Unawareness - A Gentle Introduction to both the Literature and the Special Issue*

Burkhard C. Schipper†

February 25, 2014

Abstract

This article provides a brief survey of the literature on unawareness and introduces the contributions to the special issue on unawareness in Mathematical Social Sciences. First, we provide a brief overview both about epistemic models of unawareness and models of extensive-form games with unawareness. Instead of introducing the approaches in full detail, we illustrate the main differences and similarities with the help of examples. Finally, we discuss the contributions to the special issue on unawareness.

1 Introduction

Unawareness refers to the lack of conception rather than the lack of information. Under lack of information, a decision maker does not know which event occurred while under lack of conception the agent may not even have spent any thought on an event. It may be helpful to extend Knight's well-known distinction between risk and ambiguity to unawareness: under risk, the decision maker conceives of the space of all relevant contingencies and is able to assign probabilities them. Under ambiguity, the agent still conceives of the space of all relevant contingencies but has difficulties to evaluate them probabilistically. Under unawareness, the agent cannot even conceive all relevant contingencies.

The words "aware" and "unaware" are used in many contexts with many different connotations that sometimes deviate from the notion used in formal models of unawareness. Occasionally "aware" is used in place of "knowing" as in the sentence "I was aware of the red traffic light." On the other hand we interpret "aware" to mean "generally taking into account", "being present in mind" (Modica and Rustichini [1999] p. 274), "thinking about" (Dekel et al. [1998b]) or "paying attention to" as in the sentence "Be aware of sexually transmitted diseases!" In fact, the last sentence resonates closely with the etymology of "aware" since it has its roots in the old English "gewær" (which itself has roots in the German "gewahr") emphasizing to be "wary".[1]

---

*I very much thank Simon Grant for detailed comments on an earlier draft. I am extremely grateful the editors of Mathematical Social Sciences for initiating and supporting the special issue on unawareness. Finally, I grateful for financial support from the NSF SES-0647811.

†Department of Economics, University of California, Davis. Email: bcschipper@ucdavis.edu.

[1] `http://www.etymonline.com/index.php?term=aware`

In psychiatry (see for instance, Green et al. [1993]), lack of self-awareness means that a patient is oblivious to aspects of an illness that is obvious to social contacts. This is arguably closer to "not knowing". But it also implies that the patient lacks introspection of her/his lack of knowledge of the illness. It turns out that lack of negative introspection will play a crucial role in modeling unawareness. In neuroscience, being aware is taken as making/having/enjoying some experience and being able to specify the content of consciousness (Zeman [2002], pp. 16). While the precise connotations of all those uses of unawareness are different, they have in common that the agent is unable to conceive something.

Describing properties of awareness and unawareness informally with words like "knowing", "not knowing", "lack of conception", "not thinking about it" etc. does not make awareness amenable to rigorous analysis. As a remedy, various epistemic models of unawareness have been proposed both in computer science and economics.[2] In computer science the original motivation was mainly the modeling of agents who suffer from different forms of logical non-omniscience, while in economics the motivation focused more narrowly on the lack of conception. These differences in motivations in addition to different potential applications lead to slightly different modeling approaches in computer science and economics.

The purpose of this article is twofold. First, we like to provide a brief overview both about epistemic models of unawareness (Section 2) and models of extensive-form games with unawareness (Section 3). We will focus on the main differences and similarities with the help of examples. Section 2 draws from material in Schipper [2014b]. Second, we give a brief overview about the contributions to the special issue (Section 4).

2 Epistemic Models of Unawareness

2.1 Why not using Aumann structures?

One may wonder why standard approaches to modeling information would not suffice for modeling unawareness. Consider a nonempty space of states S. An event $E \in 2^S$ like "penicillium rubens has antibiotic properties" corresponds simply to a subset of states. We denote by $K : 2^S \longrightarrow 2^S$ the knowledge operator. For the event $E \in 2^S$, the set $K(E)$ represents the event that the agent knows the event E. For such an interpretation, the operator should satisfy some properties. Here we just require an extremely basic property called necessitation, $K(S) = S$, that is satisfied by standard notions of knowledge, probabilistic beliefs, or ambiguous beliefs in the literature. We also introduce an unawareness operator on events, $U : 2^S \longrightarrow 2^S$. For the event $E \in 2^S$, the set $U(E)$ shall represent the event that the agent is unaware of the event E. Again, for such an interpretation to make sense, we need to impose properties of unawareness. The first property, called Plausibility, says that if an agent is unaware of an event then she does not know the event and she does not know that she does not know the event. Formally, for any event $E \in 2^S$, $U(E) \subseteq \neg K(E) \cap \neg K \neg K(E)$, where $\neg E := S \setminus E$. This rules out what is known as negative introspection. It is inspired by Modica and Rustichini [1994, 1999] who defined awareness of an event as knowing the event or knowing that one does not know the event. A second property is KU-introspection, for any event $E \in 2^S$, $KU(E) \subseteq \emptyset$. The agent never knows that she is unaware of an event E. While we may know that in principle there

[2] For a bibliography, see `http://www.econ.ucdavis.edu/faculty/schipper/unaw.htm`.

could exist some events that we are unaware of (an issue that is taken up by the contributions of Halpern and Rêgo [2013] and Walker [2014] of this special issue), we cannot know that we are unaware of a specific event E. Finally, we require that if an agent is aware that she is unaware of an event, then she should be aware of the event. This property called AU Reflection is stated more formally in the contrapositive as if an agent is unaware of an event then she is unaware she is unaware of that event, that is, $U(E) \subseteq UU(E)$, for any event $E \in 2^S$. Dekel et al. [1998a] observed that these conditions lead to a contradiction.

Observation. *If a state-space model satisfies Plausibility, KU-introspection, AU-reflection, and Necessitation, then $U(E) = \emptyset$, for any event $E \in 2^S$.*

Proof. $U(E) \overset{AU-refl.}{\subseteq} U(U(E)) \overset{Plaus.}{\subseteq} \neg K(\neg K(U(E))) \overset{KU-intro.}{=} \neg K(S) \overset{Nec.}{=} \emptyset.$ □

This shows that the ("standard") state-space approach is incapable of modeling unawareness. To model unawareness, we need more structure than what standard Aumann structures, Kripke structures or Harsanyi type spaces have to offer. In the following we will survey briefly such structures capable of modeling unawareness. Rather than introducing all approaches in full detail, we will illustrate the main differences and similarities with the help of a speculative trade example that has been previously studied by Heifetz et al. [2006, 2013a], who compare predictions under unawareness to the "No speculative trade" theorems without unawareness (Aumann [1976], Milgrom and Stokey [1982]).

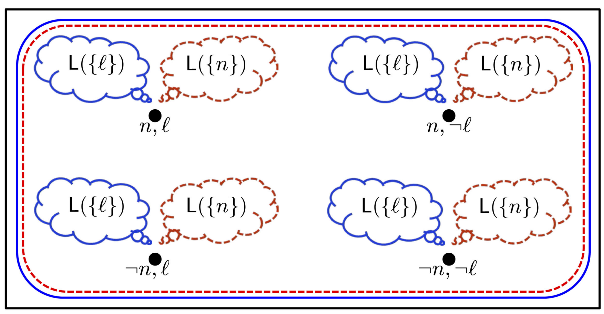
Speculative Trade Example. There are two agents, an owner o of a firm, and a potential buyer b of the firm. The status quo value of the firm is \$100 per share. The owner of the firm is aware of a potential lawsuit that reduces the value of the firm by \$20 per share. The owner does not know whether the lawsuit will occur. The buyer is unaware of the lawsuit and the owner knows that the buyer is unaware of the lawsuit. The buyer, however, is aware of a potential innovation that increases the value of the firm by \$20 per share. The buyer does not know whether the innovation will occur. The owner is unaware of the innovation and the buyer knows that the owner is unaware of the innovation.

A question of interest to economists is whether speculative trade between the owner and buyer is possible. Speculative trade is trade purely due to differences in information/awareness. In this example, we may phrase the question as follows: Suppose that the buyer offers to purchase the firm from the owner for \$100 per share. Is the owner going to sell to her? For an answer, see Heifetz et al. [2006, 2013a]. We focus here just on how to model the example with different epistemic approached to unawareness.

2.2 Awareness Structures by Fagin and Halpern [1988]

Fagin and Halpern [1988] were the first to present a formal approach of modeling awareness. They augment Kripke structures with a syntactic awareness correspondence. This provides a very flexible approach for modeling logical non-omniscience. Figure 1 depicts a simple awareness structure that models the speculative trade example. Denote by ℓ the formula "the lawsuit is brought against the firm" and by n "the novel use of the firm's product is discovered". Given such atomic formulae, we can also consider conjunction of formulae like "the lawsuit is brought against the firm and the novel use of the firm's product is discovered", negation of a formula like "the lawsuit is not brought against the firm", knowledge of a formula like "the buyer knows that the novel use of the firm's product has been discovered", and awareness of a formula like "the buyer is aware of the novel use of the firm's product". There is a space $S = \{(n, \ell), (\neg n, \ell), (n, \neg \ell), (\neg n, \neg \ell)\}$ of four states. For simplicity, we name each state by formulae that are true or false at that state. For instance the upper right state $(n, \neg \ell)$, n is true and ℓ is false. For each agent, there is an awareness correspondence that maps states into formulae that the agent is aware of at that state. At every state, the owner is aware of formulae in the set $\mathsf{L}(\{\ell\})$ involving the lawsuit (i.e., $\mathsf{L}(\{\ell\})$ denotes the set of formulae built from the atomic formula ℓ as discussed above) while the buyer is aware of formulae in the set $\mathsf{L}(\{n\})$ involving the novelty. The awareness correspondences are indicated by clouds, one for each agent. For each state, the blue solid cloud represents the awareness set of the owner while the red intermitted cloud represents the awareness set of the potential buyer. In the present example the awareness correspondences are very special because they are constant on S. The accessibility relations of agents is represented by possibility sets. The blue solid-lined possibility set belongs to the owner while the one with the red intermitted line is the buyer's. Each agent's information is trivial as neither can distinguish between any states.

Figure 1: An Awareness Structure for the Speculative Trade Example



This simple figure models the speculative trade example. In any state, the owner is unaware of the potential innovation but aware of the lawsuit because his awareness set never contains formulae involving n but only formulae involving ℓ. Similarly, at any state the buyer is unaware of the potential lawsuit but aware of the innovation because his awareness set never contains formulae involving ℓ but only formulae involving n. Awareness structures essentially add awareness correspondences to Kripke or Aumann structures, and it is precisely this additional structure that allows us to model unawareness. The accessibility relations show us that the owner does not know whether the lawsuit obtains because he cannot distinguish between states in which the lawsuit obtains and states where it doesn't. Analogously, the buyer does not know whether the innovation obtains because she cannot distinguish between states in which the innovation obtains from states where it doesn't obtain. But the owner knows that the buyer is unaware of the lawsuit because at every state of his possibility set, the buyer's awareness set does not contain formulae involving ℓ. (An analogous statement holds for the buyer with respect to n.) Again, the example is very special because each agent's information is constant across states.

One feature of awareness structures is that the accessibility relations model implicit knowledge rather than explicit knowledge. For instance, at every state the owner seems to know that he is unaware of the innovation because at any state his awareness set never contains formulae that involve n. Yet, in awareness structures explicit knowledge is defined as implicit knowledge and awareness. Clearly, since in neither state the owner's awareness set contains formulae with n, the owner is unaware of the novelty in any state and thus never explicitly knows that he is unaware of the novelty. Nevertheless, the implicit knowledge modeled by the accessibility relation is hard to interpret. Implicit knowledge that is not explicit knowledge is as if Isaac Newton would say "I know the theory of relativity but unfortunately I am not aware of it". In economics, we are only interested in knowledge that the agent is aware of, that can guide her decisions, and that in principle could be tested with choice experiments (see for instance Schipper [2014a] in this issue). While an outsider may be able to reason about the implicit knowledge of an agent, it is hard to see how the agent herself could reason about her implicit knowledge that is not explicit knowledge as well. Some authors in the awareness literature interpret implicit knowledge as "knowledge that the agent would have if she were aware of it". But this interpretation is flawed because if she really becomes aware of it, then maybe her explicit knowledge would not correspond anymore to her earlier implicit knowledge because her information may have changed as well.[3] More generally, awareness structures are hard to interpret as structures that the agents themselves could use to epistemically analyze the situation from their subjective perspective because if the awareness structure is presented to let us say the buyer, then he would become aware of the lawsuit. They are more appropriately interpreted from an outsider's point of view. A second feature of awareness structures is their syntactic awareness correspondences that assign sets of formulae to states. Generally, modeling approaches in economics using syntax are extremely rare. Yet, syntactic awareness correspondences provide also the flexibility that allows us to model unawareness.

2.3 Unawareness Structures by Heifetz et al. [2006]

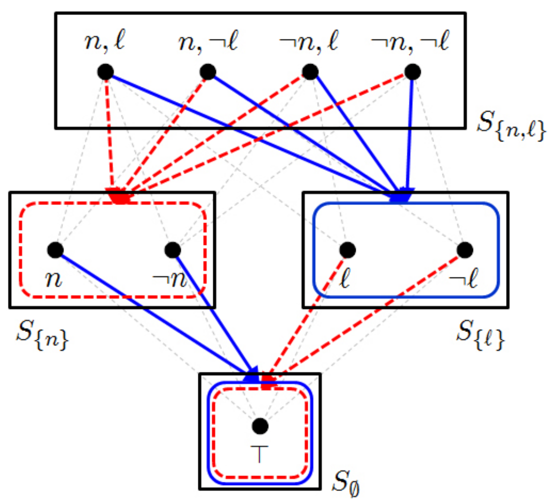
Inspired by Aumann structures, Heifetz et al. [2006] introduced an event-based approach that avoids syntax and allows for multi-agent unawareness. To circumvent the impossibility results by Modica and Rustichini [1994] and Dekel et al. [1998a], they work with a lattice of state spaces rather than a single state-space.

The unawareness structure depicted in Figure 2 models the speculative trade example. There are four spaces. Space $S_{\{n,\ell\}}$ is the richest space in which both the lawsuit and the innovation are expressible. Both spaces, $S_{\{n\}}$ and $S_{\{\ell\}}$, are less expressive than $S_{\{n,\ell\}}$. $S_{\{n\}}$ is the space in which only the innovation is expressible while $S_{\{\ell\}}$ is the space in which only the lawsuit is expressible. Finally, neither the innovation nor the lawsuit are expressible in the lowest space, $S_{\emptyset}$. We let the lattice order $\succeq$ be defined by $S_{\{n,\ell\}} \succeq S_{\{n\}} \succeq S_{\emptyset}$ and $S_{\{n,\ell\}} \succeq S_{\{\ell\}} \succeq S_{\emptyset}$. Projections from higher to lower spaces are indicated by grey dotted lines. For instance, state $(\neg n, \neg \ell) \in S_{\{n,\ell\}}$ projects to $(\neg n) \in S_{\{n\}}$. It also projects to $(\neg \ell) \in S_{\{\ell\}}$. Both $(\neg n) \in S_{\{n\}}$

[3] If one goes beyond the standard notion of knowledge and awareness generated by primitive propositions and considers more generally the problem of logical non-omniscience, then implicit knowledge may not be completely "out of mind". In such settings, implicit knowledge may capture interesting features of logical non-omniscience. So our criticism of implicit knowledge is not entirely fair because awareness structures were originally introduced to tackle the problem of logical non-omniscience more generally.

and $(\neg\ell) \in S_{\{\ell\}}$ project to $(\top) \in S_{\emptyset}$. The possibility correspondence is given by the blue solid and red intermitted arrows and soft-edged rectangles for the owner and the potential buyer, respectively. At any state in $S_{\{n,\ell\}}$ the owner's possibility set is at $S_{\{\ell\}}$. Thus, he is unaware of the innovation but aware of the lawsuit. Further, the owner's possibility set includes all states in $S_{\{\ell\}}$, which means that he does not know whether the lawsuit obtains or not. Since at every state in $S_{\{\ell\}}$ the buyer's possibility set is on S, in any state in $S_{\{\ell\}}$ the buyer is unaware of the lawsuit and the owner knows that. At any state in $S_{\{n,\ell\}}$ the buyer's possibility set is at $S_{\{n\}}$. Thus, he is unaware of the lawsuit but aware of the innovation. Further, the buyers's possibility includes all states in $S_{\{n\}}$, which means that she does not know whether the lawsuit obtains or not. Since at every state in $S_{\{n\}}$ the owner's possibility set is on S, in any state in $S_{\{n\}}$ the owner is unaware of the innovation and the buyer knows that.

Figure 2: An Unawareness Structure for the Speculative Trade Example



In comparison to awareness structures, we observe that the possibility correspondences model explicit knowledge. In fact, together with the lattice structure, they also model awareness determined by the space in which the possibility set lies. Although we used suggestive labels such as (n, ℓ) etc., unawareness structure are syntax-free. Finally, while the entire unawareness structure is the analysts model of the situation, it contains directly the "submodels" of agents.[4] For instance, the sublattice consisting of the two spaces, $S_{\{\ell\}}$ and $S_{\emptyset}$, corresponds to the model of the owner while the sublattice consisting of the two spaces, $S_{\{n\}}$ and $S_{\emptyset}$, is the buyer's model. Moreover, the sublattice $S_{\emptyset}$ is the model that both agents attribute to each other. The states in all those spaces can be interpreted as subjective descriptions of situations in the respective agent's mind.

2.4 Generalized Standard Models by Modica and Rustichini [1999]

Modica and Rustichini [1999] were the first economists to present a model of unawareness,

[4]For more on "submodels" the interested reader is referred to some very recent work by Grant et al. [2014].

the so called generalized standard model. A generalized standard models is restricted to a single agent. Thus, we cannot model the speculative trade example. While we could construct a separate generalized standard model for each of the agents, these models could not model the agent's reasoning about the other agent's awareness and knowledge. In retrospect we can understand unawareness structures of the previous section as a multi-agent generalization of Modica and Rustichini [1999]. For instance, the sublattice consisting of the two spaces $S_{\{n,\ell\}}$ and $S_{\{\ell\}}$ in Figure 2 can be viewed as a generalized standard model of the owner when we delete the possibility correspondence of the buyer.

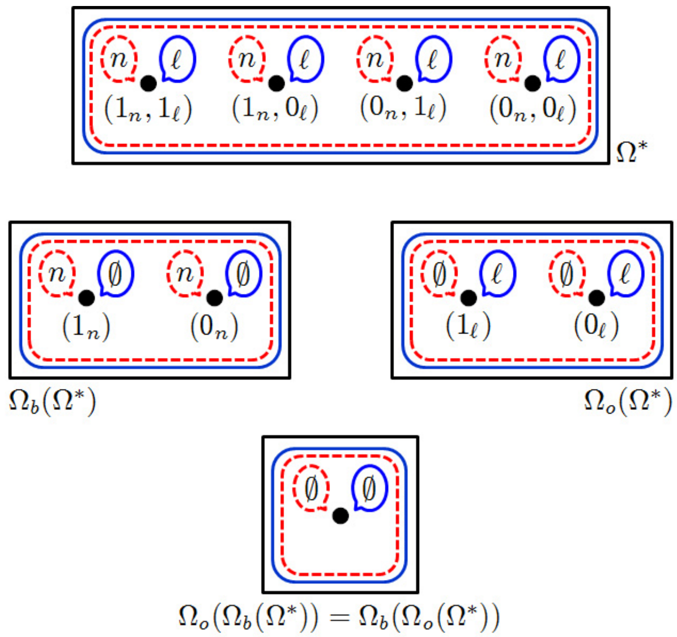
2.5 Product Models by Li [2009]

Li [2009] introduced product models by starting with a set of questions about the relevant aspects of the world that can be answered either in the affirmative or negative. Awareness then differs by subsets of questions that the agent has in mind. Such an approach to awareness is quite natural since lacking conception of some aspects of the world implies that one is not even able to ask questions about these aspects. Li [2009] introduces the product model for a single agent only. Thus, we cannot use it model the speculative trade example. Yet, in an unpublished paper, Li [2008] presents a multi-agent extension. We will illustrate the multi-agent extension with the speculative trade example. The set of questions is $\{n, \ell\}$, where we let n and ℓ stand for the questions "Is the innovation true?" and "Is the lawsuit true?", respectively. The objective state space is built from possible answers to these questions. We denote by 1_n an affirmative answer to n and by 0_n a negative answer to n (analogously for ℓ). The upmost space in Figure 3 depicts the objective state space $\Omega^* = \Omega_{\{n,\ell\}} = \{1_n, 0_n\} \times \{1_\ell, 0_\ell\}$. For instance, at the state $(1_n, 0_\ell)$ the question "Is the innovation true?" is answered in the affirmative, "The innovation is true.", while the question "Is the lawsuit true?" is answered in the negative, "The lawsuit is not true."

Li [2008, 2009] adds awareness correspondences that assign to each state the set of questions that the agent is aware of at that state. The awareness correspondences are indicated in Figure 3 by "speech bubbles" above each state. The solid blue speech bubbles belong to the owner, while the intermitted red speech bubbles are the buyer's. Both awareness correspondences are very special as they are constant on Ω^*. At every state Ω^*, the owner is aware only of questions involving the lawsuit while the buyer is only aware of questions involving the innovation. Information is modeled with possibility correspondences that assign to each state the set of states that the agent considers possible. The possibility correspondences are indicated in Figure 3 by the solid blue and intermitted red soft-edged rectangles for the owner and buyer, respectively. Again, the possibility correspondences are very special in this example as no agent can distinguish any objective states in Ω^*.

Given the awareness correspondences defined on the set of objective states Ω^*, we can construct the subjective state spaces of both agents by considering for each agent only the questions of which he is aware. At every state in Ω^*, the buyer's subjective state space is the space to the left, $\Omega_b(\Omega^*)$, while the owner's subjective state space is the space to the right, $\Omega_o(\Omega^*)$. So far, these are all the primitives of the product model. Li [2008, 2009] also defines subjective versions of the awareness and possibility correspondences. To define the subjective awareness correspondence on $\Omega_b(\Omega^*)$, we extend the objective awareness correspondence on Ω^* to the subjective states in $\Omega_b(\Omega^*)$ and $\Omega_o(\Omega^*)$ by restricting the awareness sets at the subjective

Figure 3: A Product Model for the Speculative Trade Example



states to questions available at those subjective states, respectively. Similarly, we can extend the objective possibility correspondences defined on Ω^* to subjective states by taking the projections to $\Omega_b(\Omega^*)$ and $\Omega_o(\Omega^*)$, respectively.

The subjective awareness correspondences allow us to defined another subjective state space shown as the lowest space in Figure 3. At every state in $\Omega_b(\Omega^*)$, the owner is unaware of the innovation (and the lawsuit), hence his subjective state space (in the eyes of the buyer) is $\Omega_o(\Omega_b(\Omega^*))$. Similarly, at every state in the owner's subjective state space $\Omega_o(\Omega^*)$, the buyer is unaware of the lawsuit (and the innovation); thus his subjective state space (in the eyes of the owner) is $\Omega_b(\Omega_o(\Omega^*)$. Both spaces are defined from an empty set of questions. They are identical and singleton. Again, we can extend the awareness and possibility correspondences to the lowest space as outlined above.

In retrospect, the exposition so far suggests that Li's product model bears features both of awareness structures by Fagin and Halpern [1988] and unawareness structures by Heifetz et al. [2006]. First, with awareness structures it has in common that awareness is modeled separately with an awareness correspondence although questions are used as primitive instead formulae. Since there is a one-to-one relation between questions and atomic formulae, the upmost space is analogous to the awareness structure depicted in Figure 1. Second, we see clearly that the possibility correspondence models implicit knowledge and not necessarily explicit knowledge. For instance, the possibility sets on the objective space Ω^* and the owner's (the buyer's, resp.) possibility sets on $\Omega_b(\Omega^*)$ (on $\Omega_o(\Omega^*)$, respectively) can be understood only in terms of implicit knowledge. With unawareness structures it has in common the idea of subjective states and the lattice structure.

3 Games with Unawareness

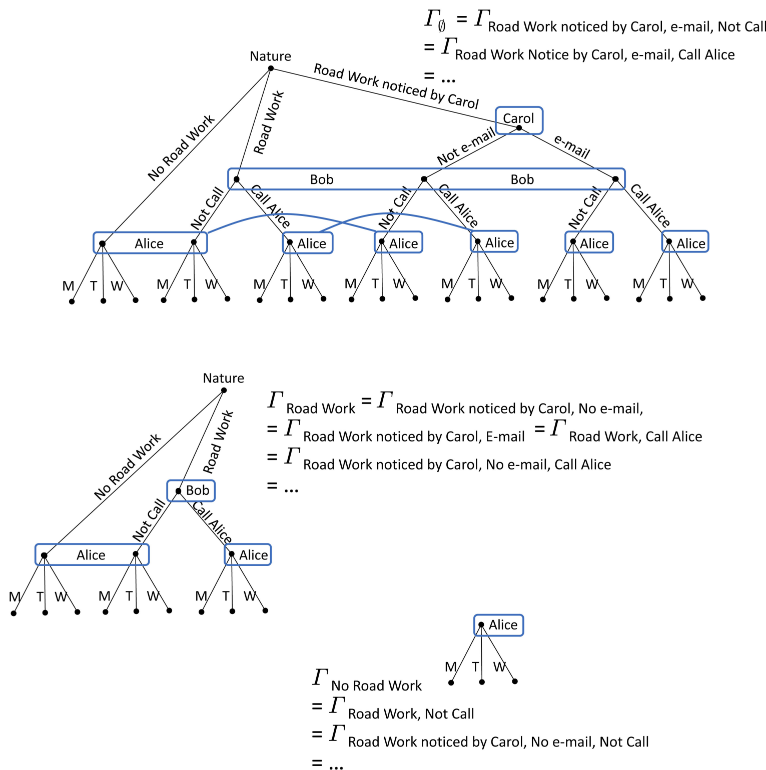
While epistemic structures reviewed in the previous section allow us to model unawareness, the main application of interest to economists, strategic interaction under asymmetric unawareness, requires us also to model actions and incentives of players. We focus here on dynamic games. Standard extensive-form games are implicitly common knowledge and common awareness among players. That is, all players are aware of all players, all actions, etc. although information about actions may differ between players and histories of the game. To allow for different levels of awareness, the game tree has to be replaced with a partially ordered forest of game trees, which is very much in analogy to the lattice of state-spaces in unawareness structures. We can start out with a comprehensive tree representing all conceivable moves and players, and then construct a forest of less expressive trees by pruning actions in an appropriate way. Intuitively, each tree of the forest corresponds to a more or less rich description of the strategic situation. Various approaches to games with unawareness have been proposed in the literature that differ by details of how the forest of game trees is constructed and how information and awareness is modeled. Although the following example due to Feinberg [2012] may not be the most compelling one in the context of unawareness, it allows us to easily highlight the main differences and similarities between the approaches.

Example of Unforeseen Road Construction. There are three players. Alice is a baker, Bob is a coffee-shop owner, and Carol is Alice's worker. Alice has to deliver pastries to Bob on Mondays or with penalties on Tuesdays or Wednesdays unless some unforeseen contingencies prevent her from doing this. Bob happens to notice one-day unforeseen roadwork on Monday morning that is likely to delay delivery if Alice's attempts to deliver on Monday. Bob believes that Alice is unaware of the roadwork. He can decide to call Alice and renegotiate a later delivery. Carol notices the upcoming roadwork already on Sunday and emails Alice. Bob is unaware of Carol and her actions. It is not hard to consider various incentives for calling or not calling etc. Yet, since we focus this review on modeling unawareness, we do not specify the payoffs here (see Feinberg [2012] for more details).

3.1 The Approach by Feinberg [2012]

At each decision node of a game, the player moving at that node entertains a view of the game. Such a view may miss certain actions or players of the original game, in which case the player at that decision node is unaware of those missing actions or players. Figure 4 depicts three different views of the unforeseen roadwork game. The tree at the bottom, $\Gamma_{\text{No Roadwork}}$, is the view of Ann at her decision node 'No Roadwork'. If there is no roadwork, then she is unaware of roadwork, does not conceive of Bob calling her about roadwork, and does not come up with the idea that Carol may email her about it. The tree in the middle, Γ_{Roadwork}, represents the view of the game by Bob after noticing roadwork. The upmost tree, $\Gamma_{\text{Roadwork Noticed by Carol, Email, No Call}}$, corresponds to Ann's view when she has been made aware of the roadwork by Carol via email. She holds this view irrespective of whether Bob calls her or not but she fully understands that Bob will notice the roadwork in the morning. Moreover, in case he doesn't call her, she understands that he could her.

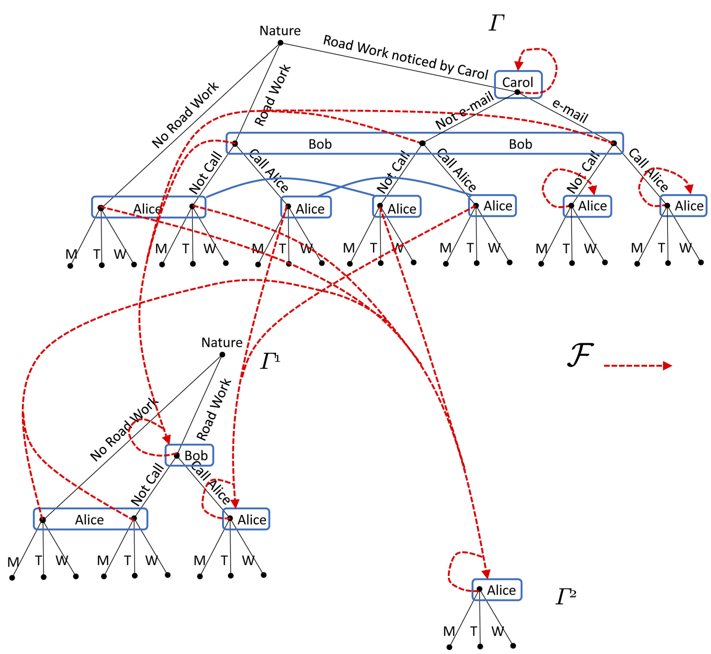
Figure 4: Unforeseen Roadwork à la Feinberg [2012]



Feinberg [2012] models explicitly with infinite sequences of views how each player at a decision node considers views of players at any decision node, their views of views at other decision nodes etc. That is, the model involves an infinite number of copies of the trees depicted in Figure 4. For instance, the tree in the bottom represents also how Bob thinks about Alice's view if he does not call upon noticing the roadwork, how Alice thinks that Bob think's about Alice's view if she receives Carol's email but no call from Bob, how Carol thinks about Alice's view if she does not email her and Bob does not call her, how Carol thinks about Bob's thought about Alice's view if he does not call, etc.

While each game tree in the 'forest of game trees' appears to be a standard extensive-form game, the interpretation of some information sets becomes non-standard. In standard games, an information set can be interpreted as the set of histories that the player considers possible at a history of this information set. Such an interpretation does apply to some but not all information sets in Feinberg's model. Consider for instance Alice's information set after the history "No Roadwork" in the upmost tree. After this history, Alice is supposed to be unaware of roadwork. Thus, it does not make sense to interpret this information set as her state of mind in which she considers "No Roadwork", "Roadwork and No Call", and "Roadwork Noticed by Carol, No Email, and No Call" possible. Similar to awareness structures by Fagin and Halpern [1988], information sets in Feinberg [2012] model implicit knowledge rather than explicit knowledge usually considered in game theory. Only when the player is also aware of all histories in her/his information set, can we interpret it as the set of histories that she/he considers possible.

Figure 5: Unforeseen Roadwork à la Halpern and Rêgo [2014]

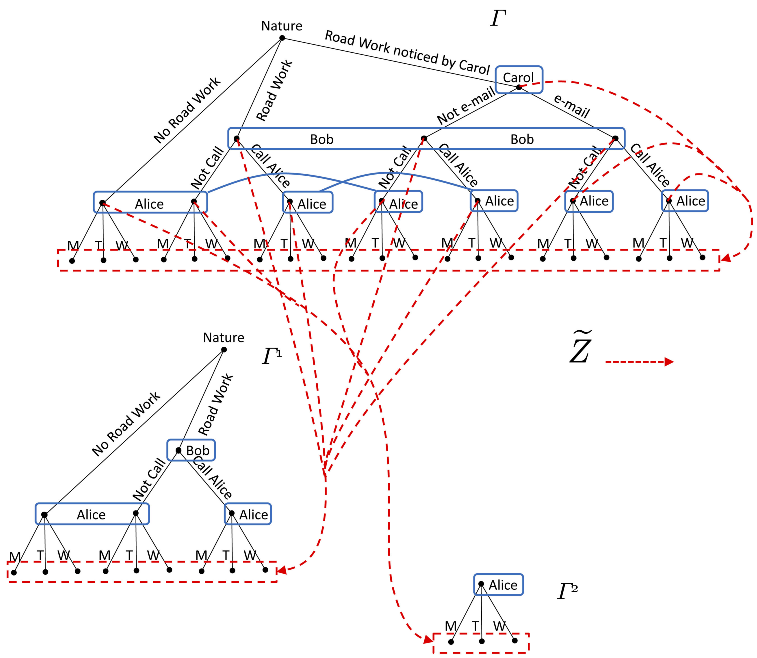


3.2 The Approach by Halpern and Rêgo [2014]

Halpern and Rêgo [2014]'s approach to extensive-form games with unawareness also involves a forest of game trees. Figure 5 shows the three games of the roadwork example. Not surprisingly, the three trees are the same as in the the previous setting. Yet, instead of modeling explicitly each player's view at any of her decision nodes about views of others etc., Halpern and Rêgo [2014] use an awareness correspondence $\mathcal{F}$ that maps each game and history into a game and information set in that game. We view this mapping somewhat analogous to awareness correspondences in awareness structures. In Figure 5 the awareness correspondence $\mathcal{F}$ is indicated by red intermitted arrows. For instance, Alice's awareness in the upmost tree at her history "No Roadwork" is given by her information set in the tree of the bottom. Note that this captures parsimoniously also higher order awareness. For instance, Carol is aware of the situation depicted in the upmost tree. This is indicated by the red intermitted arrow at her decision node. Since she is aware of this situation, she thinks that Bob is aware of the situation depicted in the middle tree because at any of Bob's histories in the upmost tree, the awareness correspondence assigns him his information set in the middle tree. Moreover, she realizes that Bob thinks that Alice's awareness is given as in the tree at the bottom because at any of her histories in the tree considered by Bob (i.e., the middle tree), the awareness correspondence assigns her the information set and tree in the bottom.

Apart from the awareness correspondence, the the forest of trees in Halpern and Rêgo [2014] looks pretty standard in the sense that each game tree is a standard extensive-form game. Yet, as in Feinberg [2012]'s approach the interpretation of some information sets becomes non-standard as they actually model implicit knowledge and not necessarily explicit knowledge; see our discussion in Subsection 3.1.

Figure 6: Unforeseen Roadwork à la Grant and Quiggin [2013]



3.3 The Approach by Grant and Quiggin [2013]

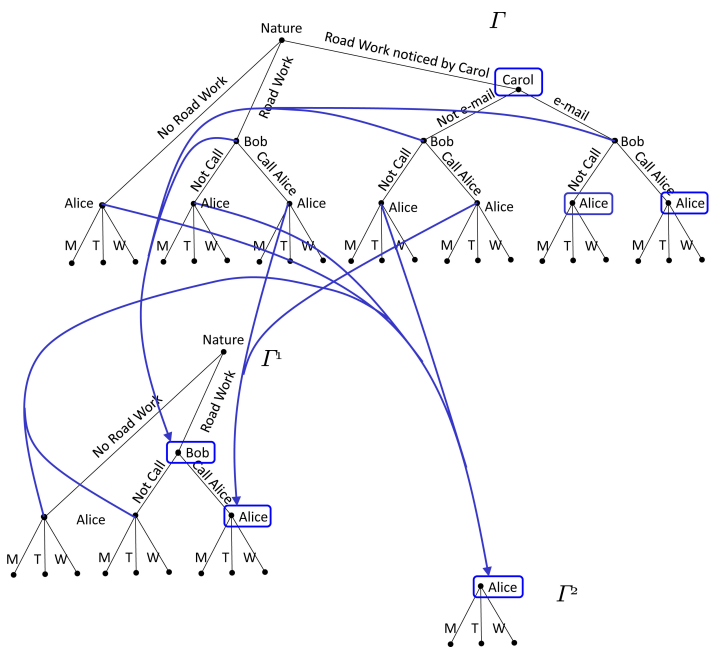
Figure 6 depicts the roadwork example using a model put forward by Grant and Quiggin [2013]. As in the other approaches, it uses a forest of game trees. Yet, instead of the awareness correspondence in Halpern and Rêgo [2014], the authors use what they call a perception mapping $\widetilde{Z}$ that maps each tree and history to the set of terminal histories in a weakly less expressive tree. The interpretation is that the value of the perception map at a player's history is the set of (terminal) histories that this player is aware of. For instance, at history "No Roadwork" in the upmost tree, Alice is only aware of (terminal) histories in the tree at the bottom. This is indicated by the red intermitted arrows in Figure 6.

The same critique applies to the interpretation of information sets in Grant and Quiggin [2013] as for Feinberg [2012] and Halpern and Rêgo [2014]. The information sets model implicit knowledge and not necessarily explicit knowledge; see our discussion in Subsection 3.1.

3.4 The Approach by Heifetz et al. [2013b]

The last approach I like to discuss allows as in standard games allow for the interpretation of information sets as sets of histories that the player considers possible. Similar to unawareness

Figure 7: Unforeseen Roadwork à la Heifetz et al. [2013b]



structures by Heifetz et al. [2006], information sets are used both to model the knowledge and awareness of players. Heifetz et al. [2013b] define a forest of game trees that is partially ordered by set inclusion of decision nodes. For the unforeseen roadwork example the forest of game trees is depicted in Figure 7. The trees are as in the previous three approaches. What differs are the information sets as well as how awareness is modeled. Consider for instance Alice's state of mind after the history "No Roadwork". Since she is unaware of roadwork, she does not consider histories "No Roadwork", "Roadwork, No Call", or "Roadwork Noticed by Carol, No Email, No Call". Rather, her state of mind is given by her information set in the tree at the bottom. That is, the information set at a node of a tree may consists of nodes in a less expressive tree. In this approach, information sets model explicit knowledge. Note that this approach also captures parsimoniously higher order awareness. For instance, Carol is aware of the situation depicted in the upmost tree because her information set is in the upmost tree. She also realizes that Bob is aware of the situation depicted in the middle tree because at any of Bob's decision nodes in the upmost tree, his information set is in the middle tree. Moreover, she thinks that Bob thinks that Alice's awareness is given as in the tree at the bottom because Ann's information set at any of her decision nodes in the middle tree is located in the tree at the bottom. Note that the forest of trees is "closed" under information sets in the sense that for every tree and any decision node in that tree, the information set at this decision node is contained in some tree of the forest of trees.

3.5 Solution Concepts for Games with Unawareness

Halpern and Rêgo [2014] defined Nash equilibrium in their framework of extensive-form games with unawareness. Rêgo and Halpern [2012] extend sequential equilibrium, a refinement of Nash equilibrium, to their framework because it is known that Nash equilibrium is a quite weak solution concept even in standard extensive-form games. For instance, it does not eliminate incredible threats. Feinberg [2012] extends assessments, the main "ingredient" of sequential equilibrium, to his framework of dynamic games with unawareness. Grant and Quiggin [2013] introduce sequential equilibrium in their framework as well. Although mathematical definitions of such standard solution concepts can be extended to dynamic games with unawareness, their application to strategic situations under unawareness may no longer be conceptually appropriate. Most commonly used solution concepts are refinements of Nash equilibrium. A profile of strategies, one for each player, is a Nash equilibrium if each player's strategy is a best response to the opponents' strategies. It presumes that strategies are mutual knowledge among players. This is often informally motivated with interactive learning of the equilibrium convention: If players interact in the game repeatedly, then eventually they will learn somehow about the strategies used by opponents. Such a motivation cannot apply to games with unawareness in general. Games with unawareness model situations where some players may be unaware of some actions. Thus they couldn't have learned previously about such actions. If such an action is played during the play of the game, then it is far from clear where the players' knowledge of the new equilibrium convention should come from. Therefore, equilibrium notions in strategic situations with unawareness may make sense only in special situations such as when players' awareness along the equilibrium path never changes, or when becoming aware also implies magically also mutual knowledge of the new equilibrium convention.

To avoid the conceptual problems of equilibrium under unawareness, Heifetz et al. [2013b] extend extensive-form rationalizability to dynamic games with unawareness. Extensive-form rationalizablity iteratively eliminates possible beliefs systems of players about opponents' strategies. It does not presume equilibrium. Nevertheless it is a strong solution for standard dynamic games because it entails forward-induction. In contrast to backward induction, which assumes that players' future behavior will be rational, forward induction also attributes rationality to players' past behavior if possible. Rather than simply excusing unexpected behavior of opponents as mistakes, a player who uses forward-induction tries to rationalize opponents' past behavior to form predictions about their future behavior. This is important under unawareness because if a rational player "becomes aware", she is by definition surprised. If becoming aware is a result of an opponent' action, then she should consider the opponent's intention for making her aware (rather than discounting it as a mistake) and should use this information to form beliefs about opponents' future play. Heifetz et al. [2011] introduce prudent rationalizability, an outcome refinement of extensive-form rationalizability and an extensive-form analogue to iterated admissibility, for dynamic games with unawareness. Meier and Schipper [2012] define the normal-form games associated to dynamic games with unawareness and characterize both extensive-form rationalizability and prudent rationalizability in dynamic games with unawareness by iterated conditional strict (weak, resp.) dominance in the associated normal-form.

4 Introduction to the Special Issue

The special issue on unawareness comprises of five articles spanning advances in the epistemic foundations of unawareness to economic applications. Halpern and Rêgo [2013] tackle the problem of awareness of unawareness. In an earlier approach, Halpern and Rêgo [2009] allowed for knowledge of unawareness of a formula. But formulae expressing that an agent considers it possible that she is aware of all formulae and also considers it possible that she is not aware of all formulae were not satisfiable in their earlier approach. This is a serious limitation for applications as it is very natural to consider agents who may be uncertain about whether or not they are aware of everything. Halpern and Rêgo [2013] fixes this important issue. Interestingly, the remedy is to introduce a kind of hybrid structure between awareness structures of Fagin and Halpern [1988] and unawareness structures of Heifetz et al. [2006]. Recall that in awareness structures the awareness correspondence associates subsets of formulae with different states but all formulae are defined at each state, while in unawareness structures potentially different subsets of formulae are defined at different states (see Halpern and Rêgo [2008], Heifetz et al. [2008]). This difference between the formalisms is immaterial as long as we are "just" interested in modeling reasoning about knowledge and propositionally determined awareness and do not care about the important conceptual issue of whether structures can be viewed from an agent's subjective perspective. In Halpern and Rêgo [2013] the difference between formulae defined at a state and the formulae that an agent is aware of at that state is of conceptual significance for a second reason. Roughly these formulae correspond to the labels that the agent could be aware that she is unaware of. Halpern and Rêgo [2013] introduce extended awareness structures, and axiom system, and prove soundness and completeness. Thus, they characterize their extended awareness structures that allow for modeling awareness of unawareness by properties of awareness and knowledge. They then explore the connection between awareness and unawareness structures by showing that the quantifier-free fragment of their logic is characterized by exactly the same axioms as the logic of Heifetz, Meier, and Schipper [2008]. Moreover, they show that under minimal assumptions they can dispense with Fagin and Halpern [1988]'s syntactic notion of awareness as this notion of awareness is essentially equivalent to the one used in Modica and Rustichini [1999] and Heifetz, Meier, and Schipper [2006, 2008]. This article is a very nice contribution as it not only provides a solution to modeling awareness of unawareness but also clarifies the relationship between the awareness literatures in computer sciences and economics. Due to an editorial mishap, the article appeared already in an earlier volume of the journal. But it was submitted and edited for the special issue.

Walker [2014] presents an alternative approach to modeling awareness of unawareness. Instead of quantifiers, he uses propositional constants such as "agent i is aware of everything" and "agent j is aware of everything that agent i is aware of". Such an approach is less expressive than Halpern and Rêgo [2013]. For instance, we cannot express that an agent i knows that there is no more than one proposition that agent j is aware of but agent k is not. Nevertheless, the approach allows for modeling awareness of unawareness in many relevant examples and thus is a viable simple alternative to more expressive but also more complicated structures. Walker [2014] introduces modified awareness structures with propositional constants, a novel two-stage semantics, an appropriate axiom system, and proves soundness and completeness. It is a nice self-contained contribution to the literature.

Schipper [2014a] presents a decision theoretic view on unawareness structures by Heifetz et al. [2006]. He complements unawareness structures with decision theoretic primitives like preference relation over acts. An act is a function from states to real numbers. These decision theoretic primitives allow him to characterize properties of the possibility correspondence of unawareness structures by corresponding properties of a decision maker's preference relation. In some sense, it emphasizes the fact that the knowledge modeled by possibility correspondences in unawareness structures reflects explicit knowledge that could in principle be revealed with choice experiments. The article had been submitted to Mathematical Social Sciences before the special issue on unawareness was conceived. It has been edited by Simon Grant, the former co-editor of Mathematical Social Sciences.

The special issue contains also one of the first papers on games with unawareness. Halpern and Rêgo [2014] present their framework for extensive-form games with unawareness and define Nash equilibrium. I already discussed their approach in Subsection 3.2.

Finally, Li et al. [2014] present a nice application of unawareness to the disclosure of verifiable information. The setting is a duopoly in which sellers have private information about the adverse effect of their products, while consumers are unaware of existence of such adverse effect unless they are informed by the sellers. Goods can be described along two dimensions, one of them describing a potential adverse effect. The products of the two firms differ in both dimensions. Each firm decides whether to make the consumers aware of the adverse effect dimension by providing verifiable information on this dimension for the products of both firms. Li et al. [2014] show that even though sellers compete for consumers, they may not be made aware of adverse effects if it leads to smaller market size. This is in contrast to unraveling results in standard verifiable information games without unawareness. The article is a nice contribution to the growing literature that explores implications of unawareness is various economic, political, and social contexts. In particular, it complements other work on verifiable information under unawareness by Heifetz et al. [2011] and Schipper and Woo [2013].

We hope that the special issue on unawareness can inspire readers to further research on unawareness. The past literature focused on foundations. We have now tools in place to do epistemic analysis, decision theory, and game theory with asymmetric unawareness. These tools should allow researchers to revisit information economics and explore systematically the implications of unawareness. We expect that in future many applications will be developed.

References

R. Aumann. Agreeing to disagree. *Annals of Statistics*, 4:1236–1239, 1976.

E. Dekel, B. Lipman, and A. Rustichini. Standard state-space models preclude unawareness. *Econometrica*, 66:159–173, 1998a.

E. Dekel, B. Lipman, and A. Rustichini. Recent developments in modeling unforeseen contingencies. *European Economic Review*, 42:523–542, 1998b.

R. Fagin and J. Halpern. Belief, awareness, and limited reasoning. *Artificial Intelligence*, 34: 39–76, 1988.

Y. Feinberg. Games with unawareness. Stanford University, 2012.

S. Grant and J. Quiggin. Inductive reasoning about unawareness. *Economic Theory*, 54:717–755, 2013.

S. Grant, J.J. Kline, P. O'Callaghan, and J. Quiggin. Sub-models and base-space knowledge for interactive unawareness. University of Queensland, 2014.

J. Green, F.C. Goldstein, B.E. Sirockman, and R.C. Green. Variable awareness of deficits in alzheimer's disease. *Neuropsychiatry, Neuropsychology, and Behavioral Neurology*, 6:159–165, 1993.

J. Halpern and L.C. Rêgo. Interactive unawareness revisited. *Games and Economic Behavior*, 61:232–262, 2008.

J. Halpern and L.C. Rêgo. Reasoning about knowledge of unawareness. *Games and Economic Behavior*, 67:503–525, 2009.

J. Halpern and L.C. Rêgo. Reasoning about knowledge of unawareness revisited. *Mathematical Social Sciences*, 65:73–84, 2013.

J. Halpern and L.C. Rêgo. Extensive games with possibly unaware players. *Mathematical Social Sciences*, forthcoming, 2014.

A. Heifetz, M. Meier, and B.C. Schipper. Interactive unawareness. *Journal of Economic Theory*, 130:78–94, 2006.

A. Heifetz, M. Meier, and B.C. Schipper. A canonical model for interactive unawareness. *Games and Economic Behavior*, 62:304–324, 2008.

A. Heifetz, M. Meier, and B.C. Schipper. Prudent rationalizability in generalized extensive-form games. University of California, Davis, 2011.

A. Heifetz, M. Meier, and B.C. Schipper. Unawareness, beliefs, and speculative trade. *Games and Economic Behavior*, 77:100–121, 2013a.

A. Heifetz, M. Meier, and B.C. Schipper. Dynamic unawareness and rationalizable behavior. *Games and Economic Behavior*, 81:50–68, 2013b.

J. Li. Interactive knowledge with unawareness. University of Pennsylvania, 2008.

J. Li. Information structures with unawareness. *Journal of Economic Theory*, 144:977–993, 2009.

S.X. Li, M. Peitz, and X.J. Zhao. Vertically differentiated duopoly with unaware consumers. *Mathematical Social Sciences*, forthcoming, 2014.

M. Meier and B.C. Schipper. Conditional dominance in games with unawareness. University of California, Davis, 2012.

P. Milgrom and N. Stokey. Information, trade and common knowledge. *Journal of Economic Theory*, 26:17–27, 1982.

S. Modica and A. Rustichini. Awareness and partitional information structures. *Theory and Decision*, 37:107–124, 1994.

S. Modica and A. Rustichini. Unawareness and partitional information structures. *Games and Economic Behavior*, 27:265–298, 1999.

L.C. Rêgo and J. Halpern. Generalized solution concepts in games with possibly unaware players. *International Journal of Game Theory*, 41:131–155, 2012.

B.C. Schipper. Preference-based unawareness. *Mathematical Social Sciences*, forthcoming, 2014a.

B.C. Schipper. Awareness. In H. van Ditmarsch, J.Y. Halpern, W. van der Hoek, and B. Kooi, editors, *Handbook of Logics for Knowledge and Belief*. College Publications, London, 2014b. to be published.

B.C. Schipper and H.Y. Woo. Political awareness, microtargeting of voters, and negative electoral campaigning. University of California, Davis, 2013.

O. Walker. Unawareness with "possible" possible worlds. *Mathematical Social Sciences*, forthcoming, 2014.

A. Zeman. *Consciousness. A user's guide.* Yale University Press, 2002.