Hotelling Model of Product Differentiation

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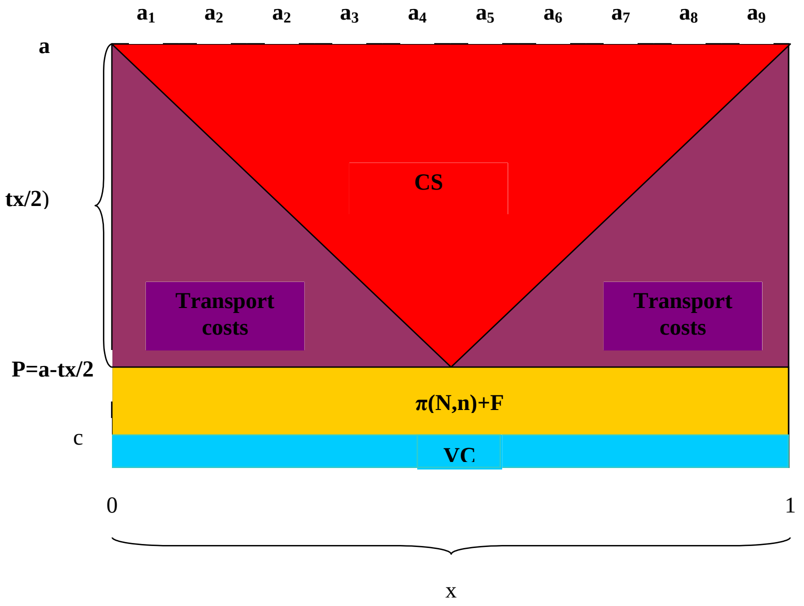
1 Monopoly (Heterogenous Products)

In Hotelling (Linear City or spatial) model, everyone is "a little different" so that each person has a different response to price changes. Hence, they don't just all go to the cheaper firm as in Bertrand Competition with homogenous goods.

Notation: FC =fixed cost of each of n outlets, N =number of consumers, a=reservation value of the all consumers, c=marginal cost. Here CS=consumer surplus. The tan colored region is revenue minus variable costs. Note that fixed costs are not drawn.



Here every consumer has the same valuation before travel costs are taken into account. t is the cost per unit of travel and x is the distance travelled. When travel cost tx is taken into account consumers become differentiated in their willingness to buy. Now the maximal price that any consumer i is willing to pay is $p_i = a - tx_i$, which depends upon how far i has to travel. As is generally the case, a monopolist must charge all consumers the same price. We assume that the monopolist sells to everyone. If the whole market is $\bar{x}$ long, then the monopolist charges so that y is the one whose willingness to pay is equal to the sum of his travel costs and the price that the monopolist is charging: $a = p + tx$.

1.1 Profit Maximizing Variety

As always, we want to maximize profit.

$$\pi(Q) = (P - c) \cdot Q - FC$$

But, now we need to adapt the profit function to this problem.

For simplicity, we are going to assume that all N consumers are served, which means that the highest price which can be charged is equal to the value of the consumer minus the highest travel cost$=t\bar{x}$. Minimizing travel costs thus maximizes the price the firm can charge. Travel costs are minimized if varieties are spread evenly. With n shops evenly spread apart, the farthest that any customer needs to travel is $\frac{\bar{x}}{2n}$,

$$p(n) = a - \frac{t\bar{x}}{2n}$$

where $\bar{x}$ is the greatest distance, which I will normalize to 1. Note that when distance is normalized, x becomes the fraction of consumers that are served. Combining price and quantity, we can fill in the usual profit max equation. We multiply the average profit the number of N to get profit function.

$$\pi(n) = \left(a - \frac{t\bar{x}}{2n} - c\right) N - nFC$$

Now, all we have to do is decide how many stores to have. Since we sell to everyone and the number of varieties n affects the price, in choosing n we are in effect choosing a price that maximizing profits, with the difference that:

1. we can only make discrete choices here.
2. we have to take into account fixed costs also in this choice.

From standard economics, profit is maximized if we produce up to last unit that marginal profit is greater than zero. If we were at Q then the profit of the next unit would be $\pi(Q+1)$. We would build that $Q+1$ unit if[1]:

$$\pi(Q+1) - \pi(Q) \geq 0$$

In the Hotelling model, profit maximization amounts to a condition on the number of outlets.

Thus, we build up to the ast $n+1$ outlet that still gives an increase in profits:

$$\pi(n+1) \geq \pi(n)$$

This amounts to:

$$\pi(n+1) = \left(a - \frac{t\bar{x}}{2(n+1)} - c\right) N - (n+1)FC \geq \left(a - \frac{t\bar{x}}{2n} - c\right) N - nFC = \pi(n)$$

$$\frac{t\bar{x}}{2(n+1)} N + FC \leq \frac{t\bar{x}}{2n} N$$

$$\frac{FC}{t\bar{x}N} \leq \frac{1}{2n(n+1)}$$

$$n(n+1) \leq \frac{t\bar{x}N}{2FC}$$

The profit maximizing number of varieties is the last $n+1$ such that $(n+1)\cdot n \leq \frac{Nt}{2FC}$.

1.2 Socially Optimal Variety

If you look at the change in surplus when we increased the number of outlets from 1 to 2, you will see that though some consumer surplus CS became the producer surplus PS, the change in total surplus is only determined by the net change in transportation costs (purple areas lined with green). That's because social surplus is the sum of CS and PS. Hence, social surplus is unchanged when some CS becomes PS or vis versa. The only thing that changes social surplus is the net change in transportation costs and fixed costs. To maximize social surplus, we want to minimize social cost,i.e., travel cost and fixed costs.

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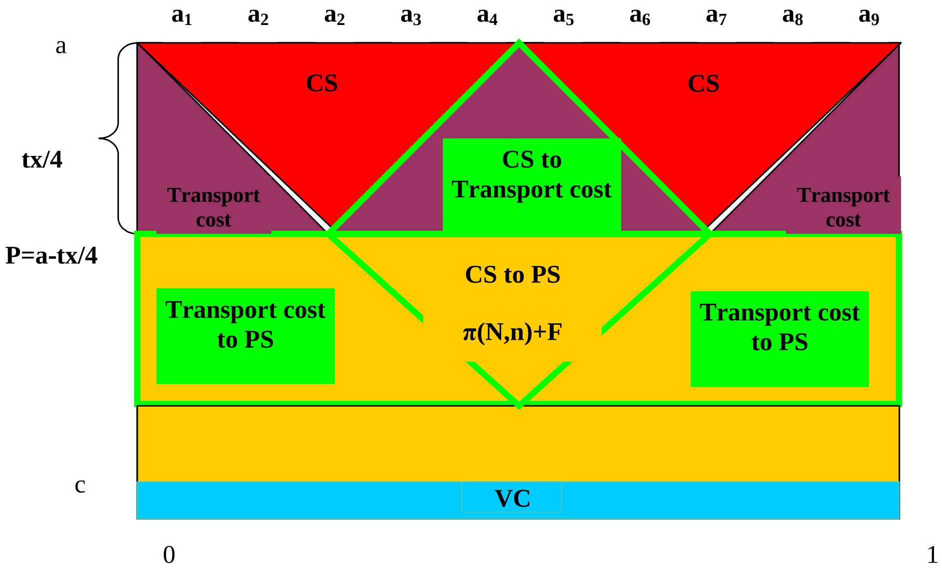
[1] We could have also used Q instead of $Q+1$ but then we couldn't use the "marginal" concept and we would have to delve into negative numbers of the first unit.



If everyone is served, the social optimal number of outlets is the cost minimizing number of outlets, where the marginal or next unit is still resulting in a cost decrease. So, we look for the last $n+1$ outlet that decreases costs.

$$C\left(n+1\right)-C\left(n\right)\leq 0$$

$$C\left(n+1\right)=\frac{t\bar{x}N}{4\left(n+1\right)}+\left(n+1\right)FC\leq\frac{t\bar{x}N}{4n}+nFC=C\left(n\right)$$

1.3 Costs

Lets develop the calculation of costs through an example.

1.3.1 Social Costs When There Is One Outlet

If there were a single outlet, the best place to put the outlet is in the middle, therefore, maximal consumer travel costs is $\frac{t\bar{x}}{2}$. The average travel cost between

1. the lowest travel cost consumer at distance 0, who has zero travel costs, and
2. the highest travel cost consumer at distance $\bar{x}$, who has travel costs $\frac{t\bar{x}}{2}$,

is then $\frac{0+\frac{t\bar{x}}{2}}{2}=\frac{t\bar{x}}{4}$, the middle person's travel cost.

1.3.2 Social Costs When There Are Two Outlet

If there are 2 outlets, the best place to put them is $\frac{1}{4}\bar{x}$ and $\frac{3}{4}\bar{x}$ which again, minimizes travel costs. The maximal travel costs is then $\frac{t\bar{x}}{4}$. The average travel cost between

1. the lowest travel cost consumer at distance 0, who has zero travel costs, and
2. the highest travel cost consumer at distance $\bar{x}$, who has travel costs $\frac{t\bar{x}}{4}$,

is $\frac{0+\frac{t\bar{x}}{4}}{2}=\frac{t\bar{x}}{8}$. The pattern is then $\frac{t\bar{x}}{4n}$ where n is the number of outlets. Multiplied by the number of consumers, the total travel cost is then:

$$C\left(n\right)=\frac{t\bar{x}N}{4n}+nFC$$

Total costs starts high because of fixed costs and approach some minimum. The last $n+1$ where costs stops decreasing is the above expression.

$$\frac{t\bar{x}N}{4\left(n+1\right)}+\left(n+1\right)FC\leq\frac{t\bar{x}N}{4n}+nFC$$

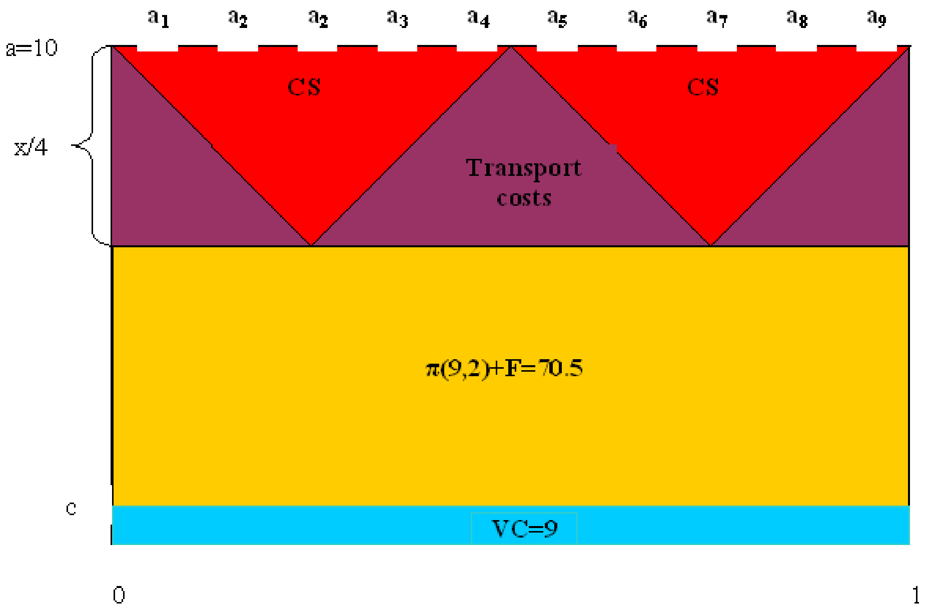
$$n\left(n+1\right)\leq\frac{t\bar{x}N}{4FC}$$

The optimal number of firms from social perspective is the last $n+1$ such that : $(n+1)\cdot n\leq\frac{N\bar{x}t}{4FC}$.

1.4 Example:

There are only 9 equally distributed people in the mile long x̄=1 town of Couch Potato Texas. Gresebomm's Fries is a gourmet french fry stand franchise. His costs consists $MC=1$ for lard and $FC=3$ for the stand. Everyone loves a serving of Gresebomm's fries about as much as they love \$10, but all suffer a travel cost from walking of $t=\$2$ per mile. From the above equations, this means $a=10, t=2, N=9, c=1, FC=3, \bar{x}=1$.

$$\pi\left(n\right)=\left(a-\frac{t\bar{x}}{2n}-c\right)N-nFC$$



What is the profit maximizing number of outlets?

outlets	Price	VC	Total FC	Total
1	$10-\frac{2}{2\cdot1}$	$9\cdot1$	$1\cdot3$	69.
2	$10-\frac{2}{2\cdot2}$	$9\cdot1$	$2\cdot3$	70.5
3	$10-\frac{2}{3\cdot2}$	$9\cdot1$	$3\cdot3$	69

Another way to put the same thing using profit equation explained above:

$\pi\left(9,1\right)=\left(10-\frac{2}{2\cdot1}-1\right)9-1\cdot3=69$
$\pi\left(9,2\right)=\left(10-\frac{2}{2\cdot2}-1\right)9-2\cdot3=70.5$
$\pi\left(9,3\right)=\left(10-\frac{2}{3\cdot2}-1\right)9-3\cdot3=69$

It would open 2 outlets.

Where would these be located? At equal distances to minimize travel costs.

What price would it charge?

$$P = a - \frac{t\bar{x}}{2n} = 10 - \frac{2}{3 \cdot 2} = 9.5$$

What would be the socially optimal number of outlets?

outlets	Total Trans Cost	VC	Total FC	Total
1	$\frac{2}{2 \cdot 2 \cdot 1} \cdot 9$	$9 \cdot 1$	$1 \cdot 3$	-16.5
2	$\frac{2}{2 \cdot 2 \cdot 2} \cdot 9$	$9 \cdot 1$	$2 \cdot 3$	-17.25
3	$\frac{2}{3 \cdot 2 \cdot 2} \cdot 9$	$9 \cdot 1$	$3 \cdot 3$	-19.5

Note that variable cost is constant for any number of outlets, so it won't affect the socially optimal number of outlets. Another way to put the same thing using cost equation explained below $-C(n) = -\frac{t\bar{x}N}{4n} - nFC$:

$S(9,1) = \left(-\frac{2}{2 \cdot 2 \cdot 1} - 1\right) 9 - 1 \cdot 3 = -16.5$
$S(9,2) = \left(-\frac{2}{2 \cdot 2 \cdot 2} - 1\right) 9 - 2 \cdot 3 = -17.25$
$S(9,3) = \left(-\frac{2}{3 \cdot 2 \cdot 2} - 1\right) 9 - 3 \cdot 3 = -19.5$

Thus, the socially optimal number of outlets is 2. Social surplus includes firm and consumer surplus. Thus, to maximize it, we just want to minimize travel costs. The difference between what is socially optimal and what is profit maximizing comes from the fact that the firm cares about the travel costs of the consumer only in so far as the firm can induce the consumer to travel and purchase. The firm doesn't care about the consumer's travel costs per see, since that's not a part of its costs. Travel costs are not borne by it, but only sets a bound on how much it can charge. Society cares about total travel AND fixed costs. The average travel costs of someone travelling from $\frac{\bar{x}}{2n}$ away is $\frac{1}{2}\left(0 + \frac{t\bar{x}}{2n}\right)$. When there are N such people, their total travel cost is: $\frac{t\bar{x}N}{4n}$.

THE FOLLOWING IS OPTIONAL.

1.5 Finding the Profit Maximizing Price when the Monopolist Does not Sell to Everyone

Since the choice of the price via the choice of n was descrete, there may be some wiggleroom for the best price, given n.

Price is profit maximizing if the loss from either increasing or decreasing the price is outweighed by the loss. (Refer to graph from section.)

From any fixed location, let r (be the fraction) of the total distance of the consumer served, where instead of using the farthest consumer at $\bar{x}$ in $p(n) = V - \frac{t\bar{x}}{2n}$, the text is using some consumer between the location of the vendor and $\bar{x}$ (p. 142). Then $p(n) = V - tr$. Since we have allowed r to be variable, we can make it a function of p:

$$r = \frac{(V - p)}{t}$$

This is a sort of "quantity demanded" function. Profit for N customers from both directions is then:

$$\pi = \max_{p} \left\{ 2N(p - c) \cdot \frac{(V - p)}{t} \right\}$$

There is no fixed costs, because we take number of stores and location to be given.

FOC:

$$2N\left(\frac{(V - p)}{t} - \frac{(p - c)}{t}\right) = \frac{2N}{t}\left((V - 2p + c)\right) = 0$$

which means that $p = \frac{V + c}{2}$.

$$\pi = \frac{2N}{t}(V - c)^2$$

Next:demand functions in Hotelling from Carabas, interpretating Hotelling in terms of standard demand, the connection between hotelling and interdependent markets, competition, Salop,when there are equilibria, Bertrand as the limiting case of Hotelling