Ph. D. Preliminary exam in Industrial Organization, September 2003

Answers to questions 1 and 2

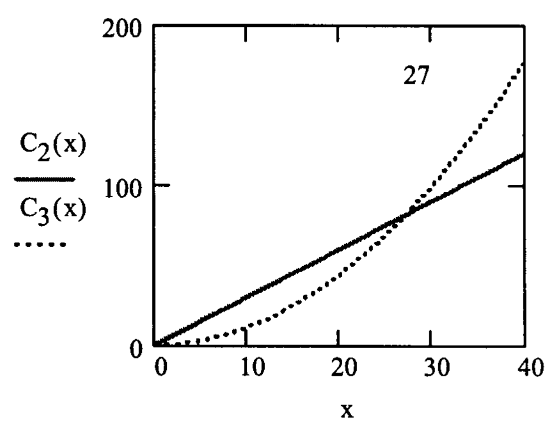
Question 1

(a) The profit function of firm *i* is $\Pi_i = q_i(83 - q_1 - q_2 - q_3) - 3q_i$. The Cournot-Nash equilibrium is given by $q_1 = q_2 = q_3 = 20$. Each firm makes a profit of 400.

(b) The profit function of the merged firm is $\Pi_m = (q_2 + q_3)(83 - q_1 - q_2 - q_3) - 3(q_2 + q_3)$ which it maximizes subject to the constraints $q_2 > 0$ and $q_3 > 0$. This is equivalent to choosing q_m to maximize $q_m(83 - q_1 - q_m) - 3q_m$ and then dividing the total output q_m in any way whatsoever between the two plants (since marginal cost is the same at the two plants). The Cournot-Nash equilibria are given by $q_1 = q_m = \frac{80}{3} = 26.67$ and $q_2 + q_3 = q_m$ with $q_2 > 0$ and $q_3 > 0$. The profit of each firm in the industry is 711.11. Since 711.11 < 2(400), the merger was not profitable.

(c) In this case, since the firm is allowed to shut down one plant and since plant 3 has a higher marginal cost than plant 2, the merged firm will maximize $q_m(83 - q_1 - q_m) - 3q_m$ and set $q_2 = q_m$ and $q_3 = 0$. Thus the post-merger Cournot-Nash equilibrium is given by $q_1 = q_m = \frac{80}{3} = 26.67$, as in part (b), with a profit per firm of 711.11. The pre-merger equilibrium is given by $q_1 = q_2 = 21,\ q_3 = 17$. The corresponding profits of firm 2 and 3 are 441 and 289, respectively. Since 711.11 < 441 + 289, once again the merger is not profitable.

(d.1) $C_2(q) := 3 \cdot q, \quad C_3(q) := \left(\frac{q}{3}\right)^2$



The cost function of the merged firm is the minimum of these two thus it is given by $(q/3)^2$ up to q = 27 and then by 3q (see graph).

Thus the profit function of the merged firm is $\Pi_m = q_m(83 - q_1 - q_m) - C_m(q_m)$ where

$$C_m(q) := \begin{cases} \left(\frac{q}{3}\right)^2 & \text{if } q \le 27 \\ \left[\left(\frac{27}{3}\right)^2 + 3\cdot(q-27)\right] & \text{if } q > 27 \end{cases}$$

(d.2) The profit function of firm 1 is: $\Pi_d(q_1, q_m) := q_1 \cdot P(q_1 + q_m) - 3\cdot q_1$.

$\Pi_{d1}(20, 40) = 400$ ▪. The profit of the merged firm is:

$$40\cdot P(60) - \left[\left(\frac{27}{3}\right)^2 + 3\cdot(40-27)\right] = 800 \text{ ▪}$$

(d.3) Now, $\frac{\partial}{\partial q_1}\Pi_d(q_1, q_m) = 0$ solve, $q_1 \rightarrow 40 - \frac{1}{2}\cdot q_m$. Thus when $q_m = 40$, $q_1 = 20$ maximizes firm 1's profits. It only remains to check if $q_m = 40$ maximizes the merged firm's profits when $q_1 = 20$. When $q_m > 27$, the revenue function of the merged firm is:

$R_{dm}(q_m) := q_m \cdot P(20 + q_m)$. Marginal revenue is: $MR_{dm}(q_m) := \frac{\partial}{\partial q_m} R_{dm}(q_m)$.

When $q_m > 27$, marginal cost for the merged firm is 3. At $q_m = 40$, we have $MR_{dm}(40) \rightarrow -17$. Since MR and MC are not equal, $q_m = 40$ does not maximize the merged firm's profits when $q_1 = 20$. Thus it is not an equilibrium.

Question 2

(a) A subgame-perfect equilibrium is a quadruple $(k_1^*, k_2^*; f_1^*(k_1,k_2), f_2^*(k_1,k_2))$ such that:

(1) for all (k_1,k_2) and for all x_2, $f_1^*(k_1,k_2) = \arg\max_{x_1} \pi_1(x_1, x_2; k_1)$ and

for all (k_1,k_2) and for all x_1, $f_2^*(k_1,k_2) = \arg\max_{x_2} \pi_2(x_1, x_2; k_2)$, that is,

$(f_1^*(k_1,k_2), f_2^*(k_1,k_2))$ is the Nash equilibrium of the second stage game, for every possible (k_1,k_2), and

(2) $k_1^* = \arg\max_{k_1} \pi_1(f_1^*(k_1,k_2), f_2^*(k_1,k_2), k_1)$ and $k_2^* = \arg\max_{k_2} \pi_2(f_1^*(k_1,k_2), f_2^*(k_1,k_2), k_2)$,

that is, (k_1^*, k_2^*) is the Nash equilibrium of the first-stage game anticipating that the second-stage game will be played according to (1).

(b) $\frac{\partial f_1^*}{\partial k_2} \neq 0$ and $\frac{\partial f_2^*}{\partial k_1} \neq 0$.

(c1) A Nash equilibrium of the single-stage game is a quadruple $\left(\hat{x}_1, \hat{x}_2, \hat{k}_1, \hat{k}_2\right)$ such that $\left(\hat{x}_1, \hat{k}_1\right) = \arg\max_{x_1, k_1} \pi_1(x_1, x_2; k_1)$ and $\left(\hat{x}_2, \hat{k}_2\right) = \arg\max_{x_2, k_2} \pi_2(x_1, x_2; k_2)$.

(c2) At a Nash equilibrium $\left(\hat{x}_1, \hat{x}_2, \hat{k}_1, \hat{k}_2\right)$ of the single-stage game it must be that

$$\frac{\partial \pi_1}{\partial k_1}\left(\hat{x}_1, \hat{x}_2, \hat{k}_1\right) = 0,\ \frac{\partial \pi_2}{\partial k_2}\left(\hat{x}_1, \hat{x}_2, \hat{k}_2\right) = 0,\ \frac{\partial \pi_1}{\partial x_1}\left(\hat{x}_1, \hat{x}_2, \hat{k}_1\right) = 0 \text{ and } \frac{\partial \pi_2}{\partial x_2}\left(\hat{x}_1, \hat{x}_2, \hat{k}_2\right) = 0 \qquad (\dagger)$$

Suppose, by contradiction, that $\hat{k}_i = k_i^*$ $(i = 1,2)$. Then, by (1) of point (a) and by the uniqueness assumption it must be that $\hat{x}_1 = f_1^*(\hat{k}_1, \hat{k}_2)$ and $\hat{x}_2 = f_2^*(\hat{k}_1, \hat{k}_2)$. Let $\pi_1^*(k_1, k_2) = \pi_1(f_1^*(k_1, k_2), f_2^*(k_1, k_2), k_1)$. Then, by (2) of point (a) it must be $\frac{\partial \pi_1^*}{\partial k_1}(\hat{k}_1, \hat{k}_2) = 0$. Now,

$$\begin{aligned} \frac{\partial \pi_1^*}{\partial k_1}(\hat{k}_1, \hat{k}_2) = {} & \frac{\partial \pi_1}{\partial x_1}(f_1^*(\hat{k}_1, \hat{k}_2), f_2^*(\hat{k}_1, \hat{k}_2), \hat{k}_1) \cdot \frac{\partial f_1^*}{\partial k_1}(\hat{k}_1, \hat{k}_2) + \\ & \frac{\partial \pi_1}{\partial x_2}(f_1^*(\hat{k}_1, \hat{k}_2), f_2^*(\hat{k}_1, \hat{k}_2), \hat{k}_1) \cdot \frac{\partial f_2^*}{\partial k_1}(\hat{k}_1, \hat{k}_2) + \qquad (\ddagger) \\ & \frac{\partial \pi_1}{\partial k_1}(f_1^*(\hat{k}_1, \hat{k}_2), f_2^*(\hat{k}_1, \hat{k}_2), \hat{k}_1) \end{aligned}$$

By (†), since $\hat{x}_1 = f_1^*(\hat{k}_1, \hat{k}_2)$ and $\hat{x}_2 = f_2^*(\hat{k}_1, \hat{k}_2)$, $\frac{\partial \pi_1}{\partial k_1}(f_1^*(\hat{k}_1, \hat{k}_2), f_2^*(\hat{k}_1, \hat{k}_2), \hat{k}_1) = 0$. For the same reason, $\frac{\partial \pi_1}{\partial x_1}(f_1^*(\hat{k}_1, \hat{k}_2), f_2^*(\hat{k}_1, \hat{k}_2), \hat{k}_1) = 0$ and thus the first term of (‡) is also equal to zero. Hence

$$\frac{\partial \pi_1^*}{\partial k_1}(\hat{k}_1, \hat{k}_2) = \frac{\partial \pi_1}{\partial x_2}(f_1^*(\hat{k}_1, \hat{k}_2), f_2^*(\hat{k}_1, \hat{k}_2), \hat{k}_1) \cdot \frac{\partial f_2^*}{\partial k_1}(\hat{k}_1, \hat{k}_2)$$

Now, by assumption (A2), $\frac{\partial \pi_i}{\partial x_j} \neq 0$ at every point, thus in particular at the point $(f_1^*(\hat{k}_1,\hat{k}_2), f_2^*(\hat{k}_1,\hat{k}_2), \hat{k}_1)$. By (b) $\frac{\partial f_2^*}{\partial k_1}(\hat{k}_1,\hat{k}_2) \neq 0$. Hence $\frac{\partial \pi_1^*}{\partial k_1}(\hat{k}_1,\hat{k}_2) \neq 0$, yielding a contradiction. The proof for firm 2 is similar.

(d) At a Nash equilibrium $(\hat{x}_1,\hat{x}_2,\hat{k}_1,\hat{k}_2)$ it must be that $\hat{x}_1 = \hat{k}_1$. In fact, if it were $\hat{x}_1 < \hat{k}_1$ then firm 1 could increase its profits by an amount equal to $c(\hat{k}_1 - \hat{x}_1)$ by reducing its capacity k_1 from $\hat{k}_1$ to $\hat{x}_1$. The same is true of firm 2. Thus the game reduces to one where firm 1 chooses k_1 to maximize $\pi_1 = k_1\,(1 - k_1 - k_2) - c\,k_1$ and firm 2 chooses k_2 to maximize $\pi_2 = k_2$ $(1 - k_1 - k_2) - c\,k_2$. Hence we get the standard duopoly equilibrium where $x_1 = k_1 = x_2 = k_2 = \frac{1-c}{3}$.

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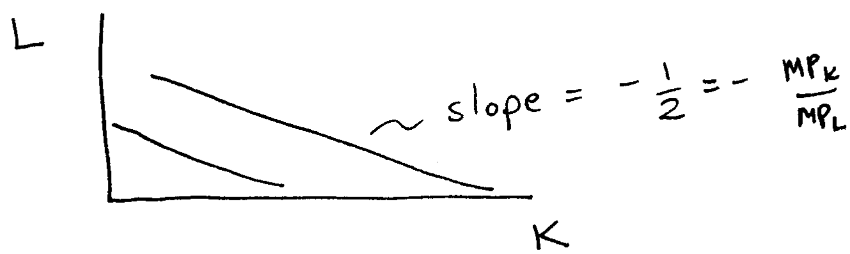
Answer to A-J Question, Question 3

a) $\dfrac{R - wL - F}{K} \le f$ or $\pi \le (f-r)K$

b) perfect substitutes:



c) Use Implicit $F^{\underline{n}}$ $Th^{\underline{m}}$:

$$\frac{dL}{dK} = -\frac{\partial\pi/\partial K}{\partial\pi/\partial L} = -\left(\frac{MR\cdot MP_K - r}{MR\cdot MP_L - w}\right)$$

$$= -\left(\frac{10MR - r}{20MR - w}\right)$$ where MR is a function of $Q=f(K,L)$, of course.

d) Note $\dfrac{dL}{dK} = 0$ when $MR = r/10$. Call the Q def'd by this Q_1.

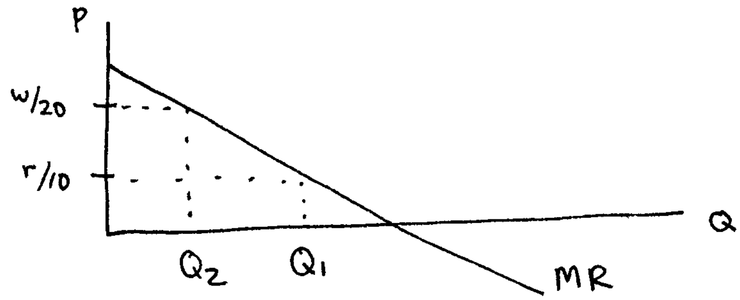
$= \infty$ when $MR = w/20$. Call the Q def'd by this Q_2.

Since you are given

$$\frac{r}{10} < \frac{w}{20},$$

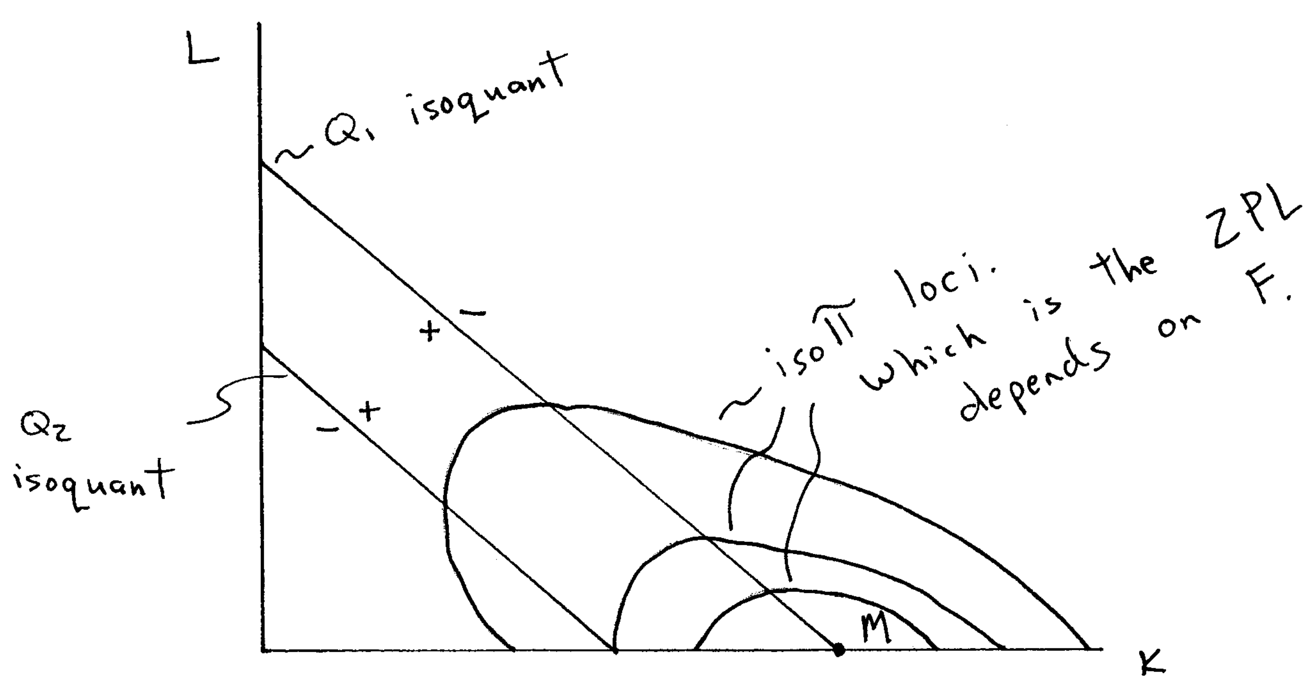
and $MR'(Q) < 0$ for linear demand,

$Q_1 > Q_2$.



So $\frac{dL}{dK} > 0$ between the Q_1 & Q_2 isoquants, and < 0 elsewhere.

Finally, $\frac{r}{10} < \frac{w}{20}$ $\Rightarrow$ will use all K and no L, so π^{max} occurs at $MR = MC$, where $MC = \frac{r}{10}$, so point M (π^{max}) is where the Q_1 isoquant meets the K axis:



e) $\pi = (f-r)K \Rightarrow R - wL - fK - F = 0$

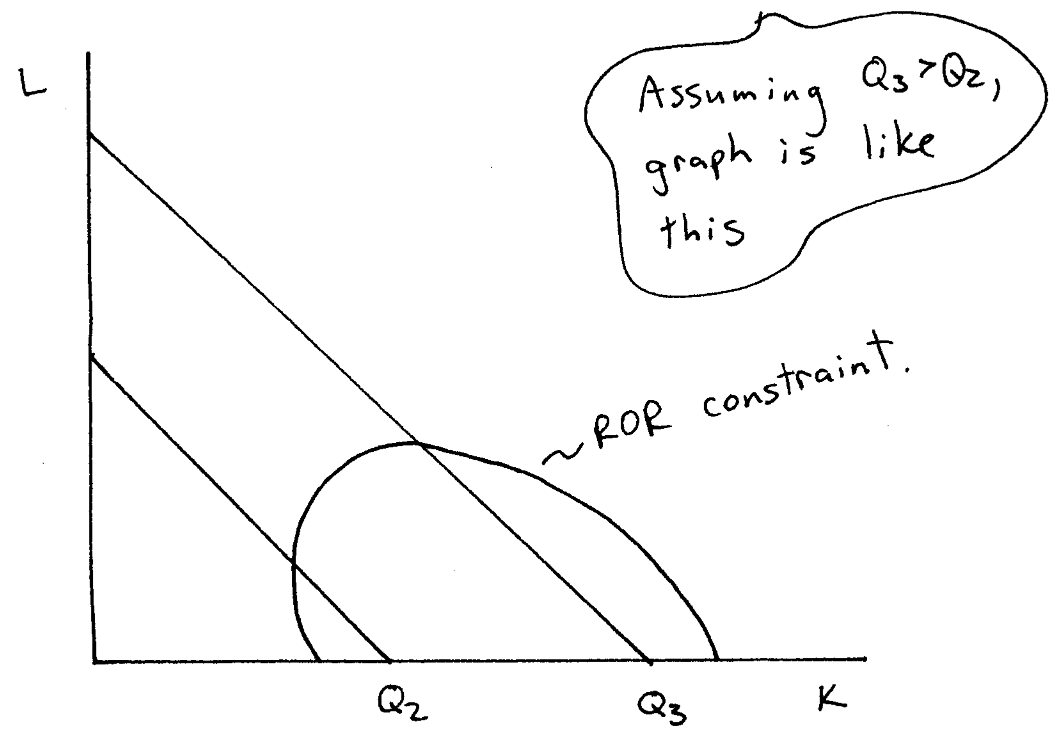
IFT $\Rightarrow$ $$\frac{dL}{dK} = -\left(\frac{MR \cdot MP_K - f}{MR \cdot MP_L - w}\right)$$

$$= -\left(\frac{10\,MR - f}{20\,MR - w}\right)$$

f) As before, let Q_3 be defid by $MR = f/10$. Q_2 is as before.

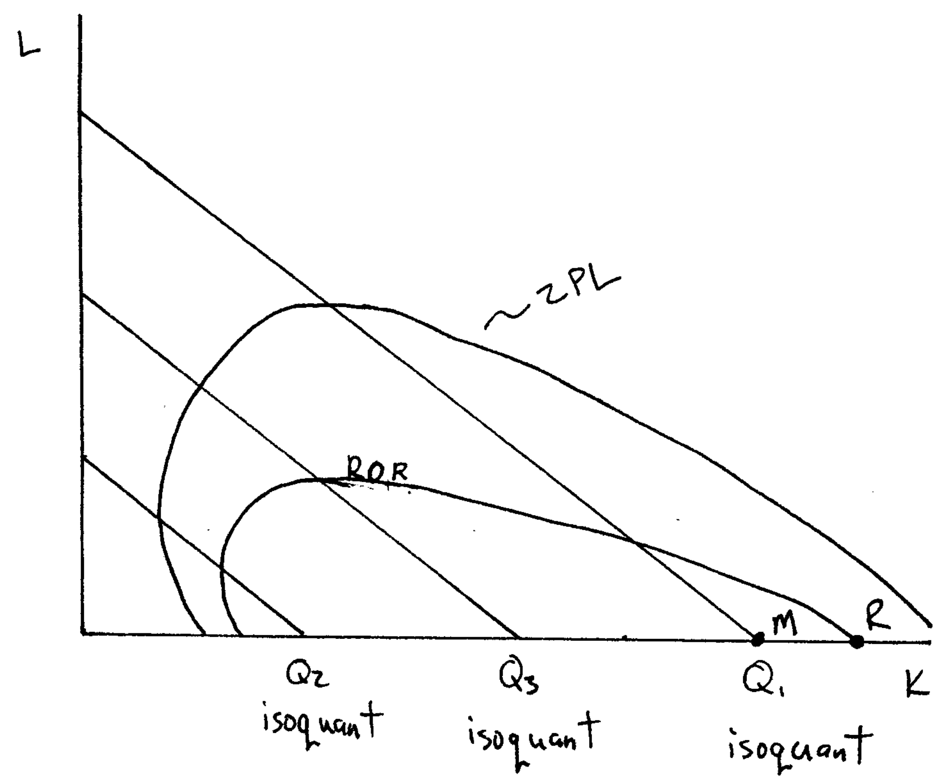
$f > r \Rightarrow$ ROR locus is <u>inside</u> ZPL.

Can find sign of $\frac{dL}{dK}$ as in (d).



g) $\frac{K}{L}$ of regulated firm is inefficiently high, given its output. Firm doesn't produce on expansion path.

h) What point will firm choose?
As always, picks pt. with most K:



Is R on the expansion path? Yes: the exp. path is the K axis.

So why doesn't the A-J effect hold? g(K,L) violates the assumption of $g(0,L) = g(K,0) = 0$ (see p. 164 of Baumol & Klevorick, which (along with the

assumption of positive MP_K & MP_L)
ensures the exp. path is in the
interior.