Answers to Questions 7 & 8 on Macro Prelim, July 2006

Q7:

2. For a change of pace, I solve this without making the substitution for i_t. The dynamic programming representation is:

$$\begin{aligned} V(k_t, z_t) &= \max_{(c_t, i_t)} \{\ln c_t + \beta E_t [V(k_{t+1}, z_{t+1})]\} \\ &+\lambda_t [z_t k_t^{\alpha} - c_t - i_t] \end{aligned}$$

with the law of motion of capital given by

$$k_{t+1} = k_t^{1-\delta} i_t^{\delta}$$

. The necessary conditions are:

$$\frac{1}{c_t} - \lambda_t = 0 \tag{6}$$

$$\beta E_t \left[\frac{\partial V(\cdot_{t+1})}{\partial k_{t+1}} \left(\delta k_t^{1-\delta} i_t^{\delta-1} \right) \right] - \lambda_t = 0 \tag{7}$$

Using the envelope theorem:

$$\frac{\partial V(\cdot_t)}{\partial k_t} = \beta E_t \left[\frac{\partial V(\cdot_{t+1})}{\partial k_{t+1}} (1-\delta) k_t^{-\delta} i_t^{\delta} \right] + \lambda_t \alpha z_t k_t^{\alpha-1} \tag{8}$$

Note that eq.(7) implies

$$\beta E_t \left[\frac{\partial V(\cdot_{t+1})}{\partial k_{t+1}} \right] = \frac{\lambda_t}{\left(\delta k_t^{1-\delta} i_t^{\delta-1} \right)} \tag{9}$$

Using this in eq.(8) yields:

$$\frac{\partial V(\cdot_t)}{\partial k_t} = \lambda_t \left[\frac{(1-\delta)}{\delta} \frac{i_t}{k_t} + \alpha \frac{y_t}{k_t} \right] \tag{10}$$

Updating and using in eq.(7) results in (also using the fact that $\left(\delta k_t^{1-\delta} i_t^{\delta-1}\right) = \delta k_{t+1} i_t^{-1}$)

$$\beta E_t \left\{ \lambda_{t+1} \left[\frac{(1-\delta)}{\delta} \frac{i_{t+1}}{k_{t+1}} + \alpha \frac{y_{t+1}}{k_{t+1}} \right] \right\} \left(\delta \frac{k_{t+1}}{i_t} \right) = \lambda_t \tag{11}$$

Note that this intertemporal efficiency condition represents the relevant tradeoffs: The LHS is the additional output due to investment: the term in parentheses represents the additional capital produced through investment while the two terms in square brackets represent the depreciation term while the second is the MPK.

This can be written as:

$$\beta E_t \left[(1-\delta) \frac{i_{t+1}}{c_{t+1}} + \alpha\delta \frac{y_{t+1}}{c_{t+1}} \right] = \frac{i_t}{c_t} \tag{12}$$

Using the resource constraint yields:

$$\zeta_0 + \zeta_1 E_t \left(\frac{y_{t+1}}{c_{t+1}} \right) = \frac{y_t}{c_t} \tag{13}$$

Where

$$\zeta_0 = 1 - \beta(1-\delta)$$
$$\zeta_1 = \beta\left[1 - \delta(1-\alpha)\right]$$

Note that $\zeta_1 < 1$. Solving this expression through recursive substitution yields:

$$\frac{y_t}{c_t} = \frac{\zeta_0}{1-\zeta_1} = \frac{1-\beta(1-\delta)}{1-\beta\left[1-\delta(1-\alpha)\right]} \tag{14}$$

As a check, note that with 100% depreciation ($\delta = 1$), this simplifies to:

$$c_t = (1-\alpha\beta)\, y_t$$

Q8:

4. Letting the money transfer at time t be denoted as $\mu\bar{M}_{t-1}$, the associated dynamic programming problem is:

$$V\left(M_{t-1}, B_{t-1}, k_t\right) = \max\left\{\left[U\left(c_t, \frac{M_{t-1}}{P_t}\right) + \beta V\left(M_t, B_t, k_{t+1}\right)\right]\right.$$
$$\left. + \lambda_t\left(w_t - c_t - k_{t+1} - \frac{M_t}{P_t}\right)\right\}$$

with w_t defined as real wealth, the law of motion is given by:

$$w_{t+1} = f\left(k_{t+1}\right) + k_{t+1}\left(1-\delta\right) + \frac{M_t}{P_{t+1}} + \frac{B_t}{P_{t+1}}\left(1+n_t\right) + \frac{\mu\bar{M}_t}{P_{t+1}}$$

The associated first-order conditions for optimal M_t, k_{t+1}, B_t are respectively:

$$U_{1,t} = \beta\left(U_{1,t+1} + U_{2,t+1}\right)\frac{P_t}{P_{t+1}} \tag{19}$$

$$U_{1,t} = \beta\left[U_{1,t+1}\left(f'\left(k_{t+1}\right) + 1 - \delta\right)\right] \tag{20}$$

$$U_{1,t} = \beta U_{1,t+1}\left(1+n_t\right)\frac{P_t}{P_{t+1}} \tag{21}$$

Note that, unlike the model studied in class, money chosen in period t affects the marginal utility of real balances in the following period (see eq. (19).

A steady-state equilibrium is defined by the time-invariant 4-tuple

(c^*, k^*, m^*, n^*) that solve the necessary conditions and the resource constraint. The steady-state level of real balances is best defined in terms of the beginning of period level of the money stock since this enters into the utility function: In equilibrium $M_{t-1} = \bar{M}_{t-1}$ and define $m^* = \frac{\bar{M}_{t-1}}{P_t}$. This implies that

$$\frac{P_t}{P_{t+1}} = \frac{1}{(1+\mu)}$$

In steady-state, the necessary conditions become:

$$U_{1,t}^* = \beta\left(U_{1,t+1}^* + U_{2,t+1}^*\right)\frac{1}{(1+\mu)} \tag{22}$$

$$1 = \beta\left(f'\left(k^*\right) + 1 - \delta\right) \tag{23}$$

$$1 = \beta\left(1+n^*\right)\frac{1}{(1+\mu)} \tag{24}$$

From eq. (23) we have immediately that money growth has no effect on k^*. Given the resource constraint, this implies that c^* is also independent of the monetary growth rate. The effect of money growth on real balances can be determined by taking the total differential of eq. (22). Making the assumption that preferences are separable in consumption and real balances (and that the utility function for real balances is strictly concave) directly proves that $\frac{dm^*}{d\mu} < 0$. The reason is that since money growth determines the inflation rate, increases in the money growth rate implies that real balances are a poor asset - so real balances fall in steady-state. From eq. (24), we see immediately that nominal interest rates are increasing in the money growth - this is due to the Fisher effect.