Answers to the July 2007 Macro Prelim

1. What is dynamic efficiency? Can the steady-state equilibrium in the Ramsey model be dynamically inefficient? Why or why not?

ANSWER: An allocation is dynamically inefficient if the capital stock exceeds the Golden Rule level. With non-negative discounting, a Ramsey steady state cannot be dynamically inefficient. If it were, the representative agent could increase utility by reducing saving and consuming some of the capital. But that contradicts the assumption that Ramsey consumers maximize utility.

2. Assume that consumption growth is random and has the following representation:

$$\frac{c_{t+1}}{c_t} = (1 + g_c)\,\varepsilon_{t+1}$$

where it is assumed that $\ln \varepsilon_t$ is distributed normally with $E(\ln \varepsilon_t) = -\sigma_c^2/2$ and $Var(\ln \varepsilon_t) = \sigma_c^2$. In an economy populated by identical agents with constant relative risk aversion utility, use the Euler equation associated with real bonds to derive an expression for the equilibrium real interest rate. Use the approximation that $\ln(1 + x) \approx x$ to simplify the expression. Discuss the implications of this expression and, in particular, discuss the impact that uncertainty has on the real interest rate. Also discuss the implications that your result has for the risk-free rate puzzle. (Recall that if $\ln z \sim N(\mu, \sigma^2)$, then $E(z) = \exp[\mu + \sigma^2/2]$.)

ANSWER: The Euler equation associated with a one period bond is:

$$1 = (1 + r_t)\,\beta E\left[\frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}}\right] = (1 + r_t)\,\beta\,(1 + g_c)^{-\gamma}\,E\left[\varepsilon_{t+1}^{-\gamma}\right]$$

where the last expression is derived by using the assumed process for consumption growth. Under the assumption that $\ln \varepsilon_t$ is normally distributed, this implies that -$\gamma \ln \varepsilon_t$ is distributed normally with mean of $E(-\gamma \ln \varepsilon_t) = \gamma\sigma_c^2/2$ and $Var(-\gamma \ln \varepsilon_t) = \gamma^2\sigma_c^2$. Then $E\left[\varepsilon_{t+1}^{-\gamma}\right] = \exp\left[\gamma\sigma_c^2/2 + \gamma^2\sigma_c^2/2\right] = \exp\left[\gamma\sigma_c^2/2\,(1+\gamma)\right]$. Therefore we have:

$$1 = (1 + r_t)\,\beta\,(1 + g_c)^{-\gamma} \exp\left[\gamma\sigma_c^2/2\,(1 + \gamma)\right]$$

Taking logs and defining $\beta = (1 + \rho)^{-1}$ and rearranging we have

$$r_t = \rho + \gamma g_c - \gamma\,(1 + \gamma)\,\sigma_c^2/2$$

The first two terms are consistent with the Keynes-Ramsey condition: if agents discount rate increases, they must be compensated for foregoing current consumption so interest rate increases; if consumption growth increases, then current consumption is relatively more scarce so interest rates increase. If uncertainty over future consumption increases, this reduces the certainty equivalent of future consumption so current consumption is relatively more abundant. Consequently, interest rates are lower.

With regard to the risk free rate puzzle, the above expression implies a larger risk free rate than observed. For instance, Mehra and Prescott estimated $g_c = 0.018$ and $\sigma_c^2 = (0.036)^2$. If $\rho = 0.01$, then $\gamma = 3$ implies a risk-free rate of around 6.5%...much higher than the estimate of around 1%. Also, if γ is increased in order to help with the equity premium puzzle, the problem is exacerbated.

3. Consider a simple stochastic growth model with 100% depreciation in which infinitely long-lived agents have instantaneous utility given by $\ln c_t$ and output is given by a Cobb-Douglas production function with labor input constant and equal to one. In this setting it is easy to show that agents consume a constant fraction of their income regardless of the serial correlation properties of the shock. That is, consumption is the same whether the shock is perceived to be long-lasting (i.e. exhibits positive serial correlation) or transitory (the shock is i.i.d.) - this is similar to a simple Keynesian consumption function. Do you see this as a contradiction to the implications of the permanent income hypothesis?

ANSWER: There is no contradiction. The fact that savings is constant in the economy regardless of the persistence of the shock stems from three auxilliary assumptions. First, the shock is due to the rate of return on capital, not to income directly, so this implies conflicting income and substitution effects. Second, since capital completely depreciates each period, a persistent shock is irrelevant. These two combined with log utility imply the income and substitution effects with persistent shocks negate each other so that savings is constant.

4. Consider a version of the Solow growth model with a constant-elasticity-of-substitution production function,

$$Y = A\left\{a(bK)^{\psi} + (1-a)[(1-b)L]^{\psi}\right\}^{1/\psi},$$

where $0 < a < 1$, $0 < b < 1$, and $\psi < 1$. The elasticity of substitution between capital and labor is $1/(1-\psi)$. There is no population growth or technical progress, and capital depreciates at rate δ.

(a) Verify that this production function has constant returns to scale. (easy)

(b) Derive the intensive form $y = f(k)$, where $y \equiv Y/L$, $k \equiv K/L$. (easy)

$$y = f(k) = A\left\{a(bk)^{\psi} + (1-a)(1-b)^{\psi}\right\}^{1/\psi}$$

(c) Verify that the marginal product of capital can be expressed as

$$f'(k) = Aab^{\psi}\left\{ab^{\psi} + (1-a)(1-b)^{\psi}k^{-\psi}\right\}^{(1-\psi)/\psi}.$$

ANSWER:

$$
\begin{aligned}
f'(k) &= A\frac{1}{\psi}\left\{a(bk)^{\psi}+(1-a)(1-b)^{\psi}\right\}^{\frac{1}{\psi}-1}\times(\psi ab^{\psi}k^{\psi-1}),\\
&= Aab^{\psi}\left\{a(bk)^{\psi}+(1-a)(1-b)^{\psi}\right\}^{\frac{1-\psi}{\psi}}k^{\psi-1},\\
&= Aab^{\psi}\left\{a(bk)^{\psi}+(1-a)(1-b)^{\psi}\right\}^{\frac{1-\psi}{\psi}}(k^{\frac{\psi(\psi-1)}{1-\psi}})^{\frac{1-\psi}{\psi}},\\
&= Aab^{\psi}\left\{a(bk)^{\psi}+(1-a)(1-b)^{\psi}\right\}^{\frac{1-\psi}{\psi}}(k^{-\psi})^{\frac{1-\psi}{\psi}}\\
&= Aab^{\psi}\left\{ab^{\psi}+(1-a)(1-b)^{\psi}k^{-\psi}\right\}^{(1-\psi)/\psi}.
\end{aligned}
$$

(d) Can this production function sustain endogenous growth? Hint: Consider two cases, one in which the elasticity of substitution is greater than 1 and another in which it is less than 1.

This depends on how $f'(k)$ behaves as k grows large, and the limit depends on the elasticity of substitution. If the elasticity is less than 1, ψ is less than 0 and $\lim_{k\to\infty} f'(k) = 0$. In this case, the model can't sustain endogenous growth. On the other hand, if the elasticity of substitution is greater than 1, $0 < \psi < 1$ and $\lim_{k\to\infty} f'(k) = Aa^{1/\psi}b > 0$. In this case, the production function is asymptotically Ak. Since the marginal product asymptotes to a positive value, growth in the capital stock continues to increase output per worker in the long run. Hence the model can potentially sustain endogenous growth if the elasticity of substitution is greater than 1.

5. Consider a consumer with preferences

$$U = \int_0^{\infty} e^{-\rho t}u(c_t)dt,$$

where ρ is the subjective discount rate, c is consumption, and $u(c) = \ln c$. She receives an exogenous flow of income y and can borrow or lend freely at a constant interest rate r, subject to a no-Ponzi-game condition that rules out infinite debt. Her flow budget constraint is

$$\dot{a} = r(a + y - c),$$

where a represents financial wealth and a_0 is the initial value.

(a) Derive the first-order conditions for optimal consumption.
(b) At what rate does consumption grow? Interpret its sign.
(c) Derive the optimal decision rule for consumption. Provide an interpretation for the case $\rho = r$.

This is essentially a continuous time version of Hall's consumption model. It can be solved by adapting his solution method.

The current-value Hamiltonian is

$$H = \log(c) + \lambda(ra + y - c).$$

The FOC are

$$\begin{aligned} \frac{\partial H}{\partial c} &= \frac{1}{c} - \lambda = 0, \\ \frac{\partial H}{\partial a} &= r\lambda = \rho\lambda - \dot{\lambda}, \\ \frac{\partial H}{\partial \lambda} &= ra + y - c = \dot{a}. \end{aligned}$$

After combining the first two conditions, we get

$$\frac{\dot{c}}{c} = r - \rho.$$

Hence consumption grows at a constant rate $r - \rho$. The slope of the consumption profile reflects the countervailing forces of interest accumulation and discounting. If the interest rate is greater, it pays to postpone consumption in order to benefit from interest accumulation. Hence consumption is backloaded and growth is positive. When the discount rate is higher, impatience outweighs interest accumulation, so it pays to frontload consumption and have a declining consumption path.

To find the decision rule, integrate the consumption growth condition with respect to time to get

$$c_t = c_0 e^{(r-\rho)t}.$$

All that remains is to pin down initial consumption. To do that, integrate the flow budget constraint with respect to time and use the NPG condition to obtain a present-value budget constraint,

$$\int_0^\infty e^{-rt} c_t dt = a_0 + h_0,$$

where

$$h_0 \equiv \int_0^\infty e^{-rt} y_t dt$$

is the present value of labor income. Substituting $c_t = c_0 e^{(r-\rho)t}$ into the present-value budget constraint delivers

$$\begin{aligned} a_0 + h_0 &= \int_0^\infty e^{-rt} c_0 e^{(r-\rho)t} dt, \\ &= c_0 \int_0^\infty e^{-\rho t} dt. \end{aligned}$$

The integral on the right-hand side is easy.

$$\begin{aligned}\int_0^\infty e^{-\rho t} dt &= (-\rho^{-1} e^{-\rho t} + \text{ constant})|_{t=0}^{\infty}, \\ &= \rho^{-1}.\end{aligned}$$

Hence, initial consumption is

$$c_0 = \rho(a_0 + h_0),$$

and the consumption decision rule is

$$c_t = \rho(a_0 + h_0)e^{(r-\rho)t}.$$

When $\rho = r$, this simplifies to

$$c_t = r(a_0 + h_0)$$

for all t. I.e., the agent consumes the annuity value of intial total wealth, measured as the sum of initial financial assets plus human capital.

6.Consider the following stochastic growth model. Agents' preferences are given by

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t \ln c_t \right]$$

while output is a function of beginning of period capital and a technology shock. That is,

$$y_t = z_t k_t^\alpha$$

It is assumed that z_t is $i.i.d.$ and the depreciation rate of capital is 100%.

a. Solve for the optimal consumption function in this economy (use the conjecture and verify method).

ANSWER: this is straightforward: $c_t = (1 - \alpha\beta)\, z_t k_t^\alpha; k_{t+1} = \alpha\beta z_t k_t^\alpha$.

b. Suppose equity is introduced into this economy and assume that the dividend associated with equity is equal to consumption. Determine the equilibrium price of equity and discuss the relationship between this price and the capital stock. (Work directly from the necessary condition associated with equity.)

ANSWER: The Euler equation for equity is:

$$\frac{q_t}{c_t} = \beta E \left[\frac{1}{c_{t+1}} (q_{t+1} + c_{t+1}) \right] = \beta E \left[\frac{q_{t+1}}{c_{t+1}} + 1 \right]$$

The equity price function can be determined in two ways (at least): One is to solve the above expression forward. The other is to make the conjecture that the price of equity will be a constant fraction of output. Either way yields the answer:

$$q_t = \frac{\beta}{1-\beta} c_t = \frac{\beta}{1-\beta} (1 - \alpha\beta)\, y_t$$

Note that this price is greater than the price of capital. (Also note that if the production function was linear, i.e. $\alpha = 1$, then equity would sell at the same price as capital.) The reason for this is due to the fact that capital's return is the MPK, not consumption directly. If the MPK is a constant, then the assets are identical.

c. Suppose one and two-period (real) bonds are introduced into this economy. What is the equilibrium behavior of the term premium?

ANSWER: The prices of one- and two-period bonds are determined by the following Euler equations evaluated at market clearing quantities:

$$p1_t \frac{1}{c_t} = \beta E \left[\frac{1}{c_{t+1}} \right] \tag{1}$$

$$p2_t \frac{1}{c_t} = \beta E \left[\frac{1}{c_{t+1}} p1_{t+1} \right] \tag{2}$$

Many students next made a **fundamental** and **critical** error: they stated that, since the technology shock is *i.i.d.*, then this implies the expectations term in eq.(1) is a constant. This would be correct IF consumption was a function of the technology shock alone but it is not. Using the equilibrium policy function for consumption in this expression yields:

$$p1_t = y_t^{1-\alpha} (\alpha\beta)^{-\alpha} \beta E \left[\frac{1}{z_{t+1}} \right]$$

here the expectational term is a constant since it is a function of z_t only. Note that the one period bond price will be procyclical or, alternatively, the one period interest rate will be countercyclical. Define the term premium as (make sure you understand this..we did this many times in class)

$$TP_t = E_t \left(\frac{p1_{t+1}}{p2_t} \right) - \left(\frac{1}{p1_t} \right)$$

This can be rearranged and expressed as:

$$TP_t = -A_t Cov_t \left(\frac{1}{c_{t+1}}, p1_{t+1} \right)$$

where $A_t = \left(\frac{1}{p2_t} \right) \left[E_t \left(c_{t+1}^{-1} \right) \right]^{-1} > 0$.Since both consumption and $p1_t$ are procyclical, then the covariance term is negative and the term premium is positive. This is precisely what the $CCAPM$ would predict: since selling a two-period bond after one period produces a return that pays out well in states in which consumption is already high and, conversely, pays out poorly when consumption is low, this asset exacerbates the volatility in agents' marginal utility of consumption and, therefore, will carry a positive risk premium.