Δ units of time.

   (c) What is the continuous-time formulation of the above equation? *Hint: remember that* $\lim_{h\to 0}(f(x+ha) - f(x))/h = f'(x)a$.

2. Using words, *briefly* define the following concepts, and explain how we have put them to use in the course (Rendahl):

   (a) A complete metric space.

   (b) A norm, and in particular, the sup-norm.

   (c) The contraction mapping theorem.

   (d) The envelope theorem.

3. Consider a Lucas-tree type economy in which the endowment is growing stochastically and assume that agents have CRRA preferences. For tractability, assume that the endowment growth rate follows a two-state Markov process. Suppose one- and two-period bonds are traded in this economy and define the term premium as the difference between the expected return from selling a two-period bond after holding it for one period and the return from a one-period bond. Provide an intuitive explanation for why the term premium is positive if and only if consumption growth exhibits negative serial correlation.

   ANSWER: Let $p_{1,i}$ denote the price of a one-period bond if the growth rate is $i = 1, 2$. It is easy to establish that $p_{1,1} > (<) \, p_{1,2}$ as $\pi > (<) \frac{1}{2}$. That is, if states are positively auto-correlated ($\pi > \frac{1}{2}$) then the one-period bond price will be lower in the high growth rate state. This is because the expected growth rate of consumption is greater in state 2 than in state 1 so agents will attempt to borrow to smooth consumption which drives the price of bonds down (increase in supply) or, stated alternatively, increases the one-period interest rate. The same reasoning applies if $\pi < \frac{1}{2}$. A two-period bond will sell for the price of a one-period bond after holding it for one period so the return is determined by the state in period $t+1$. Suppose $\pi > \frac{1}{2}$ and assume that the low growth rate occurs in period $t+1$. This implies that the MU of consumption will be high and that the return from selling the two-period bond will be high. The same logic applies if the high growth rate occurs; together this implies that covariance between the MU of consumption and the return from selling the two-period bond after one-period is positive. From the CCAPM, this implies a negative risk premium - which here is the same as the term premium. So only if $\pi < \frac{1}{2}$ will the covariance between the MU of consumption and return on the risky asset be negative as required for a positive term premium.

Longer Answer Questions. *(Each question is worth 20 points.)*

4. Consider the following (very simple) consumption-savings problem with habits

$$V(a_0, c_{-1}) = \max_{\{a_{t+1}, c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t - \gamma c_{t-1}) \tag{1}$$

$$subject\ to\ c_t + a_{t+1} = w + (1+r)a_t, \quad t = 0, 1, \ldots \tag{2}$$

with a_0 and c_{-1} given, and $\gamma \in [0, 1)$.

(a) Provide the Bellman equation to the above problem (no proof needed).

(b) Derive the first order conditions, apply the envelope theorem, and derive the Euler equation.

Now, let us modify the problem in the following way,

$$V(a_0, \tilde{a}_0, \tilde{c}_{-1}) = \max_{\{a_{t+1}, c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t - \gamma \tilde{c}_{t-1}) \tag{3}$$

$$\text{subject to} \quad c_t + a_{t+1} = w + (1+r)a_t, \quad t = 0, 1, \ldots \tag{4}$$

$$\tilde{c}_t = H(\tilde{a}_t, \tilde{c}_{t-1}), \quad and \quad \tilde{a}_{t+1} = G(\tilde{a}_t, \tilde{c}_{t-1}) \tag{5}$$

with a_0, $\tilde{a}_0$, and $\tilde{c}_{-1}$ given, and $\gamma \in [0, 1)$. $G(\cdot)$ and $H(\cdot)$ are, simply, exogenously provided functions.

(c) What is the Bellman equation associated with the problem above (no proof needed).

(d) Derive the first order conditions, apply the envelope theorem, and derive the Euler equation.

Let us now assume that the economy is populated by a *continuum* of individuals, all starting their lives with the same level of a_0 and $\tilde{a}_{-1}$. Moreover, $\tilde{a}_t$ and $\tilde{c}_t$ now happens to be *average* – or, as everyone is identical, *representative* – savings and consumption levels, respectively.

(e) What is the relationship between the functions $G(\cdot)$, $H(\cdot)$, and the individual's policy functions for c_t and a_{t+1}? (For simplicity, denote the individual's policy functions as $a' = g(a, \tilde{a}, \tilde{c}_{-1})$ and $c = h(a, \tilde{a}, \tilde{c}_{-1})$.)

(f) The economy above is commonly referred to as "catching up with the Jones's". Can you (very briefly) explain why? Why is the Euler equation associated with this economy so different from the one with standard habits? Which one of these two economies ("habits" and "keeping up with the Jones's") do you believe is Pareto-optimal, and which one is not?

5. Let z be a random variable that takes on values in $Z = \{0, 1\}$. Here, 1 denotes employment and 0 unemployment. Consider an arbitrary process of consumption $\{c_t(z^t)\}_{t=0}^{\infty}$ where $c_t : Z^{t+1} \to \mathbb{R}_+$. For this question, the probability of an unemployed individual finding a job is endogenous and depends on her search effort. For simplicity we will assume that the agent directly can influence, and therefore choose, the probability of finding a job, denoted by p_t; that is $P(z_{t+1} = 1|z_t = 0) = p_t$. Let us, for simplicity, assume that once a job is found, it lasts for perpetuity and is unaffected by the agent's choice of p; that is $P(z_{t+1} = 1|z_t = 1) = 1$, for all values of p_t.

   We can summarize these assumptions concisely by

$$\lambda(z^{t+1}) = p_t(z^t)\lambda(z^t), \text{ if } z_t = 0, \text{ and } \lambda(z^{t+1}) = \lambda(z^t), \text{ if } z_t = 1$$

   where $p_t(z^t)$ is an agent's choice of search effort in period t, after having observed history z^t. $\lambda(z^{t+1})$ denotes as usual *the probability of history z^{t+1} occurring*.

   Lastly, an agent's preferences are given by

$$\sum_{t=0}^{\infty} \sum_{z^t \in Z^{t+1}} \beta^t \{u(c_t(z^t)) - v(p_t(z^t))\}\lambda(z^t)$$

   Notice that the agent gains disutility $v(p_t(z^t))$ of searching at "intensity" $p_t(z^t)$. For simplicity we assume that $v(0) = 0$.

   (a) After any history z^t, define a *continuation value* in this economy. Denote this $\vec{V}(z^t)$.

   (b) If $z_t = 0$, derive a necessary condition for the agents' search effort in period t. *Hint: Notice that if $z_t = 0$, $\vec{V}(z^t) = u(c_t(z^t)) - v(p_t(z^t)) + \beta(p_t(z^t)\vec{V}((z^t, 1)) + (1 - p_t(z^t))\vec{V}((z^t, 0)))$*

   Now, let us assume there is a government providing unemployment insurance and taxing labor income. More precisely, the government will collect *all* labor income once the agent is employed, and always provide some sort of benefits equal to $c_t(z^t)$. The government does so in order to maximize her own (expected present value) revenue, and such that the agent is given some minimum (expected present value) utility, V_0. However, the government must also recognize that the agent will freely decide how much she searches. The government's optimization problem is then given by

$$J(V_0, z_0) = \max_{\{c_t(z^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^{t+1}} \beta^t \{z_t w - c_t(z^t)\}\lambda(z^t) \tag{6}$$

$$\textit{subject to} \quad V_0 = \sum_{t=0}^{\infty} \sum_{z^t \in Z^{t+1}} \beta^t \{u(c_t(z^t)) - v(p_t(z^t))\}\lambda(z^t) \tag{7}$$

   and also subject to the necessary condition derived in (b).

   (c) Provide the Bellman equation associated with the optimization problem above (no need for a proof). *Hint: A good starting point is to derive the Bellman equation for $z_0 = 1$, and then for $z_0 = 0$.*

6. Consider a simple, representative agent RBC model in which output, y_t, is produced via a standard Cobb-Douglas production function:

$$y_t = z_t k_t^{\alpha} h_t^{1-\alpha}$$

where k_t denotes beginning-of-period capital, h_t is labor, and z_t is an $i.i.d.$technology shock. The depreciation rate of capital is 100%. In each period, agents make consumption and labor decisions in order to maximize lifetime expected utility:

$$E_0\left[\sum_{t=0}^{\infty}\beta U\left(c_t, h_t\right)\right]$$

Within this environment, consider two variations defined by the functional form for $U(\cdot)$.

(a) In Economy A, agents have preferences given by:

$$U\left(c_t, h_t\right) = \ln c_t - \frac{1}{2}h_t^2$$

In this economy, do the following

i. Express the maximization problem as social planner problem and write down the associated Bellman equation.

ii. Solve for the equilibrium policy functions describing consumption, investment and labor.

(b) In Economy B, agents have preferences given by:

$$U\left(c_t, h_t\right) = \ln\left(c_t - \frac{1}{2}h_t^2\right)$$

i. Express the maximization problem as social planner problem and write down the associated Bellman equation.

ii. Solve for the equilibrium policy functions describing consumption, investment and labor.

(c) Compare the equilibrium behavior in both economies and provide an explanation for the differences.

ANSWER: For both cases the social planner problem can be expressed as:

$$\begin{aligned} V\left(k_t, z_t\right) &= \max\left[U\left(c_t, h_t\right) + \beta E\left(V\left(k_{t+1}, z_{t+1}\right)\right)\right] \\ & \quad \textit{subject to } z_t k_t^{\alpha} h_t^{1-\alpha} = c_t + k_{t+1} \end{aligned}$$

With associated necessary conditions describing the labor/leisure tradeoff and the consumption/investment tradeoff.

For Economy A these are:

$$h_t = (1-\alpha)\frac{z_t k_t^{\alpha} h_t^{-\alpha}}{c_t} \tag{8}$$

$$\frac{1}{c_t} = \alpha\beta E\left[\frac{z_{t+1}k_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha}}{c_{t+1}}\right] = \alpha\beta E\left[\frac{y_{t+1}}{c_{t+1}k_{t+1}}\right] \tag{9}$$

Since we have log preferences for consumption, it is reasonable to conjecture that savings (i.e. k_{t+1}) will be a constant fraction of output. That is, we conjecture that $k_{t+1} = \theta y_t$ which implies that

$c_t = (1-\theta)\, y_t$. Using this in eq.(9) yields $\theta = \alpha\beta$ or $k_{t+1} = \alpha\beta y_t$ and the conjecture is verified. For labor, mulitply both sides of eq.(8) by h_t to yield:

$$h_t^2 = (1-\alpha)\frac{y_t}{c_t} = \frac{1-\alpha}{1-\alpha\beta}$$

which implies that labor is constant. The implication is that the income and substitution effects for labor perfectly offset each other.

For Economy B, the necessary conditions are:

$$h_t = (1-\alpha)\, z_t k_t^{\alpha} h_t^{-\alpha} \tag{10}$$

$$\frac{1}{\left(c_t - \frac{1}{2}h_t^2\right)} = \alpha\beta E\left[\frac{z_{t+1}k_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha}}{\left(c_{t+1} - \frac{1}{2}h_{t+1}^2\right)}\right] = \alpha\beta E\left[\frac{y_{t+1}}{\left(c_{t+1} - \frac{1}{2}h_{t+1}^2\right)k_{t+1}}\right] \tag{11}$$

Note critically that consumption does not enter into the labor/leisure tradeoff, that is, labor is entirely determined by the MPL. In essence, this is eliminating the income effect on labor supply. Again multiply both sides of this expression by h_t to yield:

$$h_t^2 = (1-\alpha)\, y_t \tag{12}$$

Use this result in the intertemporal Euler equation:

$$\frac{1}{\left(c_t - \frac{(1-\alpha)y_t}{2}\right)} = \alpha\beta E\left[\frac{y_{t+1}}{\left(c_{t+1} - \frac{(1-\alpha)y_{t+1}}{2}\right)k_{t+1}}\right] \tag{13}$$

For consumption and capital, we can again make the conjecture that both are a constant fraction of output which yields the same result as in Economy A: $k_{t+1} = \alpha\beta y_t$. So the difference between the two economies is in the response of labor. Because the income effect is eliminated with the preferences in Economy B, output will be more variable since it is affected by both the productivity shock (impulse mechanism) and the labor response (amplification mechanism).

7. Consider the Mehra-Prescott model in which the economy is populated by infinitely lived representative agents that have time-separable preferences characterized by constant relative risk aversion. At birth, each agent is given a share of equity which provides ownership to the endowment process. The endowment, x_t, grows over time; the endowment growth rate follows a two-state Markov process with possible realizations (λ_1, λ_2) and a symmetric transition probability matrix with diagonal elements π. Assume that $\lambda_1 < \lambda_2$ and $\pi = 1/2$. In this economy, agents trade equity (with price of q_t) and one-period bonds (with net interest rate of r_t). Consumption and portfolio decisions are made in order to maximize:

$$E_0\left[\sum_{t=0}^{\infty}\beta U(c_t)\right]$$

Given this environment, do the following:

   (a) Define a recursive competitive equilibrium.

   (b) Characterize the equilibrium behavior of equity prices and interest rates.

(c) Prove that the equity premium in this economy is positive. In general (i.e. for all parameter values consistent with equilibrium), is the equity premium in the Mehra-Prescott economy always positive? Explain.

ANSWER: A recursive equilibrium is defined by three functions: $V(x_t, \lambda_t)$ defined by the usual Bellman equation and two price functions: $q(x_t, \lambda_t)$ and $r(x_t, \lambda_t)$ .Because of the assumption of CRRA, these price functions are homogenous of degree one and zero respecively. Let θ denote the price-dividend ratio, so the two price functions are: $\theta(\lambda_t)$ and $r(\lambda_t)$ which are defined by the necessary conditions associated with the Bellman equation when evaluated at market clearing quantities $c_t = x_t, z_t = 1$, and $b_t = 0$ (where z_t denotes equity holdings and b_t denotes bond purchases). Since λ_t takes on only two values, this implies the equilibrium will be defined by 4 values: $(\theta(\lambda_i), r(\lambda_i)); i = 1, 2$. From the necessary conditions (and using γ to denote agents' measure of relative risk aversion), we have

$$\theta(\lambda_i) = \beta E\left[\lambda_{t+1}^{1-\gamma}(\theta(\lambda_{t+1}) + 1)\right]$$

$$(1 + r(\lambda_i))^{-1} = \beta E\left[\lambda_{t+1}^{-\gamma}\right]$$

Note that the expectations term in both expressions will be a constant implying that the price-dividend ratio and interest rates will be as well. This implies that the price of equity is proportional to the dividend. Moreover, this implies:

$$\frac{\theta + 1}{\theta} = \frac{1}{\beta E\left(\lambda_{t+1}^{1-\gamma}\right)}$$

The equity premium in state i is defined by:

$$ep_i = E\left[\frac{q_{t+1} + x_{t+1}}{q_t}\right] - (1 + r_t)$$

Using the previous results, this can be written as (where the i subscript has been dropped since all terms on the right-hand side are constant):

$$ep = \frac{E(\lambda_{t+1})}{\beta E\left(\lambda_{t+1}^{1-\gamma}\right)} - \frac{1}{\beta E\left[\lambda_{t+1}^{-\gamma}\right]}$$

Using the definition of covariance, this expression becomes:

$$ep = -\frac{Cov(\lambda_{t+1}, \lambda_{t+1}^{-\gamma}}{\beta E\left(\lambda_{t+1}^{1-\gamma}\right) E\left[\lambda_{t+1}^{-\gamma}\right]}$$

The term in the denominator is positive while, by inspection, the covariance term is negative - this implies the equity premium will indeed be positive. This makes perfect sense based on the intuition of the CCAPM: since both the price and dividend are positively related to the endowment, equity is a risky asset since it pays out well (poorly) in those states of the world in which consumption is high (low).

(a)

$$V(a, c_{-1}) = \max_{c,a'}\{u(c - \gamma c_{-1}) + \beta V(a', c)\} \tag{3}$$

$$\text{subject to} \quad c + a' = w + (1+r)a \tag{4}$$

(b) The Euler equation is given by

$$u'(c - \gamma c_{-1}) - \gamma\beta u'(c' - \gamma c) = \beta(1+r)(u'(c' - \gamma c) - \gamma\beta u'(c'' - \gamma c')) \tag{5}$$

(c)

$$V(a, \tilde{a}, \tilde{c}_{-1}) = \max_{c,a'}\{u(c - \gamma\tilde{c}_{-1}) + \beta V(a', \tilde{a}', \tilde{c})\} \tag{6}$$

$$\text{subject to} \quad c + a' = w + (1+r)a \tag{7}$$

$$\tilde{c} = H(\tilde{a}, \tilde{c}_{-1}), \quad \text{and} \quad \tilde{a}' = G(\tilde{a}, \tilde{c}_{-1}) \tag{8}$$

(d) The Euler equation is given by

$$u'(c - \gamma\tilde{c}_{-1}) = \beta(1+r)u'(c' - \gamma\tilde{c}) \tag{9}$$

(e) Let $a' = g(a, \tilde{a}, \tilde{c}_{-1})$ and $c = h(a, \tilde{a}, \tilde{c}_{-1})$ denote the individual's policy functions. In equilibrium we know that $a = \tilde{a}$ and $c = \tilde{c}$, so $g(\tilde{a}, \tilde{a}, \tilde{c}_{-1})$ must equal $G(\tilde{a}, \tilde{c}_{-1})$ and $h(\tilde{a}, \tilde{a}, \tilde{c}_{-1})$ must equal $H(\tilde{a}, \tilde{c}_{-1})$.

(f) In the economy, an individual's utility from consumption depends on the (lagged) average consumption level of everyone else. Therefore, an individual is "punished" if he consumes less than his peers, and is "rewarded" if he consumes more. He simply tries to keep up with the Jones's.

Although the utility of consumption ultimately depends on the individual's past consumption, this is not internalized as the individual takes average consumption as given. There is therefore an externality present, and individual's in general consume too much. While the problem with habits is Pareto-optimal from a societal perspective, the keeping up with the Jones's solution is not.

Question 5 *(a)* A continuation value is defined as

$$\vec{V}(z^t) = \sum_{s=0}^{\infty} \sum_{z^{t+s} \in Z^s \times z^t} \beta^s \{u(c_{t+s}(z^{t+s})) - v(p_{t+s}(z^{t+s}))\}\lambda(z^{t+s}, z^t) \tag{10}$$

Where $\lambda(z^{t+s}, z^t) = \frac{\lambda(z^{t+s})}{\lambda(z^t)}$

(b) A necessary condition for $p_t(z^t)$ is given by the first order conditions to

$$\max_{p_t(z^t)}\{u(c_t(z^t)) - v(p_t(z^t)) + \beta(p_t(z^t)\vec{V}((z^t, 1)) + (1 - p_t(z^t))\vec{V}((z^t, 0)))\} \tag{11}$$

which equals $v'(p_t(z^t)) = \beta(\vec{V}((z^t,1)) - \vec{V}((z^t,0)))$.

(c) If $z = 1$ the Bellman equation is given by

$$J(V,1) = \max_{c,V'}\{u(c) + \beta J(V',1)\} \tag{12}$$

$$\text{s.t} \quad V = u(c) + \beta V' \tag{13}$$

If $z = 0$, the Bellman equation is given by

$$J(V,0) = \max_{c,p,V'_e,V'_u}\{u(c) - v(p) + \beta(pJ(V'_e,1) + (1-p)J(V'_u,0)\} \tag{14}$$

$$\text{s.t} \quad V = u(c) - v(p) + \beta(pV'_e + (1-p)V'_u) \tag{15}$$

$$v'(p) = \beta(V'_e - V'_u) \tag{16}$$