NPV and Demand Growth

David Ong

January 26, 2008

1 Net Present Value

Let PV=Present Value, C=asset price, and $NPV = PV - C$

end of year	1	2	3	....
PV	$\frac{\pi_1}{1+r}+$	$\frac{\pi_2}{1+r^2}+$	$\frac{\pi_3}{1+r^3}$	

Suppose your rich (but weird) uncle always likes to surprise you with \$100 for your birthday. However, you needed cash now to buy a car. Your somewhat less weird aunt understands and is willing buy off that stream of income for the next 10 years with its present value. How much can you sell it to her for when the interest rate is r=10% and your birthday is one year away?

$$PV = \frac{100}{1.1} + \frac{100}{(1.1)^2} + \frac{100}{(1.1)^3} \cdots \frac{100}{(1.1)^{10}}$$

You can just calculate this or use the following equation (optional).

$$PV = 100\left(\frac{1}{1.1} + \frac{1}{(1.1)^2} \cdots \frac{1}{(1.1)^{10}}\right) = 100\left(\sum_{t=1}^{\infty}\left(\frac{1}{(1+r)}\right)^k - \sum_{k=10}^{\infty}\left(\frac{1}{(1+r)}\right)^k\right)$$

$$= 100\left(\sum_{k=1}^{\infty}\left(\frac{1}{(1+r)}\right)^k - \frac{1}{(1+r)^{10}}\sum_{k=1}^{\infty}\left(\frac{1}{(1+r)}\right)^k\right)$$

Using the formula for the sum of a geometric series,

$$\sum_{k=1}^{\infty} s^k = \frac{1}{1-s}$$

and letting $s = \frac{1}{(1+r)}$ we have

$$\sum_{k=1}^{\infty}\left(\frac{1}{(1+r)}\right)^k = \frac{1}{1-\frac{1}{1+r}} = \frac{1+r}{r}$$

$$PV = 100\left(\left(1-\frac{1}{(1+r)^{10}}\right)\frac{1+r}{r}\right)$$

This is the difference between the present value of \$100 for an infinite number of years and \$100 for an infinite number of years, starting in 10 years. We use the infinite series because we know the sum and therefore, can use it to give a closed form solution.

$$PV = 100\left(\left(1-\frac{1}{(1.1)^{10}}\right)\frac{1.1}{.1}\right)$$
$$PV = 675.9$$

Try to find some intuitive check for this calculation.

2 Demand Growth and Discounting

2.1 Linear Costs

The following algebraic treatment should clear-up some misunderstandings of what things meant that I observed in homeworks.

Let demand: $Q = \alpha - \beta P$. If demand grew at rate g, then the new demand in t **periods of growth (not the same the number of periods of discounting)** will be $Q_t = (1+g)^t(\alpha - \beta P_t)$ .Draw a graph to see what this looks like. Notice the rotation along the vertical intercept. Solving for the inverse demand curve:

$$P_t = \left(\frac{\alpha}{\beta} - \frac{Q_t}{(1+g)^t\beta}\right)$$

Let $a = \frac{\alpha}{\beta}$, $\mathbf{b}_t$=$\frac{1}{(1+g)^t\beta}$. Then,

$$P_t = (a - bQ_t)$$

Using the solution quantity for a monopoly above, where the profit maximizing quantity for this inverse demand when $MC = c$ was:

$$Q_t = \frac{a-c}{2b} = \frac{\frac{\alpha}{\beta}-c}{\left(\frac{2}{(1+g)^t\beta}\right)} = \frac{1}{2}(g+1)^t(\alpha - c\beta)$$

$$Q_t = \frac{1}{2}(\alpha - c\beta)(1+g)^t$$

We can get the quantity demanded Q^* from the original demand by setting the growth rate $g = 0$.

$$Q^* = \frac{1}{2}(\alpha - c\beta)$$

So, what we see here is something that we could not have assumed before: the quantity demanded for the increased demand is just the original quantity demanded multiplied by the compounded growth.

$$Q_t = Q^* \left(1+g\right)^t$$

Recall from the general monopoly problem solved in 1.3.1 of "Monopoly Profit Maximization and Sunk Costs" that profit was a function of Q.

$$\pi = b \cdot Q^2 - FC$$

We can find the profit from the grown demand by substituting in $Q_t = Q^* \left(1+g\right)^t$,

$$\pi_t = \mathbf{b}_t \cdot Q_t^2 - FC = b_t \left(Q^*\right)^2 \left(1+g\right)^{2t} - FC$$

$$\pi_t = \mathbf{b}_t \cdot Q_t^2 - FC = \frac{1}{\left(1+g\right)^t \beta} \left(Q^*\right)^2 \left(1+g\right)^{2t} - FC$$

$$\pi_t = \frac{1}{\beta} \left(Q^*\right)^2 \left(1+g\right)^t - FC$$

Notice that the part that was affected by growth $\left(1+g\right)^t$ is actually separable from the quantity Q^*, meaning that we don't have to recalculate quantity.

To see how the profit with grown demand compares to the original profit, let $g = 0$ and let π^* stand for the original profit without fixed costs.

$$\pi^* = \frac{1}{\beta} Q^{*2}$$

Then the new profit is just the original profit without the fixed costs times the growth $\left(\left(1+g\right)^t\right)^2$ minus fixed costs :

$$\pi_t = \pi^* \left(1+g\right)^t - FC$$

Note that the only thing that changes with each period is the $\left(1+g\right)^t$ term. Lets do a simple example, where $\alpha = 10, \beta = 2, c = 2$, $FC = 3$ and $g = \frac{1}{4}$. Then, the optimal quantity without demand growth is:

$$Q^* = \frac{1}{2}\left(\alpha - c\beta\right) = \frac{1}{2}\left(10 - 2 \cdot 2\right) = 3$$

The optimal quantity with demand growth for period t is:

$$Q_t = Q^* \left(1 + \frac{1}{4}\right)^t = 3\left(\frac{5}{4}\right)^t$$

The profit for this quantity in the t period is:

$$\pi_t = \frac{1}{\beta} Q^* \left(1+g\right)^t - FC$$

$$\pi_t = \frac{1}{2}\left(3\right)^2 \left(\frac{5}{4}\right)^t - 3$$

$$\pi_t = \frac{9}{2}\left(\frac{5}{4}\right)^t - 3$$

The PV of such each period profit when the interest rate is 10% will be:

	end of period 1	**end of period 2**	**end of period 3**
Demand	$Q_0 = 10 - 2P$	$Q_1 = 10 - 2P$	$Q_2 = \left(\frac{5}{4}\right) Q_1$
Profit in that period	$\pi_0 = \frac{9}{2} - 3$	$\pi_1 = \frac{9}{2}\left(\frac{5}{4}\right)^1 - 3$	$\pi_2 = \frac{9}{2}\left(\frac{5}{4}\right)^2 - 3$
PV of "per period" profit	$\pi_0 = \frac{\frac{9}{2} - 3}{(1.1)}$	$\pi_1 = \frac{\frac{9}{2}\left(\frac{5}{4}\right) - 3}{(1.1)}$	$\pi_2 = \frac{\frac{9}{2}\left(\frac{5}{4}\right)^2 - 3}{(1.1)^2}$

The **NPV of the "stream" of profits** from the investment for all t **periods of growth-not t periods of discounting**–when the cost of the investment is C:

$$NPV = -C + \frac{\pi_0}{1+r} + \frac{\pi_1}{\left(1+r\right)^2} + \frac{\pi_2}{\left(1+r\right)^3} + ... \frac{\pi_{t-1}}{\left(1+r\right)^t}$$

Notice demand growth rate of g (above) is not the same as a profit growth rate of g (below):

$$NPV \neq -C + \frac{\left(1+g\right)\pi_0}{1+r} + \frac{\left(1+g\right)^2 \pi_1}{\left(1+r\right)^2} + \frac{\left(1+g\right)^3 \pi_2}{\left(1+r\right)^3} + ... \frac{\left(1+g\right)^t \cdot \pi_{t-1}}{\left(1+r\right)^t}$$

If the $C = 10$, then the NPV of this example would be:

$$NPV = -10 + \frac{\frac{9}{2} - 3}{(1.1)} + \frac{\frac{9}{2}\left(\frac{5}{4}\right) - 3}{\left(1.1\right)^2} + \frac{\frac{9}{2}\left(\frac{5}{4}\right)^2 - 3}{\left(1.1\right)^3} - 3.\,438\,2$$

Therefore, the investment is **NOT worth it**.

2.2 Quadratic Costs

Again, **if costs are quadratic**,of the form: $TC\left(Q\right) = cQ + dQ^2 + FC$,Then as shown in the "Monopoly Profit Maximization and Sunk Costs" section 1.3.2:

$$Q^* = \frac{a-c}{2\left(b+d\right)}$$

and

$$\pi = (b + d) \cdot Q^{*2} - FC$$

Again, the inverse demand hasn't changed and can be written as:

$$P_t = \left(\frac{\alpha}{\beta} - \frac{Q_t}{(1+g)^t \beta} \right)$$

Let $a = \frac{\alpha}{\beta}$, $b_t = \frac{1}{(1+g)^t \beta}$, where **t=number of periods of growth**,

$$Q_t = \frac{\frac{\alpha}{\beta} - c}{2\left(\frac{1}{(1+g)^t \beta} + d\right)}$$

$$\pi_t = (b_t + d) \cdot Q_t^2 - FC$$

Unfortunately, when the cost is quadratic, we **can't** pull out the $(1+g)^t$ from term Q_t so that we have

$$Q_t = Q^* (1+g)^t$$

If you find the above calculation tricky, you may just want to do the period by period calculation.