Answer Key for the September 2007 Macro Prelim

1. Imagine a representative consumer with time separable, logarithmic utility,

$$V = (1-\beta)E_t \sum\nolimits_{j=0}^{\infty} \beta^j \ln C_{t+j}, \tag{1}$$

where $0 < \beta < 1$. Consumption is an element of an exogenous random vector X_t which evolves according to

$$X_{t+1} = \mu + AX_t + \varepsilon_{t+1}, \tag{2}$$

where $\varepsilon_t \sim iid\ N(0, \Omega)$. Suppose that A has a single unit eigenvalue, with all other eigenvalues less than 1 in magnitude. For convenience, assume that $\ln C_t$ is the first element of the vector X_t. Assume 'sufficient discounting.' (This is intentionally vague; the context should become clear as you develop your answer.) Derive the consumer's value function. (Hint: There are at least two ways to solve this, a recursive and a nonrecursive approach. The recursive approach is a lot easier. The nonrecursive approach turns into an algebraic quagmire.)

ANSWER: The Bellman equation is

$$V(X_t) = (1-\beta)U(C_t) + \beta E_t V(X_{t+1}).$$

The max operator drops out of the rhs because this is an endowment economy. From (1) and (2), it is obvious that the value function is linear in X_t. This follows from the fact that expected utility is linear in expectations of $\ln C_{t+j}$ and that those expectations are linear in X_t. Thus, I conjecture that the value function is linear in X_t

$$V(X_t) = b_0 + b_1 X_t,$$

where b_0 is a scalar and b_1 is a row vector conformable with X. We just need to solve for the undetermined coefficients b_0 and b_1.After substituting the conjecture into the Bellman equation, we get

$$\begin{aligned} b_0 + b_1 X_t &= (1-\beta)U(C_t) + \beta E_t[b_0 + b_1 X_{t+1}], \\ &= (1-\beta)e_1 X_t + \beta[b_0 + b_1 E_t X_{t+1}], \end{aligned}$$

where e_1 is a selector vector (i.e., a row vector with a 1 in the first position and zeros everywhere else). Next, substitute the expectation of next period's X:

$$b_0 + b_1 X_t = (1-\beta)e_1 X_t + \beta[b_0 + b_1(\mu + AX_t)].$$

After a bit of algebra, we get

$$b_0 + b_1 X_t = \beta(b_0 + b_1\mu) + [(1-\beta)e_1 + \beta b_1 A]X_t.$$

Equating powers on the left and right-hand sides implies

$$\begin{aligned} b_0 &= \beta(b_0 + b_1\mu), \\ b_1 &= (1-\beta)e_1 + \beta b_1 A. \end{aligned}$$

Solve the second condition for b_1:

$$\begin{aligned} b_1 - \beta b_1 A &= (1-\beta)e_1, \\ b_1(I - \beta A) &= (1-\beta)e_1, \\ b_1 &= (1-\beta)e_1(I-\beta A)^{-1}. \end{aligned}$$

After substituting the solution into the condition for b_0, we get

$$b_0 = \beta e_1 (I - \beta A)^{-1}\mu.$$

This verifies the conjecture. Hence the value function is

$$V(X_t) = e_1(I-\beta A)^{-1}[\beta\mu + (1-\beta)X_t].$$

2. Consider a two-period overlapping generations model in which people work when young and retire when old. Consumers maximize

$$\ln(c_{1t}) + (1+\theta)^{-1}\ln(c_{2t+1}),$$

subject to the flow budget constraints

$$\begin{aligned} c_{1t} + s_t &= w_t, \\ c_{2t+1} &= (1+r_{t+1})s_t. \end{aligned}$$

c_1 and c_2 represent consumption when young and old, respectively, and w_t is labor income, s_t is savings, r_{t+1} is the real interest rate earned on saving, and $(1+\theta)^{-1}$ is the subjective discount factor.

Firms are competitive and produce output by combining labor and capital using a Cobb-Douglas production function. Labor is supplied inelastically, so employment at t is the same as the number of young people, N_t. The population grows at an exogenous rate n, so that $N_{t+1} = (1+n)N_t$. Output per worker is $Y/N = F(K/N, 1)$, or $y = f(k)$.

Equilibrium in the goods market requires that saving equal investment. Assuming no depreciation, investment is $K_{t+1} - K_t$. Aggregate savings consists of positive saving of the young minus dissaving of the old. Since the old own the capital stock, this is $N_t s_t - K_t$. Hence the law of motion for capital is

$$K_{t+1} - K_t = N_t s_t - K_t,$$

or

$$(1+n)k_{t+1} = s_t.$$

a. Draw a phase diagram for k and solve the model's steady state.

b. Next, modify the model to introduce a fully-funded social security system. I.e., imagine that a lump-sum tax is levied on young workers and invested on their behalf, and that benefits are paid when they are old out the of principle and accumulated interest. How does that alter the steady-state level of k?

ANSWER:

a. The FOC for the optimal consumption implies

$$\frac{c_{2t+1}}{c_{1t}} = \frac{1+r_{t+1}}{1+\theta}. \tag{3}$$

Using the flow budget constraints, we can also write this in terms of savings and factor prices as

$$\frac{(1+r_{t+1})s_t}{w_t - s_t} = \frac{1+r_{t+1}}{1+\theta}.$$

Hence the savings function is

$$s_t = \frac{w_t}{2+\theta}.$$

(Log utility makes the interest rate disappear as an argument of the savings function.)

Similarly, the firms profit-maximization conditions imply

$$\begin{aligned} w_t &= (1-\alpha)k_t^{\alpha}, \\ r_{t+1} &= \alpha k_{t+1}^{\alpha}. \end{aligned}$$

Substituting the condition for w_t into the savings function delivers an expression for s_t in terms of k_t

$$s_t = \frac{1-\alpha}{2+\theta}k_t^{\alpha}.$$

The equilibrium law of motion for k_t can then be expressed as

$$\begin{aligned} (1+n)k_{t+1} &= s_t \\ k_{t+1} &= \frac{1-\alpha}{(1+n)(2+\theta)}k_t^{\alpha}, \end{aligned}$$

The phase diagram relating k_t and k_{t+1} has the same shape as the Cobb-Douglas production function. The steady-state capital stock is the intersection of that curve with the 45 degree line. Because the Cobb-Douglas production function satisfies the Inada conditions, the dynamics point inward toward the steady state. To solve for the steady-state capital stock, set $k_{t+1} = k_t$ and re-arranges terms,

$$k_{ss} = \left(\frac{1-\alpha}{(1+n)(2+\theta)}\right)^{\frac{1}{1-\alpha}}.$$

b. To introduce a fully-funded social security system, we just need to modify the flow budget constraints. Suppose that young workers are taxed in an amount τ, which is then invested on their behalf, with benefits equal to $(1+r)\tau$ paid out when they are old. The flow budget constraints then become

$$\begin{aligned} c_{1t} + s_t &= w_t - \tau_t, \\ c_{2t+1} &= (1 + r_{t+1})(s_t + \tau_t). \end{aligned}$$

Notice that if we define total savings s_t^* as the sum of private and public savings $s_t + \tau_t$, the flow budget constraints are the same as before.

$$\begin{aligned} c_{1t} + s_t^* &= w_t, \\ c_{2t+1} &= (1 + r_{t+1})s_t^*, \end{aligned}$$

with total savings replacing private savings. Assuming that the social security tax is no larger than private savings in the original equilibrium, a fully-funded scheme would have no influence on outcomes. This follows because it doesn't alter anyone's objectives or constraints. Private savings would fall by the amount of the tax, but total savings would remain the same. Hence the steady-state capital stock would not be affected.

3. Consider an actual economy in which the (gross) growth rate of consumption follows the $AR(1)$ process

$$\lambda_t = 0.408 + 0.6\lambda_{t-1} + \varepsilon_t$$

where ε_t is a white noise disturbance term with $\sigma_e^2 = (0.02)^2$. Use this process to calibrate and solve the Mehra-Prescott version of the Lucas asset pricing model in which the growth rate of consumption is assumed to follow a two-state Markov process. That is, $c_{t+1} = c_t \lambda_{t+1}$ where

$$\lambda_t = \begin{cases} \lambda_1 = \lambda - \delta \\ \lambda_2 = \lambda + \delta \end{cases}$$

and the one-period transition probability matrix is symmetric with diagonal elements of π. Assume that agents have constant relative risk aversion utility with the risk aversion parameter = 2; moreover, agents' discount factor, $\beta = 0.5$. Calibrate the parameters of the Markov process using the estimated $AR(1)$ for consumption growth and compute the following:

a. Determine the conditional expected return on equity.
b. Determine the conditional interest rates.
c. Determine the unconditional equity premium.

Answer: The AR(1) process for consumption growth implies the following mean and variance:

$$E(\lambda_t) = \frac{0.408}{1 - 0.6} = 1.02$$

$$sd(\lambda_t) = \left(\frac{(0.02)^2}{1 - 0.6^2} \right)^{0.5} = 0.025$$

$$Corr(\lambda_t, \lambda_{t-1}) = 0.6$$

We know from the 2-state Markov process that:

$$E(\lambda_t) = \lambda$$

$$sd(\lambda_t) = \delta$$

$$Corr(\lambda_t, \lambda_{t-1}) = 2\pi - 1$$

So the calibration exercise yields: $\lambda_1 = 0.995, \lambda_2 = 1.045, \pi = 0.8$.

The asset pricing equations are (using the functional form for utility and imposing the equilibrium condition that $c_t = x_t$ where x_t denotes the endowment:

Equity prices:

$$q_t x_t^{-\gamma} = \beta E_t \left[x_{t+1}^{-\gamma} (q_{t+1} + x_{t+1}) \right] \tag{4}$$

Bond prices:

$$p_t = \frac{\beta E_t \left(x_{t+1}^{-\gamma} \right)}{x_t^{-\gamma}}$$

Turning first to equity prices, we use the fact that the equilibrium equity prices will be a function of both the level of the endowment and the current growth rate; moreover, this function will be homogeneous of degree one in x_t. Hence $q_t = x_t \theta_i$ where θ_i denotes that the price-dividend ratio will be a function of the growth rate only. Using this in eq.(4) and simplifying yields the two equations in the two unknowns, (θ_1, θ_2).

$$\theta_1 = \beta \left[\pi \lambda_1^{1-\gamma} (\theta_1 + 1) + (1 - \pi) \lambda_2^{1-\gamma} (\theta_2 + 1) \right]$$

$$\theta_2 = \beta \left[\pi \lambda_2^{1-\gamma} (\theta_2 + 1) + (1 - \pi) \lambda_1^{1-\gamma} (\theta_1 + 1) \right]$$

Or, using the numerical values

$$\theta_1 = 0.9 \left[0.8 \frac{1}{0.995} (\theta_1 + 1) + 0.2 \frac{1}{1.045} (\theta_2 + 1) \right]$$

$$\theta_2 = 0.9 \left[0.8 \frac{1}{1.045} (\theta_2 + 1) + 0.2 \frac{1}{0.995} (\theta_1 + 1) \right]$$

This is easily solved to yield (rounding the values):

$$\theta_1 = 7.82, \theta_2 = 7.35$$

(Does it make sense that the stock price dividend is greater in the low growth rate state?)

Now we answer the questions:

(a) The conditional expected return on equity is:

$$E_t\left(er_{t+1}\right)=\frac{E_t\left(q_{t+1}+x_{t+1}\right)}{q_t}=\frac{E_t\left[\lambda_{t+1}(\theta_{t+1}+1\right]}{\theta_t}$$

Or, in the two states:

$$E_1\left(er_{t+1}\right)=\frac{0.8\left(0.995\right)\left(\theta_1+1\right)+0.2\left(1.045\right)\left(\theta_2+1\right)}{\theta_1}$$

$$E_2\left(er_{t+1}\right)=\frac{0.2\left(0.995\right)\left(\theta_1+1\right)+0.8\left(1.045\right)\left(\theta_2+1\right)}{\theta_2}$$

Using the values for (θ_1,θ_2) determined earlier yields (we need the high - and somewhat artificial given the earlier rounding - precision)

$$E_1\left(er_{t+1}\right)=1.12096; E_2\left(er_{t+1}\right)=1.18854$$

(b) Turning to bond prices, these are determine by:

$$p_1=\beta\left[\pi\lambda_1^{-\gamma}+\left(1-\pi\right)\lambda_2^{-\gamma}\right]$$

$$p_2=\beta\left[\pi\lambda_2^{-\gamma}+\left(1-\pi\right)\lambda_1^{-\gamma}\right]$$

Using the numerical values yields:

$$p_1=0.892086, p_2=0.841139$$

Or, interest rates are:

$$r1=\frac{1}{p_1}-1=0.120968$$

$$r2=\frac{1}{p_2}-1=0.188864$$

(c) The unconditional equity premium is the average of the conditional equity premia:

$$E\left(ep\right)=\frac{1}{2}\left[\left(E_1\left(er_{t+1}\right)-\frac{1}{p_1}\right)+\left(E_2\left(er_{t+1}\right)-\frac{1}{p_2}\right)\right]=-0.000167$$

In this case, the equity premium is negative....why???

4.Consider a competitive economy with an infinite number of identical firms and households. Each period, firms hire labor at the wage rate, w_t, in order to maximize profit. The production function for a representative firm is:

$$y_t=\zeta_t h_t^{\alpha}$$

where y_t denotes output, h_t is labor, and ζ_t is an i.i.d. technology shock with $E\left(\zeta_t\right)=1$. It is assumed that $\alpha<1$.

1. Household income in each period is determined by labor income and the proceeds from equity purchased in the previous period. These proceeds consist of current profits and potential capital gains as determined by the current price of equity. (Initially, each household is endowed with one unit of equity denoted $\bar{z}$.) Each period, income is allocated between consumption and equity; these choices are made in order to maximize the discounted stream of expected utility given by:

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\gamma}}{1-\gamma} + A(1-h_t) \right] \right\}$$

It is assumed that $0 < \beta < 1$ and $0 < \gamma < 1$; the remaining parameter, $A > 0$.Given this environment, do the following:

a. Define the representative firm's and household's maximization problems. Derive and interpret the associated necessary conditions.

b. Define a recursive competitive equilibrium in this economy.

c. Rather than solve directly for the competitive equilibrium, one can instead solve an associated social planner problem. For this economy, what is the relevant social planner problem? Show that the necessary conditions associated with this problem do indeed correspond to those in (a).

d. Characterize the equilibrium behavior of labor, the wage rate, and the price of equity.

Answer: For the individual, the maximization problem is:

$$\begin{aligned} v(z_{t-1}, \zeta_t) &= \max_{(c_t, h_t, z_t)} \left\{ \frac{c_t^{1-\gamma}}{1-\gamma} + A(1-h_t) + \beta E\left[v\left(z_t, \zeta_{t+1}\right)\right] \right\} \quad (5) \\ \textit{subject to} &: w_t h_t + z_{t-1}(\pi_t + q_t) = c_t + q_t z_t \end{aligned}$$

where z_t denotes equity holdings in period t and $\pi_t = \zeta_t h_t^\alpha - w_t h_t$ denotes profits. Note the value function, $v(z_{t-1}, \zeta_t)$, is a function of the individual's beginning of period equity holdings and the current technology shock. Prices are also part of the state vector, but since these will be functions of the aggregate state (ζ_t), I suppress these in the state vector.

The associated necessary conditions are (using λ_t to denote the Lagrange multiplier for the budget constraint):

$$\begin{aligned} c_t &: c_t^{-\gamma} - \lambda_t = 0 \\ h_t &: -A + \lambda_t w_t = 0 \\ z_t &: \beta E\left[\frac{\partial V\left(z_t, \zeta_{t+1}\right)}{\partial z_t} \right] - \lambda_t q_t = 0 \end{aligned}$$

Or, combining terms and using the envelope theorem:

$$A c_t^\gamma = w_t$$

$$c_t^{-\gamma} q_t = \beta E\left[c_{t+1}^{-\gamma}\left(\pi_{t+1} + q_{t+1}\right)\right] \tag{6}$$

These have the standard interpretations.

The firms maximization problem is static and is simply:

$$\max_{h_t}\left(\zeta_t h_t^{\alpha} - w_t h_t\right)$$

which leads to the necessary condition:

$$\alpha \zeta_t h_t^{\alpha-1} = w_t \tag{7}$$

b. Definition of a recursive competitive equilibrium

A recursive competitive equilibrium can be defined by a set of functions: a value function that defines the household's problem (5), a wage function, $w(\zeta_t)$, an equity price function, $q(\zeta_t)$, a set of decision rules for households, $c(z_{t-1}, \zeta_t), h(z_{t-1}, \zeta_t)$, a decision rule for firms, $H(\zeta_t)$ where this function is the per-capita quantity of labor demanded, and an aggregate per capita decision rule for consumption $C(\zeta_t)$.

a. The household's problem (as noted above for the value function.)

b. The necessary condition for households: i.e. $q_t = q(\zeta_t)$ in eq. (6) when evaluated at market clearing quantities.

c. The necessary condition for firms: i.e. $w_t = w(\zeta_t)$ in eq.(7).

d. The consistency of individual and aggregate decisions:

$c(1, \zeta_t) = C(\zeta_t), h(1, \zeta_t) = H(\zeta_t)$. (that is, market clearing requires $z_t = 1 \forall t$.

e. The aggregate resource constraint: $C(\zeta_t) = \zeta_t [H(\zeta_t)]^{\alpha}$

c. Social planner problem

Since there is no capital, the social planner problem is simply to maximize utility each period given by:

$$\max_{c_t, h_t}\left[\frac{c_t^{1-\gamma}}{1-\gamma} + A(1-h_t)\right]$$

$$subject\ \ to\ c_t = \zeta_t h_t^{\alpha}$$

The necessary condition is:

$$A c_t^{\gamma} = \alpha \zeta_t h_t^{\alpha-1} \tag{8}$$

This is clearly the same as in the competitive economy when the household's and firm's necessary conditions are combined.

d. Characterization of equilibrium

Using the necessary condition, eq.(8) and the resource constraint implies:

$$h_t = \Gamma \zeta_t^{\frac{1-\gamma}{1-\alpha(1-\gamma)}}$$

where $\Gamma = \left(\frac{\alpha}{A}\right)^{\frac{1-\gamma}{1-\alpha(1-\gamma)}}$. Given the restrictions on (α, γ) this establishes that labor is an increasing function of the technology shock. The wage rate is given by eq.(7) so that, in equilibrium:

$$w_t = \alpha \Gamma^{(\alpha-1)} \zeta_t^{\frac{\gamma}{1-\alpha(1-\gamma)}}$$

so that wages are also increasing in the technology shock.

Equity prices are given by

$$q_t = \frac{\beta E\left[c_{t+1}^{-\gamma}\left(\pi_{t+1} + q_{t+1}\right)\right]}{c_t^{-\gamma}}$$

In the numerator, all variables are functions of next period's technology shock. Given the assumption of i.i.d. shocks, this implies the numerator is a constant. Since consumption is an increasing function of the technology shock, then equity prices will be as well.