Repeated Games Model of Collusion

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0.1 Games

We need game theory in order to model the strategic interaction of imperfect competition. Generally, for every game, there are two representations: matrix or "**normal**" form (e.g., prisoners dilemma) and game tree or "**extensive**" form (chain store game). The basic difference being that in the latter case, we can represent the 2nd player knowing what the first player did, in which case it would be a sequential move game (e.g., Stackelberg). If not, then it would be a simultaneous move game and that can be represented in either matrix or game tree form,e.g., Cournot, Bertrand competition.

Player 1\ Player 2	**strategy 1 of Player 2**	**strategy 2 of Player 2**
strategy 1 of Player 1	Payoffs of 1, Payoffs of 2	Payoffs of 1, Payoffs of 2
strategy 2 of Player 1	Payoffs of 1, Payoffs of 2	Payoffs of 1, Payoffs of 2

For each of these representations there is an equilibrium concept, which is a prediction of play. For normal form, the equilibrium concept is **NE**=a strategy situation which no one has any incentive to deviate unilaterally. The equilibrium concept for extensive form games is Subgame Perfect Equilibrium or SPE=NE in "every subgame". You just have to know that finding an **SPE** for a game with perfect information is equivalent to solving the game from the last stage, i.e., finding one the best responses of the highest numbered player (the player with the last turn) first and then seeing what the next highest number player would want to do given that best response. The crucial difference between these two representations is that some NE are not SPE, modelling the fact that some threats are incredible. For illustration consider the matrix form of the entry deterrence game:

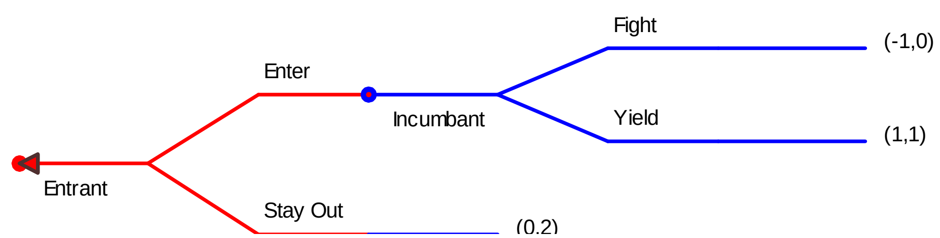
1\2	**Fight**	**Yield**
Enter	-1,0	1,1
Stay Out	0,2	0,2

Let's find the best response function of each player, by fixing the opponents strategy and finding the best response to that.

Suppose player 2 plays "Fight". Player 1's best response is to play "Stay Out". Suppose Player 2's strategy is "Yield". Player 1's best response is to play "Enter". Now, suppose Player 1 player Enter, Player 2's best response is "Yield". If Player 1 Plays "Stay Out", one of Player 2's best response is to play "Fight". The set of strategies that are mutual best responses is the set of Nash Equilibria (NE). The NE={(Stay Out,Fight), (Enter, Yield)}.

But, there is only one SPE in the extensive form (Enter, Yield), showing that Fight is an incredible threat–once the Entrant has entered, the best thing for the Incumbent to do is yield.

With imperfect information, the only difference is we use expected values of entering. (not covered)



The concept of **dominance** is applicable to both extensive form games and to normal form games. A strategy dominates another if, regardless of what the opponent does, you are strictly better off using this strategy. Defect is dominant in the one shot prisoners dilemma/Cournot game, where cooperate means producing half the monopoly quantity.

1\2	**Cooperate**	**Defect/ Cheat**
Cooperate	1,1	**-1,2**
Defect/ Cheat	**2,-1**	0,0

This has a unique dominant strategy equilibrium. Hence **Dominance Equilibrium**, where everyone is playing their dominant strategy, implies a NE, but of course, not vice versa.

As we saw with Cournot and Bertrand competition, players are jointly better off cooperating, but given that their opponent is cooperating, they are better off individually defecting. This incentive to defect can be overcome in a repeated game setting by tying present payoffs to future payoffs.

0.2 Constructing Repeated Games From One Period Games

We don't need all of the details from the Bertrand, Cournot models we have studied for the repeated games model of collusion. We just need 4 points: cooperative, unilateral defection from cooperation (only one firm cheats) and joint defection payoffs (both cheat). If π_m =monopoly profits, then the individual firm's profits when they cooperate is $\pi_i^{CC} = \frac{\pi}{2}$. The unilateral deviation payoff for firm 1 is $\pi_1^{DC} = \pi_m$ in the case of Bertrand. In the case of Cournot, we have to use the best response function to half the monopoly quantity Q_m: $BR_i\left(\frac{Q_m}{2}\right) = \frac{a-c}{2b} - \frac{1}{2}\left(\frac{Q_m}{2}\right)$ (see Cournot notes). Let π_2^{DC} be player 2's payoff from cooperating when player 1 has defected. (vice versa for player 1.) The joint defection payoff is just their respective Cournot and Bertrand competitive payoffs. Call it π_i^{DD}. The important thing to notice is that $\pi_1^{DC} > \pi_1^{CC} > \pi_1^{DD} > \pi_1^{CD}$, $\pi_2^{CD} > \pi_2^{CC} > \pi_2^{DD} > \pi_2^{DC}$ so the matrix will look like:

1\2	**Cooperate**	**Defect/ Cheat**
Cooperate	π_1^{CC}, π_2^{CC}	$\pi_1^{CD}, [\pi_2^{CD}]$
Defect/ Cheat	$[\pi_1^{DC}], \pi_2^{CD}$	$[\pi_1^{DD}, \pi_2^{DD}]$

Square brackets [] indicate best responses.

0.3 Finitely Repeated Games

Remember that collusion costs because, cheating always gives the highest payoff today, no matter what your opponent does today. You only collude because that would increase your payoff tomorrow. But, if there is no tomorrow, as in the last period, then there is no incentive to collude. You and your opponent will cheat. But, if you are both going to cheat tomorrow, no matter what you do today, there is no point in incurring the cost of cooperating today. Hence, there is no point in colluding in the 2nd to last period. . . .

0.4 Infinitely Repeated Games

A "Grim Trigger" strategy consists of two parts:

0.4.1 Grim Trigger Strategy:

1. Cooperate until defection
2. If ever ANYONE defects, defect forever after.

0.4.2 The NE when players play Grim Trigger

1. On the equilibrium path, everyone cooperates. In this NE, whenever someone defects, we move off the equilibrium path into the punishment phase of the equilibrium.[1]

[1] Note that the possible NE of the dynamic game includes the NE of the one stage game but usually has NE that spans across many one stage games. Note also, for any decisions involving dynamics (for the purposes of this class) we only deal with "present values", PV. We don't deal with future values.

2. In the punishment phase (off the equilibrium path), EVERYONE defects forever.

In this equilibrium, a player will either cooperate or defect ALWAYS: given fixed payoffs π and discount rate δ, there is either sufficient incentive to cooperate or not. The players incentives are completely determined by π, δ which are fixed. There is no "trust" or "fear" or learning. In a cooperative phase, they MAY cooperate–depending on sufficiency of incentives. But, in a defection phase, they WILL defect, because there is no incentive to cooperate when you know your opponent is going to defect. To get a clearer idea of what this means, consider this finite punishment strategy.

Forgive and forget Strategy: Punish defection for 3 periods only:

1. Cooperate until defection
2. If ever ANYONE defects, defect for 3 periods, then do 1).

The NE when players play Forgive and Forget Strategy:

1. On the equilibrium path, everyone cooperates. In this NE, whenever someone defects, we move off the equilibrium path into the punishment phase of the equilibrium.
2. In the punishment phase (off the equilibrium path), EVERYONE defects for 3 periods and then cooperates until defection.

In this equilibrium, a player will either cooperate or defect ALWAYS: given fixed π and δ, there is either sufficient incentive to cooperate or not.

Example: Comparison of Grim Trigger and Forgive and Forget Strategy in the Context of Repeated Prisoner's Dilemma The simple prisoner's dilemma has the same structure as Cournot. What matters is that everyone is better off cooperating, but it's in everyone's interest to defect, regardless of whether the opponent cooperates or defects in every stage game–when the future is not considered.

1\2	**C**	**D**
C	2,2	-3,3
D	3,-3	**1,1**

Note that $NE = (D, D)$ for the single stage game. Players only want to cooperate when there is a future: when the game is repeated. The NEs of the repeated game is to either ALWAYS collude or ALWAYS defect (which is the repeated single stage game). Suppose that there is sufficient incentive for players to collude–players collude in this equilibrium and if someone ever defects by "accident" taking the game off the equilibrium path, there will be punishment for 3 periods, after which, they can return to the equilibrium path of cooperation. To see this more clearly, let's go through the numbers:

Let $\delta = \frac{1}{1+r}$:[2]

	Period	**0**	**1**	**2**	**3**	**4**	**....**			∞
	Cooperation payoff	2	2δ	$2\delta^2$	$2\delta^3$	$2\delta^4$	$2\delta^5$	....	....	$\frac{2}{1-\delta}$
a)	**Defection payoff (Grim)**	3	1δ	$1\delta^2$	$1\delta^3$	$1\delta^4$	$1\delta^5$	....	....	$3+\frac{\delta}{1-\delta}$
b)	**Defection payoff (Grim)**	3	1δ	$1\delta^2$	$1\delta^3$	$\frac{\delta^4}{1-\delta}$	0	0	0	$3+\frac{\delta}{1-\delta}$
c)	**Defection payoff (Forgive)**	3	1δ	$1\delta^2$	$1\delta^3$	$2\delta^4$	$2\delta^5$	....	....	$3+\frac{\delta}{1-\delta}+?$
d)	**Defection payoff (Forgive)**	3	1δ	$1\delta^2$	$1\delta^3$	$2\frac{\delta^4}{1-\delta}$	0	0	0	$3+\frac{\delta}{1-\delta}+\frac{\delta^4}{1-\delta}$

$3+\frac{\delta}{1-\delta}+\frac{\delta^4}{1-\delta}$ in d) follows from a comparision between the series in d) and b).

Lets look at the δ that would support cooperation for Grim Trigger:[3]

[2] Note that the difference between a) and b) is that I have written the payoffs from the 4th period onwards as a one time payment at the 4th period and an infinite number of zero payments from the 5th period onwards. This allows easy comparison between a) and c) using b) and d) which are the same series, rewritten.

[3] Summing an infinite series:

$$\text{cooperation payoff} = \frac{2}{1-\delta} \geq 3 + \frac{\delta}{1-\delta} = \text{defection payoff}$$

$$2 \geq 3(1-\delta) + \delta$$

$$\delta \geq \frac{1}{2}$$

Again, the "equilibrium paths" –one for each type of equilibria– (which does not include "accidents") for the infinitely repeated game is to either ALWAYS collude if no one has defected, or ALWAYS defect if someone has defected.

For the "punish for 3 periods" game, cooperation requires that:

$$\text{cooperation payoff}{=}\frac{2}{1-\delta} \geq 3 + \frac{\delta}{1-\delta} + \frac{\delta^4}{1-\delta} = \text{defection payoff}$$

$$2 \geq 3(1-\delta) + \delta + \delta^4$$

$$2 \geq 3 - 3\delta + \delta + \delta^4$$

$$\delta \geq \frac{1}{2} + \frac{\delta^4}{2}$$

There is now an extra positive term on the right hand side, which means that δ on the left LHS must be greater.

Condition for cooperation (Grim)	Condition for cooperation (Forgive)
$\delta \geq \frac{1}{2}$	$\delta \geq \frac{1}{2} + \frac{\delta^4}{2}$

If LHS is true, LHS may still be false: someone may cooperate under Grim, but not under "Forgive and Forget Strategy". If RHS is true, LHS must be true. Whoever would cooperate under "Forgive and Forget Strategy" would cooperate under Grim. Therefore, you are less likely to cooperate under Forgive strategy than under Grim.

$$1 + \delta + \delta^2 + \delta^3 ....$$

Notice that:

$$1 + \delta + \delta^2 + \delta^3 .... - \left(1 + \delta + \delta^2 + \delta^3 ....\right) = 1$$

Therefore, collecting the common terms:

$$\left(1 + \delta + \delta^2 + \delta^3 ....\right)(1-\delta) = 1$$

Thus,

$$\left(1 + \delta + \delta^2 + \delta^3 ....\right) = \frac{1}{(1-\delta)}$$

The discount rate δ is equal to the inverse of the gross interest rate: $\frac{1}{R}$, which is equal to R=1+r. Thus, $\delta = \frac{1}{1+r}$. If you substituted, you will see that:

$$\left(1 + \delta + \delta^2 + \delta^3 ....\right) = \frac{1}{\left(1 - \frac{1}{1+r}\right)} = \frac{1+r}{r}$$

Therefore, $\frac{\delta}{1-\delta} = \frac{1}{1+r} \cdot \frac{1+r}{r} = \frac{1}{r}$