Feel free to impose additional structure on the problems below, but please state your assumptions clearly.

**Question 1**

(a) $U$ is here the value of unemployment.

(b) 

\[ U = b\Delta + e^{-\Delta \rho} U + (1 - e^{-\Delta \rho}) V \]  

(1)

The probability of finding a job is positively related to $\Delta$. The reason is that when time-periods are short, the probability of finding a job within a short time-period is small, and vice versa.

(c) Rearranging and taking limits we find that

\[ \delta U = b + p(V - U) \]  

(2)

**Question 4**

(a) The Bellman equation is given by

\[ V(a, c_{-1}) = \max_{c, a'} = \{u(c - \gamma c_{-1}) + \beta V(a', c)\} \]  

(3)

s.t \hspace{1cm} c + a' = w + (1 + r)a \hspace{1cm} (4)

(b) This is straightforward to verify using Blackwell's sufficient conditions. Notice however, that we need the return function to be bounded in order for these conditions to be applicable.

(c) The envelope conditions are given by

\[ V(c_{-1}, a_{-1}) = -\gamma u'(c - \gamma c_{-1}) \]  

(5)

\[ V(a, c_{-1}) = u'(c - \gamma c_{-1})(1 + r) + \beta V(c_{-1}, a_{-1})(1 + r) \]  

(6)

(d) The first order condition with respect to $a'$ is given by

\[ -u'(c - \gamma c_{-1}) + \beta V'_a(a', c) - \beta V'_c(a', c) = 0 \]  

(7)

Inserting the envelope conditions produces the Euler equation

\[ u'(c - \gamma c_{-1}) - \gamma u'(c' - \gamma c) = \beta (1 + r)(u'(c' - \gamma c) - \gamma u'(c'' - \gamma c')) \]  

(8)

**Question 5**
1. Consider the following equation

\[ U = b + e^{-\delta}(e^{-p}U + (1 - e^{-p})V) \]

(a) If \( b \) equals unemployment benefits, and \( 1 - e^{-p} \) is the probability of finding a job, what is the interpretation of \( U \)?

(b) In the equation above, one time-period is equal to one unit of time. Reformulate the equation such that one time-period equals \( \Delta \) units of time. How does the probability of finding a job change as \( \Delta \) becomes smaller? Why is this?

(c) What is the continuous-time formulation of the above equation?

2. Consider a Solow growth model in which the aggregate production function is of the form:

\[ Y(t) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha} \]

where \( Y \) denotes output, \( K \) is the capital stock, \( L \) is labor and \( A \) is labor-augmenting technology. Assume that \( L \) and \( A \) grow exogenously at the rates \( n \) and \( g \) respectively. Demonstrate that, in the steady-state, the equilibrium in this economy is consistent with Kaldor’s stylized facts of growth.

ANSWER: Kaldor’s stylized facts of growth are: 1. \( Y/L \) is growing over time, 2. \( K/L \) is constant, 3. Factor shares are constant, 4. rental rate of capital is constant, 5. wage is increasing. When the steady state is defined by the ratio \( k^* = (K/AL) \) and technology is Cobb-Douglas, these conditions are true (demonstration is trivial).

3. In order for the equilibrium in a standard Ramsey-Cass-Koopmans economy to exhibit balanced growth (which, in turn, is consistent with Kaldor’s stylized facts), it is necessary to place restrictions on agents’ preferences. Provide an intuitive argument for the nature of these restrictions. That is, you do not need to formally derive the restrictions; instead, discuss their motivation.

ANSWER: The two restrictions stem from the necessary conditions associated with an agents’ optimum. The intertemporal efficiency condition implies that, on a balanced growth path, agents intertemporal MRS is equal to the MPK (net of depreciation). Since, the MPK is the rental rate of capital, this will be constant while per-capita consumption is growing at the rate of technological progress. Hence, we require that the change in agent’s MU of consumption be constant in the growing economy. This is satisfied by assuming iso-elastic preferences. The other restriction is related to the labor-leisure decision and states that agents’ MRS between consumption and leisure is equal to the real wage. On a balanced growth path both the real wage and consumption are growing at the rate of technological progress but per-capita labor is constant. So this restricts the range of permissible preferences: if consumption
and leisure are separable, then all that is required is that the utility of consumption be isoelastic. If preferences are not separable, then the elasticity of the MU of labor with respect to consumption must be consistent with change in MU (this is enough for this answer - more precision is not required)

**Longer Answer Questions. (Each question is worth 20 points.)**

4. Let \( z \) be a random variable that takes on values in \( Z = \{0, 1\} \). Here, 1 denotes employment and 0 unemployment. Consider an arbitrary process of consumption \( \{c_t(z^t)\}_{t=0}^{\infty} \) where \( c_t : Z^{t+1} \rightarrow \mathbb{R}_+ \).

For this question, the probability of an unemployed individual finding a job is endogenous and depends on her search effort. Search effort is simply the binary choice between being a searcher, or not being a searcher. If the agent is a searcher, she sets variable \( x \) to 1, and the probability of finding a job equals \( \bar{p} \); that is \( P(z_{t+1} = 1 | z_t = 0) = \bar{p} \). If the agent decides not to search, she sets \( x \) to 0, and the probability of finding a job is zero; that is \( P(z_{t+1} = 1 | z_t = 0) = 0 \). Let us, for simplicity, assume that once a job is found, it lasts for perpetuity and is unaffected by the agent’s choice of \( x \); that is \( P(z_{t+1} = 1 | z_t = 1) = 1 \).

We can summarize these assumptions concisely by

\[
\lambda(z^{t+1}) = p(x)\lambda(z^t), \text{ if } z_t = 0, \text{ and } \lambda(z^{t+1}) = \lambda(z^t), \text{ if } z_t = 1
\]

where \( p(x) = \bar{p} \) if \( x = 1 \) and zero if \( x = 0 \). \( \lambda(z^{t+1}) \) denotes as usual the probability of history \( z^{t+1} \) occurring.

Lastly, an agent’s preferences are given by

\[
\sum_{t=0}^{\infty} \sum_{z^t \in Z^{t+1}} \beta^t \{u(c_t(z^t)) - x_t(z^t)\} \lambda(z^t)
\]

where \( x_t(z^t) \) is an agent’s choice of search effort in period \( t \), after having observed history \( z^t \).

Notice that the agent gains disutility 1 of being a searcher.

(a) After any history \( z^t \), define a **continuation value** in this economy. Denote this \( \vec{V}(z^t) \).

(b) Show that if \( z_t = 0 \), a **necessary** condition for an agent to be a searcher (i.e. to set \( x_t(z^t) = 1 \)) is given by

\[
V((z^t, 1)) - V(z^t, 0) \geq \frac{1}{\beta \bar{p}}
\]

(c) If \( z_t = 1 \), what is the optimal choice of \( x_t(z^t) \)?

(d) Now, let us assume there is a government providing unemployment insurance and taxing labor income. More precisely, the government will collect all labor income once the agent is employed, and always provide some sort of benefits equal to \( c_t(z^t) \). The government does so in order to maximize her own (expected present value) revenue, and such that the agent is given some minimum (expected present value) utility. However, the government must also recognize that the agent will freely decide whether to search or not. To make the problem simple(r), we assume that the **government** always prefers the agent to be a
searcher when unemployed (i.e. to set $x_t(z^t) = 1$ whenever $z_t = 0$, but not otherwise). The government’s optimization problem is then given by

$$J(V, z_0) = \max_{\{c_t(z^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^{t+1}} \beta^t \{ z_tw - c_t(z^t) \} \lambda(z^t)$$

subject to $V((z^t, 1)) - V(z^t, 0) \geq \frac{1}{\beta p}, \quad \forall z_t = 0, \; t = 0, 1, \ldots$ \hspace{1cm} (1)

$$V = \sum_{t=0}^{\infty} \sum_{z^t \in Z^{t+1}} \beta^t \{ u(c_t(z^t)) - x_t(z^t) \} \lambda(z^t)$$

Provide the Bellman equation associated with the optimization problem above (no need for a proof). Hint: A good starting point is to derive the Bellman equation for $z_0 = 1$, and then for $z_0 = 0$.

5. Consider the following (very simple) consumption-savings problem with habits

$$V(a_0, c_{-1}) = \max_{\{c_{t+1}, c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t - \gamma c_{t-1})$$

subject to $c_t + a_{t+1} = w + (1 + r)a_t, \quad t = 0, 1, \ldots$ \hspace{1cm} (2)

with $a_0$ and $c_{-1}$ given, and $\gamma \in [0, 1)$.

(a) Derive the Bellman equation corresponding to the optimization problem above (i.e. prove “Theorem 1”).

(b) Prove that this is a contraction mapping with modulus $\beta$.

(c) Use the envelope theorem to derive the derivative of the value function with respect to (both) state variables (no proof).

(d) Derive the Euler equation.

Now, let us modify the problem in the following way,

$$V(a_0, \tilde{c}_0) = \max_{\{\tilde{a}_{t+1}, \tilde{c}_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t - \gamma \tilde{c}_t)$$

subject to $c_t + a_{t+1} = w + (1 + r)a_t, \quad t = 0, 1, \ldots$ \hspace{1cm} (3)

$$\tilde{c}_{t+1} = H(\tilde{a}_{t+1}), \quad \tilde{a}_{t+1} = G(\tilde{a}_t)$$

with $a_0$ and $\tilde{a}_0$ given, and $\gamma \in [0, 1)$.

(e) What is the Bellman equation associated with the problem above (no need for a proof).

(f) Derive the Euler equation (no proof).

Let us now assume that the economy is populated by a continuum of individuals, all starting their lives with the same level of $a_0$. Moreover, $\tilde{a}_t$ and $\tilde{c}_t$ now happens to be average (or representative) savings and consumption levels, respectively.

(g) What is the relationship between the functions $G(\cdot), \; H(\cdot)$, and the individual’s policy functions for $c_t$ and $a_{t+1}$?
(h) The economy above is commonly referred to as “catching up with the Jones’s”. Can you explain why? Why is the Euler equation associated with this economy so different from the one with standard habits?

6. In Kydland and Prescott’s original RBC model, they made the assumption that investment does not produce capital immediately; i.e. the economy exhibited “time to build”. Consider a representative-agent (i.e. no population growth), non-stochastic version of their economy and assume that agents have preferences given by:

$$\sum_{t=0}^{\infty} \beta^t (\ln c_t + \theta \ln (1 - h_t))$$

where \(c_t\) is consumption and \(h_t\) is time spent in work activity. Aggregate output is produced using a standard Cobb-Douglas production function:

$$y_t = k_t^\alpha h_t^{1-\alpha}$$

where \(y_t\) is output and \(k_t\) denotes capital. (Note that there is no technological progress in the economy.) In each period, agents choose consumption, work effort and investment in order to maximize lifetime utility. In this economy, an investment project started at time \(t\) does not produce capital until next period. The costs associated with this project are spread out over the two-period horizon. Let \(s_{it}\) denote an investment project that is finished after \(i\) periods \((i = 1, 2)\) where households pay the fraction \(\omega_i\) of the total costs. Then total investment expenditures are given by:

$$i_t = \omega_1 s_{1t} + \omega_2 s_{2t}$$

and, of course, \(\omega_1 + \omega_2 = 1\).

The law of motion for the capital stock is given by:

$$k_{t+1} = k_t (1 - \delta) + s_{1t}$$

Given this environment, do the following:

(a) Express the associated social planner problem for this economy as a dynamic programming problem. Be explicit in identifying the states and control variables in each period (along with the laws of motion for the state variables). (Note: it is easiest to write the law of motion for capital as an additional constraint with an associated Lagrange multiplier.)

(b) Derive and interpret the necessary conditions associated with an optimum.

(c) Solve for the steady-state output-capital ratio, the investment-capital ratio, and the ratio of time spent in work activity to time spent in leisure as a function of the exogenous parameters.

ANSWER: The state vector in this economy is: \(z_t = (k_t, s_{1t})\) while the controls are given by the vector \(x_t = (c_t, h_t, s_{2t}, k_{t+1})\). The law of motion for capital is stated in the problem while the law of motion for investment-in-process is simply:

$$s_{1t+1} = s_{2t}$$
The associated DP problem is (where the definition of investment has been used in the resource constraint):

\[
V(z_t) = \max_{x_t} \left\{ \ln c_t + \theta \ln (1 - h_t) + \beta V(z_{t+1}) \right\}
\]

\[
+ \lambda_t \left[ k_t^\alpha h_t^{1-\alpha} - c_t - \omega_1 s_{1t} - \omega_2 s_{2t} \right] + \gamma_t \left[ k_t (1 - \delta) + s_{1t} - k_{t+1} \right]
\]

After applying the envelope theorem, the necessary conditions are:

\[
\frac{c_t}{1 - h_t} = \frac{(1 - \alpha)}{\theta} k_t^\alpha h_t^{-\alpha}
\]

(9)

\[
\gamma_t = \beta \lambda_{t+1} \alpha k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha} + \beta \gamma_{t+1} (1 - \delta)
\]

(10)

\[
\omega_2 \lambda_t + \beta \omega_1 \lambda_{t+1} = \beta \gamma_{t+1}
\]

(11)

The first expression is the standard necessary condition associated with the labor/leisure tradeoff. The second condition states that the shadow price associated with capital is equal to the MPK weighted by the shadow price of output and the discounted value of the capital stock remaining after depreciation. The third condition states that the expenditure cost of investment (over the two period horizon) must be equal to the discounted shadow price of a unit of capital.

In steady-state, these two conditions imply:

\[
\frac{\bar{y}}{k} = \frac{\bar{\gamma}}{\lambda} \frac{1 - \beta (1 - \delta)}{\alpha \beta}
\]

\[
\frac{\bar{\gamma}}{\lambda} = \omega_1 + \frac{1}{\beta} \omega_2
\]

Combining the two yields (and using \(\omega_1 + \omega_2 = 1\))

\[
\frac{\bar{y}}{k} = \frac{1 - \beta (1 - \delta)}{\alpha \beta} \left( \frac{1 - \omega_1 (1 - \beta)}{\beta} \right)
\]

In steady-state, \(\bar{s}_1 = \bar{s}_2\). Combining this with the law of motion of capital produces the same result in a typical growth model:

\[
\frac{\bar{i}}{k} = \delta
\]

Finally, using eq.(9) yields:

\[
\frac{\bar{h}}{1 - \bar{h}} = \frac{1 - \alpha \bar{y}}{\theta \bar{c}}
\]

From the resource constraint, we have

\[
\frac{\bar{y}}{\bar{c}} = 1 + \frac{\bar{i} \bar{k} \bar{y}}{\bar{k} \bar{y} \bar{c}}
\]

so the previous results can be used to write this expression in terms of the parameters.

7. Consider a simple stochastic, representative agent cash-in-advance economy in which the level of the endowment, \(x_t\), is an i.i.d. process while the aggregate money stock is constant. In this economy, agents face two markets. In the beginning of the period, agents visit the asset market where money, one-period nominal bonds and one-period real bonds are traded. They next visit
the goods market where money (acquired in the asset market) is used to finance consumption. At the beginning of each period agents receive the endowment (so all uncertainty is resolved) and they sell this in the goods period. The money from these sales along with any unspent money is carried over into the next period. Hence agents face two constraints associated with each market: in the asset market, beginning of period wealth is used to purchase money, nominal bonds and real bonds; in the goods market, money is used to purchase consumption goods (the CIA constraint). Nominal bonds cost $1 at time $t$ and return $(1 + n_t)$ in the following period’s asset market. Real bonds cost one unit of consumption in period $t$ and return $(1 + r_t)$ units of consumption in the asset market next period. Agents make their consumption and portfolio decisions in order to maximize:

$$E_0 \left[ \sum_{t=0}^{\infty} \beta^t U(c_t) \right]$$

where the utility function has standard properties. Given this description, do the following:

(a) Express the individual’s maximization problem as a dynamic programming problem. (Note: At the individual level, do not assume that the CIA constraint is binding.)

(b) Define a recursive competitive equilibrium in this economy. Assume that the cash-in-advance constraint is always binding.

(c) Characterize and explain the behavior of nominal and real interest rates in this economy.

(d) The Fisher relationship states that the expected real return on nominal bonds is equal to the real interest rate. Is that true in this economy? Explain.

ANSWER: Define real wealth at the beginning of period $t$ as:

$$w_t = x_{t-1} + \left( \frac{M_{t-1} - P_{t-1}c_{t-1}}{P_t} \right) + \frac{B_{t-1} (1 + n_{t-1})}{P_t} + b_{t-1} (1 + r_{t-1})$$ (12)

As is typical, this implies that wealth is determined by bond purchases made in the previous period (the last two terms). But the first two terms represent the timing convention: since the good market meets at the end of the period, it is the sales of last period’s endowment which determines today’s wealth; in addition, any money not spent in the goods market will add to wealth in period $t$. The dynamic programming problem can be written as:

$$V (w_t, x_t) = \max_{(c_t, M_t, B_t, b_t)} \left\{ U(c_t) + \beta E[V(w_{t+1}, x_{t+1})] \right\}$$

Along with the law of motion for $w_t$ implied by eq. (12). The necessary conditions are:

$$c_t : \quad U_t^* + \beta E \left[ \frac{\partial V (w_{t+1}, x_{t+1})}{\partial w_{t+1}} \left( - \frac{P_t}{P_{t+1}} \right) \right] - \gamma_t = 0$$ (14)

$$M_t : \quad \beta E \left[ \frac{\partial V (w_{t+1}, x_{t+1})}{\partial w_{t+1}} \left( \frac{1}{P_{t+1}} \right) \right] - \lambda_t \frac{1}{P_t} + \gamma_t \frac{1}{P_t} = 0$$ (15)

$$B_t : \quad \beta E \left[ \frac{\partial V (w_{t+1}, x_{t+1})}{\partial w_{t+1}} \left( \frac{1 + n_t}{P_{t+1}} \right) \right] - \lambda_t \frac{1}{P_t} = 0$$ (16)
The envelope theorem implies \( \frac{\partial V(w, x)}{\partial w} = \lambda_t \) so using this and combining the first two expressions yields:

\[ U'_t = \lambda_t \]

Since agents can always acquire money in the asset market before facing the CIA constraint, the form of wealth does not matter so that, as in most real economies, the marginal utility of consumption is equal to the shadow price of real wealth. This implies the Euler equations for both bonds are familiar:

\[ U'_0 \frac{1}{P_t} = \beta E \left[ U'_{t+1} \frac{1}{P_{t+1}} \right] (1 + n_t) \]  
\[ U'_t = \beta E \left[ U'_{t+1} \right] (1 + r_t) \]

A recursive equilibrium is defined by three functions: a value function as defined in eq. (13) and two functions for the nominal and real interest rates: \( n(x_t) \) and \( r(x_t) \) that are consistent with market clearing: \( c_t = x_t \) and \( M_t = \bar{M} \) (where this denotes the aggregate money stock).

Since, by assumption the CIA constraint is binding, this implies \( P_t = \frac{M}{x_t} \) in equilibrium. Using this in the necessary conditions provides the equations which define the two interest rates:

\[ (1 + n(x_t)) = \frac{U'(x_t) x_t}{\beta E [U'(x_{t+1}) x_{t+1}]} \]  
\[ (1 + r(x_t)) = \frac{U'(x_t)}{\beta E [U'(x_{t+1})]} \]

To characterize the behavior of interest rates, first note that, since the endowment is \( i.i.d. \), the denominator in both expressions is a constant. For real interest rates, we have immediately that real interest rates are monotonically decreasing in the endowment. This makes sense: a high endowment today increases the demand for bonds, hence the rate of return must fall in order for markets to clear. The behavior of nominal interest rates is determined by the function: \( f(x) = U'(x) x \). If this function is increasing (implying relative risk aversion is less than one), then nominal interest rates are positively related to the endowment, if the function is decreasing then nominal interest rates are decreasing in the endowment; if preferences are logarithmic, then nominal interest rates are constant. Why is this? As seen in eq. (20), an increase in the endowment causes the interest rate to fall. But expected inflation is given by:

\[ E_t \left[ \frac{P_{t+1}}{P_t} \right] = x_t E \left[ \frac{1}{x_{t+1}} \right] \]

So expected inflation increases with the current endowment (more goods today causes today’s price level to fall). Expected inflation moves one-for-one with the endowment while the real interest rate depends on the elasticity of the marginal utility of consumption (RRA). If RRA is less than one, then the fall in the real interest rate is less than the increase in expected inflation so nominal interest rates rise. If RRA is greater than one, we have the opposite scenario. While this reasoning is, in general, correct, it assumes that there is no risk premium on nominal bonds. But nominal bonds are a risky real asset; moreover, as was just discussed, the real return on nominal bonds is positively related to the endowment. (A high endowment causes a low price level so the real return on nominal bonds is high) But this carries the further implication that the covariance between agents’ MU and the real return on nominal bonds is...
bonds is negative implying a positive risk premium. One can show this formally by defining
the risk premium as:

\[ rp_t = (1 + n_t) E_t \left[ \frac{P_t}{P_{t+1}} \right] - (1 + r_t) \]

where the first term is the expected real return on nominal bonds. Using the eqs. (19), (20)
and (21) this can be written as:

\[ rp_t = -A_t \operatorname{Cov} (U'' (x_{t+1}), x_{t+1}) > 0 \]

where \( A_t = U'' (x_t) / \{ E [U'' (x_{t+1})] E [U'' (x_{t+1}) x_{t+1}] \} > 0. \)
(a) A continuation value is defined as

\[ V(z^t) = \sum_{s=0}^{\infty} \sum_{z^{t+s} \in Z^s \times z^t} \beta^s \{ u(c_{t+s}(z^{t+s})) - x_{t+s}(z^{t+s}) \} \lambda(z^{t+s}, z^t) \]  

(9)

Where \( \lambda(z^{t+s}, z^t) = \frac{\lambda(z^{t+s})}{\lambda(z^t)} \)

(b) At any maximum, a necessary condition is such that there exists no profitable deviations from the optimal plan. That is, an agent should prefer to be a searcher to not being a searcher on a period by period basis. Therefore we have

\[ u(c_t(z^t)) - 1 + \beta(\bar{p}V((z^t, 1)) + (1 - \bar{p})\bar{V}((z^t, 0))) \geq u(c_t(z^t)) + \beta\bar{V}((z^t, 0)) \]  

(10)

Rearranging provides the desired condition.

(c) If \( z_t = 1 \), the optimal choice of \( x_t \) is always equal to zero.

(d) If \( z = 1 \) the Bellman equation is given by

\[ J(V, 1) = \max_{c,\bar{V}'} \{ w - c + \beta J(\bar{V}', 1) \} \]  

s.t. \( V = u(c) + \beta\bar{V}' \)  

(11)

(12)

If \( z = 0 \) the Bellman equation is given by

\[ J(V, 0) = \max_{c,\bar{V}',\bar{V}_0} \{ -c + \beta(\bar{p}J(\bar{V}', 1) + (1 - \bar{p})J(\bar{V}_0', 0)) \} \]  

s.t. \( V = u(c) - 1 + \beta(\bar{p}\bar{V}' + (1 - \bar{p})\bar{V}_0') \)  

\[ \bar{V}' - \bar{V}_0' \geq \frac{1}{\beta\bar{p}} \]  

(13)

(14)

(15)