Insiders Trading on Unknown Unknowns

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Abstract

We study a CARA-normal asset market with a continuum of traders under asymmetric awareness of components affecting the mean or the variance of the fundamental value of the asset. While the prior literature focuses on asymmetric information, we focus on asymmetric awareness. A trader is unaware of a component of the fundamental value (e.g., a line item of the balance sheet or profit & loss of the stock's firm) if she does not conceive of it and hence does not even know that she does not know it. Traders interact in a uniform price double auction with asymmetric awareness and private information. We derive the linear symmetric Bayes-Nash equilibrium best understood as a prediction in the medium-run because every realization is consistent with every trader's awareness but in the very long run traders may statically learn that they are biased. We study market depth, price precision, price volatility, and ex ante trading volume including some comparative statics w.r.t. the distribution of traders' awareness. In a version of the model with common signals, we study the incentives of aware traders to raise the awareness of other traders. We completely characterize the nonempty set of signals and pre-disclosure market prices for which aware traders raise the awareness of unaware traders. Disclosure of awareness occurs when pre-disclosure market clearing prices would be "too low" or "too high" given the signal.

Keywords: Rational expectations, disclosure, information efficiency, unawareness, neglected risks, strategic trading, market microstructure.

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1 Introduction

There is a large literature on how information of traders aggregates in asset markets (see Vives (2008)). All of these papers assume that traders conceive of all relevant events affecting the fundamental value of the asset but may have different information about those events. That is, traders are allowed to have asymmetric information but are assumed to have common awareness of all relevant events. In this sense, the prior literature just studies the impact of known unknowns. It is conceivable though that an insider may not just have better information about some line-item of the balance sheet or profit & loss of the firm whose security is traded in an asset market but may have also private awareness of the existence of such line-items. Consequently, they have a much more fine-grained view of private and public signals about material information of the firm and are able to anticipate the possible arrival of such information. Other traders may be unaware of the existence of certain particular events affecting the profitability of the firm. Consequently they cannot even conceive of information pertaining to them including the possibility that other traders could have such information. They do not know that they do not know these events. This asymmetric awareness is studied in the current paper.

We construct a model with CARA utility and normal distributed random variables for informed trading in a market for a single asset. Such models are more or less standard (e.g., Vives (2008, Chapter 4.2.1), Kyle (1989), Hellwig (1980), Grossman and Stiglitz (1980), Grossman (1976), Admanti (1985), Diamond and Verrecchia (1981)) except that we allow for different awareness levels among informed traders. That is, some traders may conceive of some events effecting the terminal value of the stock while others do not. Asymmetric awareness is modelled with a simple version of unawareness structures (Heifetz et al. (2013b)). We allow for unawareness of events that affect the mean or the variance of the fundamental value of the asset. Traders compete with demand schedules in an uniform price double action. There is a continuum of each type of traders. Our model is an example of a large Bayesian game with unawareness and we derive Bayesian Nash equilibrium (Meier and Schipper (2014a)). The equilibrium is best understood as a prediction in the medium-run because every realization is consistent with every trader's awareness but in the very long run traders may statically learn that their beliefs are biased. The unique linear equilibrium strategies in closed form are then compared across awareness levels and their comparative statics is studied as much as it is feasible. For instance, traders who are aware of more events affecting the volatility of the fundamental value are less sensitive to price changes. In fact, those traders may have even *upward sloping demand curves* if the variance of the additional term of the fundamental value that they are aware is sufficiently larger than the variance of the terms that all are aware. These traders become *momentum traders* (in the sense of Zhou (2020)) with upward sloping demand linear functions. Market depth with traders who are aware of more components affecting the volatility of the fundamental is larger than with just traders who are unaware of such components. W.r.t. price precision, more aware traders affect price precision differently from more informed traders. It is well known that the larger the share of informed traders or the better their information, the higher is the price precision. In contrast, the larger the share of more aware traders and the higher their awareness level, the lower is price precision because higher awareness means awareness of additional components contributing to the volatility of the fundamental. Price volatility in markets with more aware traders is higher than in markets with less aware traders.

Next we study incentives to disclose awareness to other market participants. We simplify the model further by assuming that informed traders observe a *common* signal that is just interpreted by them differently depending on their individual awareness levels. By eliminating private information, we can focus purely on changes in awareness. The simplified model is akin to that of Grossman and Stiglitz (1980) except that we allow for asymmetric awareness. Yet, rather than using the model to analyze information acquisition as in the model by Grossman and Stiglitz (1980) with common awareness, we study endogenous incentives for informed traders with superior awareness to disclose their awareness to unaware traders. We allow aware traders to condition their disclosure decision not just on the (common) signals but also market clearing prices. We fully characterize the set of signals and pre-disclosure market clearing prices. For every common signal, disclosure of awareness occurs when pre-disclosure market clearing prices are either "too low" or "too high" given the signal.

Our paper contributes to the recent emerging literature on financial markets with unawareness. Heifetz et al. (2006) and Heifetz et al. (2013b) show to what extent speculative betting is possible under unawareness and prove a no-speculative betting theorem under unawareness. Galanis (2018) and Meier and Schipper (2014b) show related results. Liu (2017) studies how awareness interacts with information acquisition in a market for an asset with one big trader with superior awareness and a continuum of small traders with less awareness. He shows that under certain assumptions the loss of overall information quality is higher for moderate awareness asymmetry. Auster and Pavoni (2020) study a search problem in which many retail investors with limited awareness delegate investment decisions to a few financial intermediaries. Latter compete for investors by raising awareness of investment options. They show that under limited competition, intermediaries restrict disclosure to very risky and very safe investment opportunities. The presence of fully aware investors can have positive or negative spill-over effects on other investors depending on whether the entire market is served or not, respectively. Gui et al. (2020) study markets with naive and sophisticated retail investors. Naive investors can be exploited by a fraudulent monopolistic firm that offer too-high-to-be-true returns. Consequently, they underestimate the true return of the financial product. Depending on the fraction of naive and sophisticated investors, the firm may refrain from offering fraudulent investment opportunities because it does not want to loose the sophisticated investors. They also study competition among firms. They show that a honest firm may not want to reduce the number of naive investors too much through disclosure because the fraudulent competitor may decide to compete for sophisticated investors instead being content with the naive investors. Finally, Carvajal et al. (2021) study disclosure of awareness before initial public offerings (IPOs). An entrepreneur may disclose awareness any time before the IPO in which an investment bank sells stock to a continuum of heterogeneous traders. While prior research has shown that full disclosure of information minimizes the IPO, there are conditions under which full disclosure of awareness leads to a larger IPO as it essentially creates risk sharing opportunities.

This entire literature on financial markets with unawareness is naturally related to the mostly empirical literature on financial literacy and financial awareness. The empirical literature has focused more on awareness of assets (e.g., Guiso and Jappelli (2005)) while we focus on awareness of events affecting the value of an asset. With this focus, our work is related to notions of neglected risk. Gennaioli et al. (2012) consider a simple model in which an investor neglects the least likely of three states and which of the three states is neglected may change dramatically after observing some signal. Consequently, investors may overreact. We do not impose such a limited "budget constraint" on awareness. Our traders may discover previously neglect risks without neglecting risks they had previously considered. Finally, our model is related to the literature on financial markets in which traders neglect some informational content of prices. Zhou (2020) studies a very similar market game to ours. Different from out approach, he considers common awareness of all random variables among traders. However, traders differ in their sophistication of strategic reasoning. In particular, he considers a version of finite level-k reasoning where level-0 traders neglect the informational content of prices and level-k traders assume that other traders are level-$k-1$ traders. Momentum trading with linear strategies may also occur in his model. Moreover, the response of traders to private signals is the same irrespective of their level of sophistication. However, they differ in their response to price changes. This is similar to our model in which traders respond the same to signals irrespective of their awareness type but differ in response to prices. Eyster et al. (2019) simply assume that traders do not appreciate the information in prices to an extent fixed by an exogenous parameter. We differ from both Zhou (2020) and Eyster et al. (2019) in that we do not assume that some traders differ in their processing of information contained in prices but rather obtain it as a result. We show that asymmetric awareness of events affecting to the fundamental may lead traders react differently to prices. In our case, unaware traders in some sense overreact to prices.

Since we study a large competitive market under unawareness with a partially revealing rational expectations equilibrium, our work is related to the literature on general equilibrium under unawareness. Modica et al. (1998) study agents' inability to foresee all states of nature in an exchange economy. This may naturally lead to bankruptcy albeit unintentional. They illustrate that an equilibrium may not exist. Kawamura (2005) shows an extension to economies with production. Teeple (2021) assumes that despite being unaware, agents are on average correct, an assumption also used in the above mentioned work by Carvajal et al. (2021). With this assumption, equilibria exist despite unintended defaults and only partial delivery of asset promises via a pooling institution. In all those general equilibrium models with unawareness, the notion of equilibrium is somewhat problematic as agents may be surprised in equilibrium. It is therefore questionable in what sense the outcome can be understood as the result of an equilibrium learning and discovery process.[1] In our model, we do not assume that unaware traders are on average correct. While market clearing depends of awareness levels, no matter what the level of awareness of the market participant is, she can implicitly rationalize any observation in equilibrium because of the presence of noise traders.

[1]For a discussion of related issues with equilibrium in games with unawareness, see Schipper (2018).

2 Model

2.1 CARA-Normal Model with Unawareness

In this section, we outline a model with CARA utility and normal distributed random variables for informed trading in market for a single asset allowing for different awareness levels among informed traders. Consider a market with a single risky asset whose fundamental value is a random variable $\tilde{f} = \tilde{v} + \tilde{w} + c$ with $\tilde{v} \sim N(0, \sigma_v^2)$, $\tilde{w} \sim N(0, \sigma_w^2)$ and $c \neq 0$ being a constant. Random variables $\tilde{v}$ and $\tilde{w}$ are uncorrelated.

In the baseline model there are noise traders and a continuum of informed traders. Noise traders trade for exogenous liquidity reasons. Informed traders indexed by i come in four types distinguished by their awareness levels. Denote by M_ℓ the set of traders of awareness level ℓ and by m_ℓ the measure of traders of awareness level ℓ for $\ell \in \{1, 2, 3, 4\}$. We normalize $M_1 \cup M_2 \cup M_3 \cup M_4 = [0, 1]$ and $m_1 + m_2 + m_3 + m_4 = 1$.

0. The liquidity traders' demand for the asset is equal to a random variable $\tilde{z} \sim N(0, \sigma_z^2)$. We assume that this random variable is independent of all other primitive random variables. The expected volume of noise trading $\mathbb{E}[|\tilde{z}|] = \sqrt{\frac{2}{\pi}}\sigma_z$ is proportional to its standard deviation σ_z.

1. Any trader with awareness level 1 is only aware of fundamental random variable $\tilde{v}$ but unaware of $\tilde{w}$ and c. Thus, she perceives the fundamental to be $\tilde{f}_1 := \tilde{v}$. Trader $i \in M_1$ also receives a private information signal $\tilde{s}_i$. She understands that this signal depends on the fundamental value and an idiosyncratic noise random variable $\tilde{\varepsilon}_i$. We assume that $\tilde{\varepsilon}_i \sim N(0, \sigma_\varepsilon^2)$. Moreover, we assume that $\tilde{\varepsilon}_i$ is i.i.d. across all traders $i \in [0, 1]$ no matter their awareness level. Thus, trader $i \in M_1$ interprets the signal as $\tilde{s}_i := \tilde{f}_1 + \tilde{\varepsilon}_i$.

2. Any trader with awareness level 2 is aware of everything that traders with awareness level 1 are aware. In addition they are also aware of the constant c. Thus, they perceive the fundamental value to be $\tilde{f}_2 := \tilde{v} + c$. Compared to traders of awareness level 1, they perceive the same volatility but also a shift in the mean of the fundamental value. Each trader $i \in M_2$ also receives a private information signal that she interprets as $\tilde{s}_i := \tilde{f}_2 + \tilde{\varepsilon}_i = \tilde{v} + c + \tilde{\varepsilon}_i$. As before, $\tilde{\varepsilon}_i \sim N(0, \sigma_\varepsilon^2)$ is i.d.d.

3. Any trade with awareness level 3 is aware of everything that traders with awareness level 1 are aware. In addition they are also aware of random variable $\tilde{w}$. Thus, they perceive the fundamental value to be $\tilde{f}_3 := \tilde{v} + \tilde{w}$. Compared to traders with awareness level 1, they perceive the same mean but have the full picture of the variance of the fundamental value. Each trader in $i \in M_3$ also receives a private information signal that she interprets as $\tilde{s}_i := \tilde{f}_3 + \tilde{\varepsilon}_i = \tilde{v} + \tilde{w} + \tilde{\varepsilon}_i$. As before, $\tilde{\varepsilon}_i \sim N(0, \sigma_\varepsilon^2)$ is i.i.d.

4. Any trader with awareness level 4 is aware of everything that other traders are aware. Thus, they perceive the fundamental value to be $\tilde{f}_4 := \tilde{v} + \tilde{w} + c$. I.e., they both have the full picture of the variance and the mean of the fundamental value. Each trader in $i \in M_4$ also receives a private information signal she interprets as $\tilde{s}_i := \tilde{f}_4 + \tilde{\varepsilon}_i = \tilde{v} + \tilde{w} + c + \varepsilon_i$, where as for the other traders $\tilde{\varepsilon}_i \sim N(0, \sigma_\varepsilon^2)$ is i.i.d.

The idea that traders not being aware of all components of the fundamental value may be motivated by interpreting the fundamental value as equity of a firm whose stock is traded in the asset market. When traders form beliefs about the equity, they need to take into account all line items of the balance sheet and the profit and loss statement. These line items are then summed up (liabilities and losses with negative sign) into the equity. Thus, we consider the fundamental value as a sum of random variables. It is realistic that insiders may not just have better information about some line items but also be aware of some items that others may not even conceive of. This is our motivation for modelling unawareness of some terms in the sum of random variables making up the fundamental value.

Note that we assume that all traders receive information signals with the same precision regardless of the awareness type. We do not want to confound effects due to differences in awareness by introducing further asymmetries in information.

We assume further that all primitive random variables have finite variance. I.e., $\sigma_v^2, \sigma_w^2, \sigma_\varepsilon^2 < \infty$. Since ε_n is also i.i.d., this assumption allows us to invoke the Strong Law of Large Numbers to conclude that for all $\ell \in \{1, 2, 3, 4\}$,

$$\frac{1}{m_\ell} \int_{M_\ell} \tilde{s}_i di = \frac{1}{m_j} \int_{M_\ell} (f_\ell + \tilde{\varepsilon}_i) di \overset{a.s.}{\to} f_\ell. \tag{1}$$

We follow the notational convention to denote with a tilde a random variable and without tilde a realization of the random variable. E.g., $\tilde{v}$ is the random variable and v is a realization of $\tilde{v}$.

All informed traders have non-random initial endowments which are normalized to zero. Denote by x_i the demand of trader i and p be the price of the asset in the market faced by the trader. A trader i with awareness level ℓ perceives her ex post return from trading x_i at price p when the fundamental value is f_ℓ to be $(f_\ell - p)x_i$. Since we focus on the differences due to awareness, we assume that all traders have identical CARA utility function over returns no matter their awareness level. I.e., i with awareness level ℓ perceives her ex post utility from trading x_i at price p when the fundamental value is f_ℓ to be

$$U((f_\ell - p_n)x_i) \quad = \quad -e^{-\rho(f_\ell - p)x_i} \tag{2}$$

where $\rho > 0$ is the coefficient of absolute risk aversion. Assuming that risk aversion is unaffected by awareness level is a common assumption in the theoretical literature on unawareness (e.g., Heifetz et al. (2013a), Karni and Vierø (2013), Dominiak and Tserenjigmid (2018) etc.); see Ma and Schipper (2017) for some preliminary experimental evidence with small risks.

Consistent with unawareness type spaces introduced by Heifetz et al. (2013b), traders form beliefs about each other's awareness type subject to their own awareness. The unawareness type space is presented in Figure 1. There are four spaces of awareness types that are ordered by their richness. In the upmost space there are all four awareness types 1, 2, 3, and 4. Awareness type 4 is aware of all types. Thus, the belief of a trader of awareness type 4 over other awareness types is given simply by the prior $m^4 = (m_1, m_2, m_3, m_4)$ on the upmost space. Awareness type 2 is aware that traders can be of her own awareness type or of lower awareness type, that is awareness type 1. She is unaware of traders can be awareness type 3 and 4. Yet, her perception is correct in the sense as she views any trader

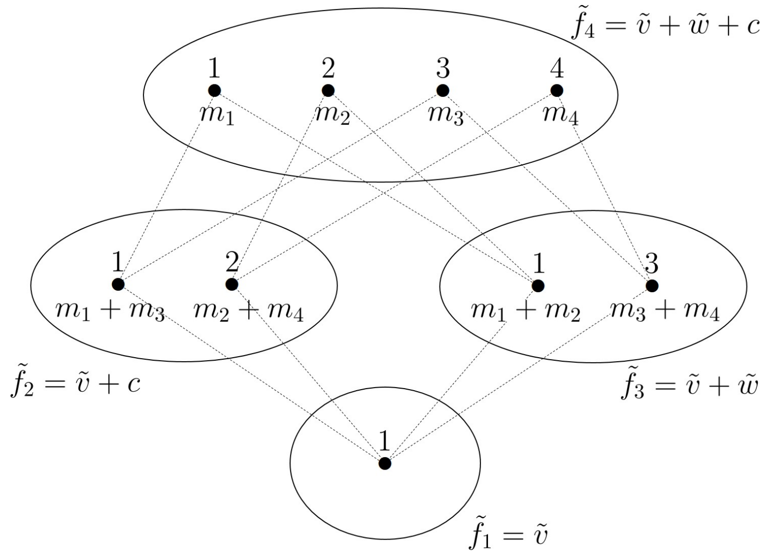

Figure 1: Unawareness Type Space

of awareness type 4 as a trader of awareness type 2, i.e., she just misses what awareness type 4 is aware of beyond her own awareness. Moreover, she views any trade of awareness type 3 as a trader of awareness type 1 because she realizes that those traders are not aware of what she is aware and is unaware of what these traders are aware beyond her own awareness. Thus, she considers $M_1^2 = M_1 \cup M_3$ and $M_2^2 = M_2 \cup M_4$. The belief of a trader of awareness type 2 over other awareness types is given by the "marginal" of the prior on awareness types 1 and 2, i.e., $m^2 = (m_1 + m_3, m_2 + m_4)$ on the left space in Figure 1. Projections from higher spaces to lower spaces are indicated with dashed lines. Similarly, awareness type 3 considers $M_1^3 = M_1 \cup M_2$ and $M_3^3 = M_3 + M_4$ and has the belief $m^3 = (m_1 + m_2, m_3 + m_4)$ on the right space. Finally, awareness type 1 is unaware of other awareness types and believes that all traders are of her own awareness type, i.e., $M_1^1 = M_1 \cup M_2 \cup M_3 \cup M_4$ and $m^1 = m_1 + m_2 + m_3 + m_4 = 1$. In Figure 1 we also print for easy reference beside each space how each awareness type perceives the fundamental value of the asset.

All distributional assumptions on primitive random variables are common knowledge among traders for those random variables of which they are aware of, respectively.

2.2 Market Game and Solution

We consider a simultaneous game with incomplete information and unawareness as follows: The players are the informed traders. Players have awareness types such that m_ℓ is the measure of players with awareness type $\ell \in \{1, 2, 3, 4\}$. Each player i is also endowed with an information signal realization s_i. (Thus, we may view (ℓ, s_i) as the "type" of player i.) Player i's strategy is an awareness-dependent map from her signals to demand functions. A demand function maps prices to quantities demanded. We write $X_\ell(i, s_i)$ for the demand function of trader i with awareness level $\ell \in \{1, 2, 3, 4\}$. $X_\ell(i, s_i)(p)$ is the quantity demanded by trader i with awareness level ℓ when the price is p. We let X_1 be defined for all $i \in M_1^1 = M_1 \cup M_2 \cup M_3 \cup M_4$, X_2 be defined for all $i \in M_2^2 = M_2 \cup M_4$, X_3 be defined for all

$i \in M_3^3 = M_3 \cup M_4$, and $X_{i,4}$ be defined for all $i \in M_4$. This is because for instance we need X_1 defined for $i \in M_2$ for any trader with awareness level 1 or 3. A strategy is not just a device for modelling what players might play for any realization of their signal but also an object of the opponents' beliefs and perception. For instance, a trader with awareness level 2 perceives any trader in $i \in M_3$ as being of awareness level 1. When such a trader forms beliefs about the behavior of latter, she considers X_1 and not X_3 since she is unaware of awareness type 3. We require that X_ℓ is continuous both in signals and prices for every i for which it is defined (when interpreting X_ℓ as mapping from signals and prices into quantities demanded for every i for which it is defined).[2]

The game proceeds as follows. At the interim stage, after each player i has received her signal realization s_i, she submits a demand schedule $X_\ell(i, s_i)$ to the auctioneer, who is not considered as a player but part of the institutional rules of the market game. At the same time, aggregate demands by noise traders are realized z according to $\tilde{z}$ and submitted to the auctioneer. The auctioneer selects an public market clearing price p using equation:

$$\sum_{\ell \in \{1,2,3,4\}} \int_{M_\ell} X_\ell(i, s_i)(p) di + z \quad = \quad 0, \tag{3}$$

if possible. If the market clearing price exists but is not unique, he chooses the one with the minimum absolute value (which exists since demand schedules are upper hemi-continuous) and the positive one if there is a negative market price with the same absolute value and both having the minimum absolute value. If the market clearing price does not exist, then there is positive excess demand at all prices or negative excess demand at all prices. In the former case, the auctioneer selects the price $p = \infty$. All buyers with bounded quantities obtain negative infinite utility. In latter case, the auctioneer selects the price $p = -\infty$ and all sellers with bounded quantities obtain negative infinite utility.

While the basic workings of the market are common knowledge among all traders, their perception of market clearing differs depending on their awareness. A trader with awareness level 4 understands that if a market clearing price exists, it satisfies market clearing equation (3). In contrast, awareness types 1, 2, and 3 perceive the market clearing price to satisfy, respectively,

$$\int_{[0,1]} X_1(i, s_i)(p) di + z \quad = \quad 0 \tag{4}$$

$$\int_{M_1 \cup M_3} X_1(i, s_i)(p) di + \int_{M_2 \cup M_4} X_2(i, s_i)(p) di + z \quad = \quad 0 \tag{5}$$

$$\int_{M_1 \cup M_2} X_1(i, s_i)(p) di + \int_{M_3 \cup M_4} X_3(i, s_i)(p) di + z \quad = \quad 0 \tag{6}$$

This is because they can only perceive that traders have their own or a lower awareness level. Recall that for instance we defined X_1 even if $i \in M_4$ because as we see i's strategy

[2] More generally, we could allow X_ℓ to be a nonempty convex-valued correspondence that is continuous in both signals and prices for i. Then in below description of market clearing, the auctioneer would need to pick a continuous selection for i and awareness level ℓ and this selection should be commonly known among all traders whose awareness level is as least as high ℓ. Since we will focus on linear demand functions anyway, we restrict the exposition to demand functions. This eases notation.

at awareness level 1 is important for the perception of market clearing for any trader of awareness type 1.

Since the perception of market clearing depends on a trader's awareness level, the market clearing price and the actual quantities demanded become awareness-dependent random variables. Each trader of awareness level 1 perceives the price as random variable $\tilde{p}_1(X_1)$. Each trader of awareness level 2 perceives the price as a random variable $\tilde{p}_2(X_2, X_1)$. Each trader of awareness level 3 perceives the price as a random variable $\tilde{p}_3(X_3, X_1)$. Finally, trader of awareness level 4 perceives the price as a random variable $\tilde{p}_4(X_4, X_3, X_2, X_1)$. With some abuse of notation, we drop the subscript from the price random variables.

Definition 1 *A Bayes-Nash equilibrium of the game with incomplete information and unawareness defined above is a profile of strategies $(X_\ell^*)_{\ell \in \{1,2,3,4\}}$ such that*

(4) for all $i \in M_4$, s_i, and X_4,

$$\begin{aligned} &\mathbb{E}[U((\tilde{f}_4 - \tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*))X_4^*(i, s_i)(\tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*))] \\ \geq\ &\mathbb{E}[U((\tilde{f}_4 - \tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*))X_4(i, s_i)(\tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*))] \end{aligned} \tag{7}$$

(3) for all $i \in M_3 \cup M_4$, s_i, and X_3,

$$\begin{aligned} &\mathbb{E}[U((\tilde{f}_3 - \tilde{p}(X_3^*, X_1^*))X_3^*(i, s_i)(\tilde{p}(X_3^*, X_1^*))] \\ \geq\ &\mathbb{E}[U((\tilde{f}_3 - \tilde{p}(X_3^*, X_1^*))X_3(i, s_i)(\tilde{p}(X_3^*, X_1^*))] \end{aligned} \tag{8}$$

(2) for all $i \in M_2 \cup M_4$, s_i, and X_2,

$$\begin{aligned} &\mathbb{E}[U((\tilde{f}_2 - \tilde{p}(X_2^*, X_1^*))X_2^*(i, s_i)(\tilde{p}(X_2^*, X_1^*))] \\ \geq\ &\mathbb{E}[U((\tilde{f}_2 - \tilde{p}(X_2^*, X_1^*))X_2(i, s_i)(\tilde{p}(X_2^*, X_1^*))] \end{aligned} \tag{9}$$

(1) for all $i \in M_1 \cup M_2 \cup M_3 \cup M_4$, s_i, and X_1,

$$\mathbb{E}[U((\tilde{f}_1 - \tilde{p}(X_1^*))X_1^*(i, s_i)(\tilde{p}(X_1^*))] \geq \mathbb{E}[U((\tilde{f}_1 - \tilde{p}(X_1^*))X_1(i, s_i)(\tilde{p}(X_1^*))] \tag{10}$$

A Bayes-Nash equilibrium is symmetric if for any ℓ equilibrium strategies $X_\ell^*(i, \cdot)$ are constant in $i \in M_\ell$.

We interpret the Bayes-Nash equilibrium as Nash equilibrium of a strategic game in which players are triples (ℓ, i, s_i), actions are demand functions, and payoff functions are the expected utility functions.

Note that since there is a continuum of traders, none of them perceives an influence on the price when deviating to another strategy X_ℓ.

In a symmetric Bayes-Nash equilibrium all traders who are perceived to have the same awareness level play the same equilibrium strategy.

The Bayes-Nash equilibrium is an instance of general Bayes-Nash equilibrium defined for Bayesian games with unawareness in Meier and Schipper (2014a) except that we allow for a continuum of players. Generally, Bayes-Nash equilibria may involve surprises in games with unawareness (Schipper (2018)). In the present case, it is best interpreted as a prediction in the medium run. Because of noise traders, any individual observation of own quantity and market price is not inconsistent with the trader's awareness level. In the medium run, she can always rationalize an observation with some realization of the noise trader's demand. In the very long run though, when she may be able to learn also about the statistical relationship between the fundamental, prices, and her own quantities. Consequently, she might find out eventually that her beliefs over these random variables were wrong. There is still no reason for her to question whether she is unaware of something. Rather, in the very long run she may "correct" her beliefs to some biased beliefs akin to an omitted variable bias. The discussion implies that under unawareness, Bayes-Nash equilibria of the market game specified above loose the property of (long-run) ex post optimality of Bayes-Nash equilibria in Kyle (1989, p. 323). Yet, since as we argued the Bayes-Nash equilibrium under unawareness is best understood as a prediction in the medium-run, it is still a useful solution concept because we should be interested in predictions of trading by players who face unawareness but have not yet had sufficient time to learn about the exact statistical relationship between the observed variables.

A Bayes-Nash equilibrium of the market game can be understood as a particular rational expectations equilibrium. The rational expectations equilibrium is only partially revealing because of noise trading. Moreover, it is well-known that fully-revealing rational equilibrium cannot be implemented strategically in market games as defined above even without unawareness (see Vives (2008, Chapter 4.2)).

The following proposition characterizes the symmetric Bayes-Nash equilibrium in linear strategies. For any random variable $\tilde{y}$ denote by τ_y the precision of y, i.e., the inverse of the variance $\tau_y := \frac{1}{\sigma_y^2}$.

Proposition 1 *There exists a unique Bayes-Nash equilibrium in linear strategies that is symmetric. It is characterized by:*

1. *The equilibrium strategies of informed traders of awareness type 1 are given by*

$$X_1^*(s_i)(p) = \beta_1 s_i - \gamma_1 p \tag{11}$$

with

$$\beta_1 = \frac{\tau_{\varepsilon_i}}{\rho} \tag{12}$$

$$\gamma_1 = \frac{\left(\tau_v + \tau_{\varepsilon_i} + \left(\frac{\tau_{\varepsilon_i}}{\rho}\right)^2 \tau_z\right)\rho}{\tau_{\varepsilon_i}\tau_z + \rho^2} \tag{13}$$

2. *The equilibrium strategies of informed traders of awareness type 2 are given by*

$$X_2^*(s_i)(p) = \alpha_2 + \beta_2 s_i - \gamma_2 p \tag{14}$$

with

$$\alpha_2 = c\frac{\tau_v \rho}{(m_2 + m_4)\tau_{\varepsilon_i}\tau_z + \rho^2} \tag{15}$$

$$\beta_2 = \beta_1 \tag{16}$$

$$\gamma_2 = \gamma_1 \tag{17}$$

3. *The equilibrium strategies of informed traders of awareness type 3 are given by*

$$X_3^*(s_i)(p) = \beta_3 s_i - \gamma_3 p \tag{18}$$

with

$$\beta_3 = \beta_1 \tag{19}$$

$$\gamma_3 = \frac{\left(\tau_{v+w} + \tau_{\varepsilon_i} + \left(\frac{\tau_{\varepsilon_i}}{\rho}\right)^2 \tau_z\right)\rho}{(m_3+m_4)\tau_{\varepsilon_i}\tau_z + \rho^2} - \frac{\tau_{\varepsilon_i}\tau_z}{(m_3+m_4)\tau_{\varepsilon_i}\tau_z + \rho^2}(m_1+m_2)\gamma_1 \tag{20}$$

4. *The equilibrium strategies of informed traders of awareness type 4 are given by*

$$X_4^*(s_i)(p) = \alpha_4 + \beta_4 s_i - \gamma_4 p \tag{21}$$

with

$$\alpha_4 = c\frac{(m_2+m_4)\tau_{v+w}\tau_{\varepsilon_i}\tau_z\rho + \tau_{v+w}\rho^3}{(m_4\tau_{\varepsilon_i}\tau_z + \rho^2)((m_2+m_4)\tau_{\varepsilon_i}\tau_z + \rho^2)} - \frac{\tau_{v+w}\tau_{\varepsilon_i}\tau_z}{m_4\tau_{\varepsilon_i}\tau_z + \rho^2}\alpha_2 \tag{22}$$

$$\beta_4 = \beta_1 \tag{23}$$

$$\gamma_4 = \gamma_3 \tag{24}$$

The proof is contained in the appendix. To characterize the symmetric Bayesian Nash equilibrium in linear strategies, we make use of the fact that informed traders with lower awareness levels do not react to informed traders with higher awareness levels since former are unaware of latter's awareness. Thus, Bayes-Nash equilibrium is constructed from lower awareness levels "upward" (Meier and Schipper (2014a, Proposition 2)). Starting with the lowest awareness, the proof for each level is then similar to the standard proof (e.g. Vives (2008, Proposition 4.1)) demonstrating that adding asymmetric unawareness to market microstructure models do not make them intractable. Assuming strategies are linear, we use the appropriate market clearing condition to derive an equation for the price random variable given equilibrium strategies. Then we use an implication of the projection theorem for normally distributed variables to obtain solutions for the conditional variance and the expectation of the fundamental (conditional on individual signal realization and the realization of the price). The expected CARA utility models takes on a mean-variance form. After deriving first-order conditions and plugging in the terms from the mean-variance form, we identify the coefficients of the linear strategies. At the next higher awareness level, traders take these strategies into account when considering market clearing and the information contained in the price. Yet, they realize that traders at the lower awareness level misinterpret their signal because they are unaware of some term(s) of the fundamental.

In above characterization of the symmetric linear Bayes-Nash equilibrium, the parameter capturing the sensitivity to the signal, β_ℓ, is identical across all awareness types $\ell \in \{1, 2, 3, 4\}$. At a first glance, this is somewhat counterintuitive as traders with different awareness levels may interpret signals differently. The larger the variance of the idiosyncratic noise in the signal and the larger the risk aversion of the traders, the less they react to the signal. All informed traders have a common understanding of the noise in the signal, no matter their awareness level. Although they misperceive components of the fundamental value and thus terms that additively make up parts of the signal, they do not realize this because they are unaware of those terms depending on their awareness level. Thus, they all react similar to the signal.

The parameter capturing the sensitivity to the price is identical across awareness types 1 and 2. This is because compared to awareness type 2, awareness type 1 just misses a constant term of the fundamental value. Yet, both perceive the variance of the fundamental to be the same. As idiosyncratic noise is washed out by the Strong Law of Large Numbers and both also share the same understanding of the noise trader's demands, both types also decode the information capturing in the price in the same fashion. (This is shown formally in the proof contained in the appendix.) The comparative statics of awareness types 1 and 2 sensitivity to price w.r.t. precisions of random variables and coefficient of risk aversion is stated more formally below. The straightforward proof is omitted.

Proposition 2 *In the symmetric linear Bayes-Nash equilibrium:*

$$\frac{\partial \gamma_1}{\partial \tau_v} > 0 \tag{25}$$

$$\frac{\partial \gamma_1}{\partial \tau_{\varepsilon_i}} > 0 \text{ iff } \tau_v < \frac{(\tau_{\varepsilon_i}\tau_z + \rho^2)^2}{\tau_z \rho^2} \tag{26}$$

$$\frac{\partial \gamma_1}{\partial \tau_z} < 0 \tag{27}$$

$$\frac{\partial \gamma_1}{\partial \rho} < 0 \text{ iff } \rho^2 \geq \tau_{\varepsilon_i}\tau_z \text{ or } \left\{\rho^2 < \tau_{\varepsilon_i}\tau_z \text{ and } \tau_v < \frac{(\tau_{\varepsilon_i}\tau_z + \rho^2)^2 \tau_{\varepsilon_i}}{(\tau_{\varepsilon_i}\tau_z - \rho^2)\rho^2}\right\} \tag{28}$$

Moreover, γ_1 is increasing in the variance of the perceived fundamental v conditional on information signal and price.

The last claim in Proposition 2 is immediate from the proof of Proposition 1 in the appendix where we derive the precision (i.e., reciprocal of the variance) of the perceived fundamental, v, conditional on signal and price,

$$\tau_v + \tau_{\varepsilon_i} + \left(\frac{\tau_{\varepsilon_i}}{\rho}\right)^2 \tau_z,$$

as a consequence of the projection theorem applied to normally distributed variables.

Being aware of the constant term in the fundamental affects nevertheless trading strategies through the introduction of an intercept. That is, while informed traders of awareness level 2 have the same sensitivity to the price and the signal as informed traders of awareness level 1, former have a shifted version of latter's linear strategy. The intercept of awareness type 2's strategy also depends on the $m_2^2 := m_2 + m_4$, the measure of informed traders perceived by awareness type 2 as being of awareness type 2. In a standard model without unawareness, we would have $\beta_2 + \frac{\alpha_2}{c} = \gamma_2$ (e.g., Vives (2008, Proposition 4.1)). This does not hold in our model since traders with awareness level 1 do not consider the fundamental's constant component c and traders with awareness level 2 realize this. Only when $m_2 + m_4 = 1$ (i.e., no traders are unaware of c), above equality of parameters holds.

The comparative statics of awareness type 2's strategy intercept w.r.t. precisions of random variables and coefficient of risk aversion is stated more formally below. The straightforward proof is omitted.

Proposition 3 *In the symmetric linear Bayes-Nash equilibrium:*

$$\frac{\partial \alpha_2}{\partial c} > 0 \tag{29}$$

$$\frac{\partial \alpha_2}{\partial \tau_v} > 0 \tag{30}$$

$$\frac{\partial \alpha_2}{\partial \tau_{\varepsilon_i}} < 0 \tag{31}$$

$$\frac{\partial \alpha_2}{\partial \tau_z} < 0 \tag{32}$$

$$\frac{\partial \alpha_2}{\partial \rho} < 0 \text{ iff } \rho^2 \geq \tau_{\varepsilon_i}\tau_z(m_2 + m_4) \tag{33}$$

$$\frac{\partial \alpha_2}{\partial m_2^2} < 0 \tag{34}$$

Awareness type 4 is also aware of the constant term c contributing to the fundamental value. Consequently, her strategy also features an intercept. The proof on the following sufficient condition for ordering the intercepts is contained in the appendix.

Proposition 4 *In the symmetric linear Bayes-Nash equilibrium, $m_2 \leq \tau_v$ implies $\alpha_4 < \alpha_2$.*

The identical sensitivity parameter to the price in the linear equilibrium strategy of both awareness types 1 and 2 differs from the parameters of awareness types 3 and 4 because latter conceive also of an additional random term of the fundamental for which the price is informative as well. We can order price sensitivity parameters as follows:

Proposition 5 *In symmetric linear Bayes-Nash equilibrium, awareness types 3 and 4 are less sensitive to price changes than awareness types 1 and 2, i.e., $\gamma_3 = \gamma_4 < \gamma_1 = \gamma_2$.*

The proof is contained in the appendix.

While $\gamma_1 = \gamma_2$ is always positive, hence awareness types 1 and 2 always have downward sloping demand functions, the functional form of γ_3 and γ_4 leaves open the possibility of upward sloping demand curves. Strategies with upward and downward sloping demand curves have been called *momentum* and *contrarian* strategies, respectively, by Zhou (2020). Although our model is static, momentum traders' asset demand is increasing in price similar to trend-chasing strategies in dynamic markets.

Proposition 6 *In the symmetric linear Bayes-Nash equilibrium with a positive share of awareness type 1 traders, informed traders of awareness type 3 or 4 are momentum traders if the variance of the additional term of the fundamental that they are aware of is sufficiently larger than the variance of the term that all awareness types are aware.*

The proof is contained in the appendix.

The comparative statics of $\gamma_3 = \gamma_4$ w.r.t. parameters of the game are given as follows: The straightforward proof is omitted.

Proposition 7 *In the symmetric linear Bayes-Nash equilibrium:*

$$\begin{aligned}
\frac{\partial\gamma_3}{\tau_{v+w}} &> 0 && (35)\\
\frac{\partial\gamma_3}{\tau_{\varepsilon_i}} &> 0 \text{ if } \tau_{\varepsilon_i}\tau_z < \rho^2 \text{ and } \tau_v < \frac{(\tau_{\varepsilon_i}\tau_z + \rho^2)^2}{\tau_z\rho^2} && (36)\\
\frac{\partial\gamma_3}{\tau_z} &< 0 \text{ if } \tau_{\varepsilon_i}\tau_z \leq \rho^2 && (37)\\
\frac{\partial\gamma_3}{\rho} &< 0 \text{ if } \tau_{\varepsilon_i}\tau_z \leq \rho^2 \text{ and sufficiently small } \tau_v && (38)
\end{aligned}$$

As usual we define:

Definition 2 (Market depth) Market depth *is the reciprocate of the mean price sensitivity in the market,*

$$\lambda_1 := \frac{1}{\gamma_1}; \text{ for } \ell = 2,3, \lambda_\ell := \frac{1}{(m_1 + m_{5-\ell})\gamma_1 + (m_\ell + m_4)\gamma_\ell}; \ \lambda_4 := \frac{1}{\sum_{\ell=1}^4 m_\ell\gamma_\ell}.$$

Intuitively, a market is deep if the effect of noise trading on the price is small. Since the reciprocate of the mean price sensitivity is the parameter on the noise trader's demand, it is a convenient measure of market depth. λ_ℓ is the market depth if the informed trader with the highest awareness level has awareness level ℓ, for $\ell = 1,2,3,4$. Alternatively, λ_ℓ can be interpreted as the market depth perceived by informed traders with awareness level ℓ (even though their could be informed traders with awareness level in comparable or higher than ℓ).

Corollary 1 *In the symmetric linear Bayes-Nash equilibrium,*

$$\lambda_1 = \lambda_2 < \lambda_3 = \lambda_4.$$

Roughly, in markets with informed traders who are aware of additional components affecting the variance of the fundamental, market depth is higher.

Another commonly used market quality parameter is price precision.

Definition 3 (Price precision) Price precision *is defined for $\ell = 1,2,3,4$ by*

$$\eta_1 := \tau_{f_1} + \beta_1^2\tau_z; \text{ for } \ell = 2,3, \eta_\ell := \tau_{f_\ell} + ((m_1+m_{5-\ell})\beta_1 + (m_\ell+m_4)\beta_\ell)^2\tau_z; \eta_4 := \tau_{f_4} + \left(\sum_{\ell=1}^4 m_\ell\beta_\ell\right)^2 \tau_z.$$

We interpret η_ℓ as the price precision in a market in which the informed trader with the highest awareness level has awareness level ℓ, for $\ell = 1,2,3,4$. Alternatively, we can interpret it as the price precision perceived by an informed trader with awareness level ℓ. When price precision grows large, prices become more and more revealing. So in some sense, it is a measure of how close we are to fully revealing rational expectations equilibrium.

Corollary 2 *In symmetric linear Bayes-Nash equilibrium,*

$$\eta_1 = \eta_2 > \eta_3 = \eta_4.$$

Note how more aware traders affect price precision differently from more informed traders. It is well known that the larger the share of informed traders and the better their information, the higher is the price precision. We find the larger the share of more aware traders and the higher their awareness level, the lower is the price precision. This is because higher awareness means awareness of additional components contributing to the variance of the fundamental.

We now turn to price volatility as measured by the ex ante variance of the prices. Depending on their awareness level, informed traders have different perceptions of the price, a random variable, since it aggregates information about the fundamental value, which in turns depends on the informed traders' awareness level. E.g., $\tilde{p}(X_2^*, X_1^*)$ is the price random variable emanating from equilibrium strategies of informed traders with awareness levels 2 and 1. This is the price random variable perceived by a trader with awareness level 2. Alternatively, we can interpret this as the actual random variable of the price when we the market consists of informed traders with awareness level at most 2.

Proposition 8 *In symmetric linear Bayes-Nash equilibrium,*

$$Var(\tilde{p}(X_1^*)) = Var(\tilde{p}(X_2^*, X_1^*)) = \lambda_1^2 \left(\left(\frac{\tau_{\varepsilon_i}}{\rho} \right)^2 \sigma_v^2 + \sigma_z^2 \right) <$$
$$Var(\tilde{p}(X_3^*, X_1^*)) = Var(\tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*)) = \lambda_3^2 \left(\left(\frac{\tau_{\varepsilon_i}}{\rho} \right)^2 \sigma_v^2 + \left(\frac{\tau_{\varepsilon_i}}{\rho} \right)^2 \sigma_w^2 + \sigma_z^2 \right)$$

We omit the proof with the straightforward calculations. Note that the higher price volatility perceived by traders with awareness levels 3 and 4 is also due to the higher market depth, which is also affect by variance of the fundamental's additional component $\tilde{w}$. We note that the ex ante expected price of traders with awareness levels 1 and 3 is zero while it is strictly positive for traders with awareness levels 2 and 4. Latter is due to the fact that they perceive also of the non-zero constant component of the fundamental.

Now suppose that all awareness types have equal measure. I.e., there is an equal share of awareness types of traders in the market. Which awareness type would trade most in expectations? For each type, the ex ante expected trading quantity follows a normal distribution since strategies are a linear combination of normally distributed variables. In order to avoid that demand and supply of a type cancels each other out, we use the absolute value of trading quantities (e.g., as in Vives (2008, p. 121)). Those follow folded normal distributions.

Definition 4 *Ex ante expected trading volume is given by*

$$\nu_1 \ := \ \mathbb{E}\left[\left| \int_{M_1} X_1^*(\tilde{s}_i)(\tilde{p}(X_1^*) d_i \right|\right] \tag{39}$$

$$\nu_\ell \ := \ \mathbb{E}\left[\left| \int_{M_\ell} X_\ell^*(\tilde{s}_i)(\tilde{p}(X_\ell^*, X_1^*) d_i \right|\right], \ \textit{for } \ell = 2, 3 \tag{40}$$

$$\nu_4 \ := \ \mathbb{E}\left[\left| \int_{M_4} X_4^*(\tilde{s}_i)(\tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*) d_i \right|\right] \tag{41}$$

The proof of the following observation is contained in the appendix:

Proposition 9 *In symmetric linear Bayes-Nash equilibrium, $\nu_1 < \nu_2$. We are unable to compare trading volume of other awareness types without further assumptions.*

The folded normal distribution of traded quantities of types 2 and 4 has not mean zero. So the error function needs to be used in when computing the expectations of absolute values of traded quantities.

We finish this section with observing some comparative statics of market quality w.r.t. changes in the measures of awareness levels. These observations are straightforward implications of prior results.

Corollary 3 *In linear symmetric Bayes-Nash equilibrium, market depth λ_3 and λ_4 increases in $m_3 + m_4$. λ_1 and λ_2 stay constant.*

Corollary 4 *In linear symmetric Bayes-Nash equilibrium, price precision η_1, η_2, η_3 and η_4 stay constant in changes of m_1, m_2, m_3, and m_4.*

Corollary 5 *In linear symmetric Bayes-Nash equilibrium, price volatility of $\tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*)$ and $\tilde{p}(X_3^*, X_1^*)$ increase in $m_3 + m_4$. $\tilde{p}(X_2^*, X_1^*)$ and $\tilde{p}(X_1^*)$ stay constant.*

3 Incentives to Raise Awareness

In this section, we consider incentives of aware traders to raise the awareness of unaware traders. Would traders of with more awareness have an incentive to raise the awareness of traders with less awareness and if yes for which signals and prices?

To answer such a question, we introduce a simplification similar to Grossman and Stiglitz (1980) (see Vives (2008, Chapter 4.2.2)), namely the signal is now commonly observed among all traders. While Grossman and Stiglitz (1980) used this assumption to facilitate studying incentives for individual information acquisition, we use a version with asymmetric awareness to study the incentives for public disclosure of awareness by insiders. Eliminating private information allows us to focus on private awareness instead and simplifies the analysis.

Assume that the fundamental value of the asset is $\tilde{f} = \tilde{v} + \tilde{w}$, where $\tilde{v} \sim \mathcal{N}(0, \sigma_v^2)$ is normally distributed with mean 0 and variance σ_v^2. The second component $\tilde{w} \sim \mathcal{N}(\bar{w}, \sigma_w^2)$ follows a normal distribution with mean $\bar{w}$ and variance σ_w^2. Random variables $\tilde{v}$ and $\tilde{w}$ are drawn independently.

There is a continuum each for two types of traders in the market: Type A agents are (A)ware that the distribution of $\tilde{f}$ is the sum of $\tilde{v}$ and $\tilde{w}$. However, type B agents are unaware of $\tilde{w}$ and consider only $\tilde{v}$. Denote $\tilde{f}_A := \tilde{v} + \tilde{w}$ and $\tilde{f}_B := \tilde{v}$ the fundamental value as perceived by types A and B, respectively. Agents of type B are comparable to agents of type 1 in the prior section. Agents of type A are comparable to agents of type 4 in the prior section (rather than type 3). This is because the additional random variable $\tilde{w}$ that agents of type A are aware of is now not assumed to necessarily have mean zero. That is, random variable $\tilde{w}$ in this section corresponds to random variable $\tilde{w} + c$ in the prior section.

We let M_A and M_B denote the sets of traders of type A and B, respectively. Analogously, we denote by m_A and m_B the measures of the two types of traders, respectively. Assume $m_A, m_B > 0$. We normalize $M_A \cup M_B = [0,1]$ and $m_A + m_B = 1$.

All agents observe a common signal s. They perceive this signal equal to the sum of their perceived fundamental value of the asset plus an independent normally distributed noise $\tilde{\varepsilon} \sim \mathcal{N}(0, \sigma_\varepsilon^2)$ with mean zero. The interpretation is that all traders observe for instance the *same* earnings announcement. However, due to the difference in their awareness, they interpret the signal in different ways. Type A traders interpret it as $s \equiv \tilde{v} + \tilde{w} + \tilde{\varepsilon}$, while type B traders interpret it as $s \equiv \tilde{v} + \tilde{\varepsilon}$ as a result of their unawareness of $\tilde{w}$. Since the signal is common, nobody has private information. "Informed" traders just differ by their awareness. The unawareness type space is similar to Figure 1 except that it just consists of two spaces (see Figure 2).

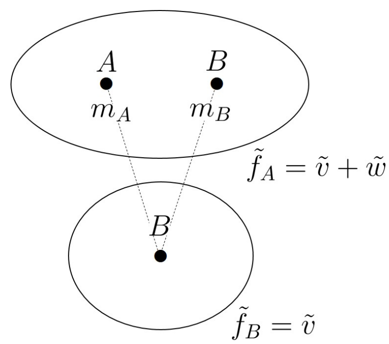

Figure 2: Unawareness Type Space in the Simplified Model

Again, we assume as in the previous section that all primitive random variables have finite variance, i.e., $\sigma_v^2, \sigma_w^2, \sigma_\varepsilon^2 < \infty$. We also continue to assume that the ex post utility function of each trader is CARA with identical coefficient of absolute risk aversion ρ (see equation (2)).

In principal, awareness traders may want to disclose awareness at any time: before observing the common signal, after observing the common signal but before the price emerges, and during the price-finding market process. While our base-line market model in the previous section is a simultaneous-move game with rational expectations Bayes-Nash equilibrium, it is typically understood as a reduced-form model for a dynamic complex market process so as to make the problem amenable to analysis. In this section, we continue with the reduced-form approach by allowing aware traders to make signal and price contingent disclosure decisions. After observing the common signal, all traders submit a demand function to the market maker. However, aware traders may also submit a sealed envelope to the market maker. This sealed envelope has a price range written on it. The market maker computes the market clearing price. However, if the market clearing price is in a range written on the envelope, the market maker is to open the envelope and broadcast the message written on the letter in it. At that point, informed traders are allowed to "update" their demand functions.[3] Subsequently, the market maker computes the market clearing price and trades

[3]It would be enough to just allow unaware traders to update their demand functions. Importantly, we assume that the noise traders's demand is unaffected by disclosure of awareness. This assumption is justified when interpreting noise traders strictly as liquidity traders who trade for exogenous reasons.

are executed.

A strategy of a player assigns to each signal a tuple of functions. For any trader of type A, $i \in M_A$, it is a tuple $(d(i,s), X_A^N(i,s), X_A^D(i,s), X(i,s))$ where $d(i,s)$ assigns to each signal to a choice function $d(i,s)(p)$ that maps prices to actions "Disclosure", D, or "Non-disclosure", N. $X_A^N(i,s)$ maps the signal to a demand function in the market game after non-disclosure, $X_A^D(i,s)$ maps the signal to a demand function in the market game after disclosure, and $X(i,s)$ maps the signal to a demand function in the market game perceived by traders of type B (before disclosure) in which also traders of type A are unaware. For any trader of type B, $i \in M_B$, the strategy is a tuple $(X_B^D(i,s), X(i,s))$ where $X_B^D(i,s)$ maps the signal to a demand function in the market game after disclosure and $X(i,s)$ maps the signal to a demand function in the market game where all traders are unaware.

With regard to liquidity traders, we assume that noise trading is a non-degenerate random variable $\tilde{z}$ with mean $\bar{z} \neq 0$.

Any unaware traders (all traders of type B before disclosure and all traders in the eyes of traders B before disclosure) perceive the market clearing price p to satisfy

$$\int_{[0,1]} X(i,s)(p)di + z \quad = \quad 0 \tag{42}$$

A trader of type A after non-disclosure perceives the market clearing price to satisfy

$$\int_{M_A} X_A^N(i,s)(p)di + \int_{M_B} X(i,s)(p)di + z \quad = \quad 0 \tag{43}$$

Finally, after disclosure any trader perceives the market clearing price to satisfy

$$\int_{M_A} X_A^D(i,s)(p)di + \int_{M_B} X_B^D(i,s)(p)di + z \quad = \quad 0 \tag{44}$$

Since the perception of market clearing depends on the traders' awareness levels, the market clearing price becomes a awareness-dependent random variable that with slight abuse of notation we denote by $\tilde{p}(X_A^D, X_B^D)$ (i.e., market game after disclosure), $\tilde{p}(X_A^N, X)$ (i.e., market game after nondisclosure as perceived by traders of type A), and $\tilde{p}(X)$ (i.e., market game as perceived by traders of type B before disclosure), respectively.

We use symmetric perfect Bayes-Nash equilibrium with linear demand schedules. Compared to the prior section, it determines also the disclosure strategy of traders of type A and the equilibrium demand schedules of traders of type B after disclosure.[4] No matter the disclosure decision by traders of type A, the market games feature Bayes-Nash equilibria. This is where the idea of perfection plays a role. Note that updating of beliefs after disclosure/non-disclosure is trivial as disclosure of awareness just changes awareness[5] because signals are common anyway.

Definition 5 *A perfect Bayes-Nash equilibrium of the game with incomplete information and unawareness defined in this section is a profile of strategies, one for each player $i \in [0,1]$, such that:*

[4] Note that since disclosure is costless, there is no coordination problem, volunteer's dilemma, or free-riding among traders of type A w.r.t. whether or not to disclose.

[5] Upon becoming aware of random variable $\tilde{w}$, traders of type B are assumed to understand that $\tilde{w} \sim \mathcal{N}(\bar{w}, \sigma_w^2)$ and $f_A = \tilde{v} + \tilde{w}$.

(A) For all $i \in M_A$ and s, the equilibrium strategy $(d^*(i,s), X_A^{N*}(i,s), X_A^{D*}(i,s), X^*(i,s))$ satisfies

(A1)

$$\begin{aligned}&\mathbb{E}[U(\tilde{f}_B - p)X^*(i,s)(p) \mid s, p = \tilde{p}(X^*)] \\ \geq\ &\mathbb{E}[U(\tilde{f}_B - p)X(i,s)(p) \mid s, p = \tilde{p}(X^*)] \text{ for all } X(i,s)\end{aligned} \tag{45}$$

(A2)

$$\begin{aligned}&\mathbb{E}[U(\tilde{f}_A - p)X_A^{D*}(i,s)(p) \mid s, p = \tilde{p}(X_A^{D*}, X_B^{D*})] \\ \geq\ &\mathbb{E}[U(\tilde{f}_A - p)X_A^{D}(i,s)(p) \mid s, p = \tilde{p}(X_A^{D*}, X_B^{D*})] \text{ for all } X_A^D(i,s)\end{aligned} \tag{46}$$

(A3)

$$\begin{aligned}&\mathbb{E}[U(\tilde{f}_A - p)X_A^{N*}(i,s)(p) \mid s, p = \tilde{p}(X_A^{N*}, X^*)] \\ \geq\ &\mathbb{E}[U(\tilde{f}_A - p)X_A^{N}(i,s)(p) \mid s, p = \tilde{p}(X_A^{N*}, X^*)] \text{ for all } X_A^N(i,s)\end{aligned} \tag{47}$$

(A4) $d^*(i,s)(p = \tilde{p}(X_A^{N*}, X^*)) = D$ if and only if

$$\begin{aligned}&\mathbb{E}[U(\tilde{f}_A - \tilde{p}(X_A^{D*}, X_B^{D*}))X_A^{D*}(i,s)(\tilde{p}(X_A^{D*}, X_B^{D*})) \mid s, p = \tilde{p}(X_A^{N*}, X^*)] \\ \geq\ &\mathbb{E}[U(\tilde{f}_A - p)X_A^{N*}(i,s)(p) \mid s, p = \tilde{p}(X_A^{N*}, X^*)]\end{aligned} \tag{48}$$

(B) For all $i \in M_B$ and s, the equilibrium strategy $(X_B^{D*}(i,s), X^*(i,s))$ satisfies

(B1)

$$\begin{aligned}&\mathbb{E}[U(\tilde{f}_B - p)X^*(i,s)(p) \mid s, p = \tilde{p}(X^*)] \\ \geq\ &\mathbb{E}[U(\tilde{f}_B - p)X(i,s)(p) \mid s, p = \tilde{p}(X^*)] \text{ for all } X(i,s)\end{aligned} \tag{49}$$

(B2)

$$\begin{aligned}&\mathbb{E}[U(\tilde{f}_A - p)X_B^{D*}(i,s)(p) \mid s, p = \tilde{p}(X_A^{D*}, X_B^{D*})] \\ \geq\ &\mathbb{E}[U(\tilde{f}_A - p)X_B^{D}(i,s)(p) \mid s, p = \tilde{p}(X_A^{D*}, X_B^{D*})] \text{ for all } X_B^D(i,s)\end{aligned} \tag{50}$$

The equilibrium is symmetric if all players with the same awareness level play the same demand schedule in the respective market games, i.e., for all s, $X_B^{D*}(i,s) = X_A^{D*}(j,s)$ for any $i \in M_B$ and $j \in M_A$, $X^*(i,s) = X^*(j,s)$ for all $i,j \in [0,1]$, $X_A^{N*}(i,s) = X_A^{N*}(j,s)$ for all $i,j \in M_A$, and $d^*(i,s) = d^*(j,s)$ for all $i,j \in M_A$, and trader $i \in M_A$ use identical disclosure strategies, i.e., $i,j \in M_A$, $d(i,s) = d(j,s)$ for all s.

The first part of the closed-form characterization of equilibrium is analogous to Proposition 1 in that we characterize Bayes-Nash equilibrium in each of the market games. The proof of this part of the characterization is simpler compared to the proof of Proposition 1 because as observed in Vives (2008, Chapter 4.2.2), once all traders observe a common signal, the price transmits no additional information. The price cannot contain more information than the joint information of traders s. Thus, for all s and p, $\mathbb{E}[f_\ell \mid s,p] = \mathbb{E}[f_\ell \mid s]$ and $Var[f_\ell \mid s,p] = Var[f_\ell \mid s]$ for $\ell \in \{A,B\}$. Yet, characterization of the perfect Bayes Nash equilibrium goes significantly beyond Proposition 1 because we also need to characterize the equilibrium disclosure decision by traders of type A.

First, we characterize equilibrium demand schedules. The proof is in the appendix.

Proposition 10 *In the unique symmetric perfect Bayes-Nash equilibria in linear demand schedules, the demand schedules in the respective market games are characterized by: For any $i \in [0,1]$, common signal s, and price p,*

$$X_A^{N*}(i,s)(p) = X_A^{D*}(s)(p) = X_B^{D*}(i,s)(p) \quad = \quad \alpha + \beta s - \gamma_A p \tag{51}$$

$$X^*(i,s)(p) \quad = \quad \beta s - \gamma_B p \tag{52}$$

with

$$\alpha = \frac{\tau_{v+w}}{\rho}\bar{w} \tag{53}$$

$$\beta = \frac{\tau_\varepsilon}{\rho} \tag{54}$$

$$\gamma_A = \frac{\tau_{v+w} + \tau_\varepsilon}{\rho} \tag{55}$$

$$\gamma_B = \frac{\tau_v + \tau_\varepsilon}{\rho} \tag{56}$$

Note that the equilibrium demand schedules only depend on awareness. The equilibrium demand schedule of aware traders does not depend on their disclosure decision. Moreover, equilibrium demand schedules of traders who have been made aware by the disclosure of aware traders have demand schedules identical to demand schedules of aware traders.

As in the prior section, aware traders are less responsive to the price than unaware traders. Unaware traders do not perceive the entire volatility of the fundamental value. That's why they "overreact" to prices. The short proof is contained in the appendix.

Proposition 11 *In unique symmetric perfect Bayes-Nash equilibrium in linear demand schedules, $\gamma_A < \gamma_B$.*

Next, we characterize the equilibrium disclosure strategy of traders of type A. The proof is contained in the appendix.

Proposition 12 *In the unique symmetric perfect Bayes-Nash equilibrium in the game with incomplete information and unawareness described in this section, we have that for all $i \in M_A$, $d^*(i,s)(p = \tilde{p}(X_A^{N*}, X^*)) = D$ if and only if $(s,p) \in K_1 \cup K_2$ defined by*

$$K_1 := \{(s,p) \mid \psi_1 p + \psi_2 \leq s \text{ and } p \leq \psi_3\} \tag{57}$$

$$K_2 := \{(s,p) \mid \psi_1 p + \psi_2 \geq s \text{ and } p \geq \psi_3\} \tag{58}$$

with

$$\psi_1 := \frac{2\sigma_v^2(\sigma_w^2 + \sigma_v^2) + 2\sigma_v^2\sigma_\varepsilon^2 + m_B\sigma_w^2\sigma_\varepsilon^2}{2\sigma_v^2(\sigma_w^2 + \sigma_v^2)} \tag{59}$$

$$\psi_2 := -\frac{(2 - m_B)\sigma_\varepsilon^2\bar{w}}{2(\sigma_w^2 + \sigma_v^2)} \tag{60}$$

$$\psi_3 := -\frac{\sigma_v^2}{\sigma_w^2}\bar{w} \tag{61}$$

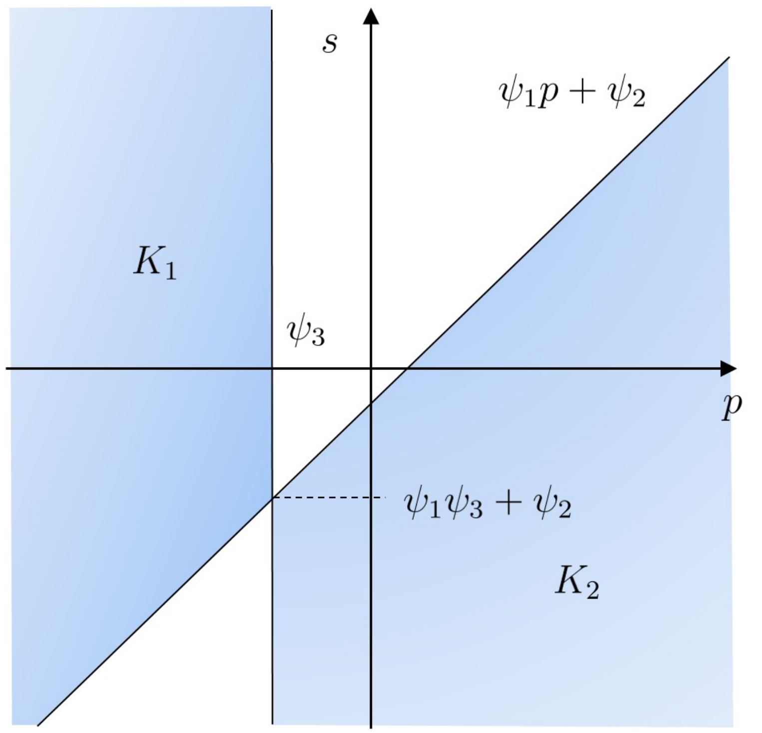

Figure 3: Sets of tuples of signal and pre-disclosure market clearing prices for which disclosure occurs

Note that $\psi_1 > 1$, $\psi_2 < 0$ if and only if $\bar{w} > 0$, and $\psi_3 < 0$ if and only if $\bar{w} > 0$. Thus, the area of signal-price tuples for which traders of type A disclose awareness in equilibrium is the union of two cones K_1 and K_2 defined by two lines. Intuitively, for every common signal, aware traders disclose their awareness if pre-disclosure prices are too high or too low. We illustrate this in Figure 3 for the case $\bar{w} > 0$. (Essentially turn the picture 180° for the case $\bar{w} < 0$.)

The proof relies heavily on the form of the conditional expected CARA utility. For every common signal s, it is quadratic in price and minimized at some price $p_{\min}(s)$. Because of its quadratic form, for every post-disclosure market price, there exists a symmetric counterpart that is the same distance from $p_{\min}s$ as the post-disclosure price at s but just in the opposite direction. Conditional expected utility at s with the post-disclosure market price equals conditional expected utility at s with the symmetric counterpart price (see Figure 4 for an illustration). Disclosure occurs when the pre-disclosure market clearing price p is in an intermediate range between the post-disclosure market clearing price and its symmetric counterpart because in such a case pre-disclosure conditional expected utility is smaller than post-disclosure conditional expected utility. Since signals are public, a trader of type A can infer from pre-disclosure market clearing price p the demands from noise traders. This allows her to fully anticipate the post-disclosure market clearing price. Call it $p^D(s, p)$ and its symmetric counterpart $q^D(s, p)$. It turns out that the post-disclosure market price does not depend on s because for any s and pre-disclosure market price p the post-disclosure $p^D(s, p)$ must be market clearing. Yet, both the post-disclosure market price $p^D(p)$ and

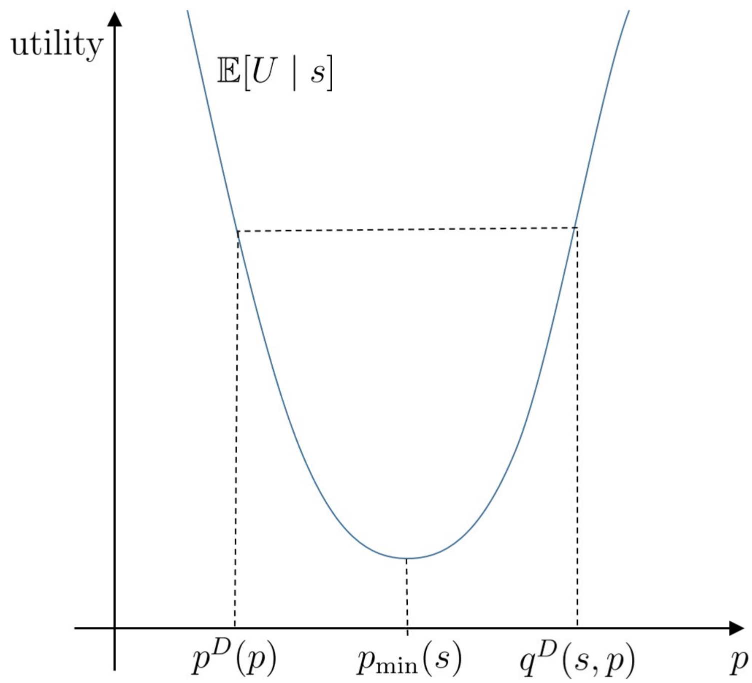

Figure 4: Illustration of quadratic conditional expected utility

its symmetric counterpart $q^D(s,p)$ depend on the pre-disclosure market price (through the inferred demands of noise traders). So disclosure occurs at s if the pre-disclosure market price p is between $p^D(p)$ and $q^D(s,p)$ where both $p^D(p)$ and $q^D(s,p)$ can be described with linear equations. Solving this linear inequality problem yields the areas defined by K_1 and K_2.

A Proofs

Proof of Proposition 1

To characterize the symmetric Bayesian Nash equilibrium in linear strategies, we make use of the fact that informed traders with lower awareness levels do not react to informed traders with higher awareness levels since former are unaware of latter's awareness. Thus, Bayes-Nash equilibrium is constructed from lower awareness levels "upward" (Meier and Schipper (2014a, Proposition 2)).

Awareness types 1: Suppose all traders of awareness type 1 use the linear equilibrium strategy for all s_i

$$X_1^*(s_i)(p) \quad = \quad \beta_1 s_i - \gamma_1 p \tag{62}$$

for any realization of the price p for some $\beta_1, \gamma_1 \in \mathbb{R}$ with $\beta_1, \gamma_1 \neq 0$.

Note that we can interpret the quantity $\tilde{x}_1^* = X_1^*(\tilde{s}_i)(p)$ as a random variable that is normally distributed since it is a linear transformation of the normally distributed random variable $\tilde{s}_i$.

Using the market clearing equation 4, we obtain

$$\begin{aligned} \int_{[0,1]} (\beta_1 \tilde{s}_i - \gamma_1 p) di + z &= 0 \\ \beta_1 \int_{[0,1]} (f_1 + \tilde{\varepsilon}_i) di - \gamma_1 p + z &= 0 \end{aligned} \tag{63}$$

Invoking the implication of the Strong Law of Large Numbers (equation (1)) and the definition of f_1 we obtain

$$\beta_1 v - \gamma_1 p + z = 0. \tag{64}$$

Solving for p we obtain

$$p = \frac{\beta_1}{\gamma_1} v + \frac{1}{\gamma_1} z. \tag{65}$$

Since this holds for any realization of v and z, we can interpret the price is a random variable that is normally distributed since it is a linear combination of normally distributed random variables. This is the price random variable that emerges from equilibrium strategies X_1^*. Thus, we write

$$\tilde{p}(X_1^*) = \frac{\beta_1}{\gamma_1} \tilde{v} + \frac{1}{\gamma_1} \tilde{z}. \tag{66}$$

Since strategies are mutually known among informed traders with awareness level 1 in equilibrium, these informed traders can deduce information contained in the realization of the price about the fundamental (which they understand as the random variable $\tilde{v}$) as well as the amount of noise trading.

Applying a linear transformation to the price random variable $\tilde{p}(X_1^*)$ and denoting by $\tilde{z}_1 := \frac{1}{\beta_1} \tilde{z}$, we can "isolate" the information contained in the price by defining

$$\tilde{h}_1 := \frac{\gamma_1}{\beta_1} \tilde{p}_1(X_1^*) = \tilde{v} + \frac{1}{\beta_1} \tilde{z} = \tilde{v} + \tilde{z}_1. \tag{67}$$

Clearly, by the definition of z_1 and properties of the variance, $\sigma_{z_1}^2 = \frac{1}{\beta_1^2} \sigma_z^2$.

Denote by $\tau_1 := \frac{1}{Var[\tilde{v} | s_i, p = \tilde{p}(X_1^*)]}$, i.e., the inverse of the variance of $\tilde{v}$ conditional on signal s_i and the price p. This is the precision of the fundamental perceived by trader i of awareness type 1 conditional on the signal realization s_i and the realization of the price $p = \tilde{p}(X_1^*)$. The fact that we condition on $p = \tilde{p}(X_1^*)$ means that we condition on the information conveyed in the price through equilibrium strategies. Clearly, by the definition of random variable h_1, $\tau_1 = \frac{1}{Var[\tilde{v} | s_i, h_1]}$.

From the projection theorem applied to normally distributed variables (e.g., DeGroot (1970, Chapter 5), Vives (2008, 10.2.1)) follows that

$$\tau_1 = \tau_v + \tau_{\varepsilon_i} + \tau_{z_1} \tag{68}$$

and

$$\mathbb{E}[\tilde{v} \mid s_i, p = \tilde{p}(X_1^*)] = \mathbb{E}[\tilde{v} \mid s_i, h_1] \quad = \quad \frac{\tau_{\varepsilon_i}}{\tau_1} s_i + \frac{\tau_{z_1}}{\tau_1} \frac{\gamma_1}{\beta_1} p. \tag{69}$$

If all informed traders but i use equilibrium strategies X_1^*, then conditional on information signal s_i and price $p = \tilde{p}(X_1^*)$, trading returns $(\tilde{v} - p)x_i$ of the informed trader i from quantity x_i is a random variable that is distributed normally. It is well-known (e.g., Danthine and Moresi (1993, pp. 979–980), Vives (2008, 10.2.4)) that in this case the expected CARA expected utility conditional on information signal s_i and and price $p = \tilde{p}(X_1^*)$ takes on the "mean-variance" form:

$$\begin{aligned}
&\mathbb{E}[U((\tilde{v} - p)x_i) \mid s_i, p = \tilde{p}(X_1^*)] \\
&= \mathbb{E}[(\tilde{v} - p)x_i \mid s_i, p = \tilde{p}(X_1^*)] - \frac{\rho}{2} Var[(\tilde{v} - p)x_i \mid s_i, p = \tilde{p}(X_1^*)] \\
&= x_i \mathbb{E}[\tilde{v} \mid s_i, p = \tilde{p}(X_1^*)] - p x_i - \frac{\rho}{2} x_i^2 Var[\tilde{v} \mid s_i, p = \tilde{p}(X_1^*)]
\end{aligned} \tag{70}$$

where the last line follows from well-known properties of the expected value and the variance.

Any strategy of informed trader i such that it's realized quantity is (ex post) optimal conditional on the price p and information signal s_i for every price p and information signal s_i is (when understood as a demand function) also interim optimal for i given just information signal s_i when other informed traders follow strategy X_1^*.

For every p and s_i, maximizing expected utility given by equation (70) w.r.t. x_i yields the first-order condition

$$x_i^* \quad = \quad \frac{\mathbb{E}[\tilde{v} \mid s_i, p = \tilde{p}(X_1^*)] - p}{\rho Var[\tilde{v} \mid s_1, p = \tilde{p}(X_1^*)]} \tag{71}$$

Since the objective function (70) is strictly concave in x_i, the first-order condition is also sufficient for the maximum. Using equations (68) and (69), we rewrite equation (71)

$$\begin{aligned}
x_i^* &= \left(\frac{\tau_{\varepsilon_i}}{\tau_1} s_i + \frac{\tau_{z_1}}{\tau_1} \frac{\gamma_1}{\beta_1} p - p \right) \frac{\tau_1}{\rho} \\
&= \frac{\tau_{\varepsilon_i}}{\rho} s_i - \frac{\beta_1 \tau_1 - \gamma_1 \tau_{z_1}}{\beta_1 \rho} p \\
&= \beta_1 s_i - \gamma_1 p
\end{aligned}$$

where last line follows from identifying coefficients with

$$\begin{aligned}
\beta_1 &= \frac{\tau_{\varepsilon_i}}{\rho} \\
\gamma_1 &= \frac{\tau_1 \rho}{\tau_{\varepsilon_i} \tau_z + \rho^2}.
\end{aligned}$$

Recall that $\tilde{\varepsilon_i}$ is i.i.d. across informed traders i. Thus, β_1 and γ_1 are identical for all traders i and equilibrium strategies are symmetric. This proves the characterization of equilibrium strategies of awareness types 1.

Awareness types 2: The step is similar to the previous except that all informed traders of awareness type 2 take into account equilibrium strategies of informed traders of awareness type 1. Suppose all traders of awareness type 2 use the linear equilibrium strategy for all s_i

$$X_2^*(s_i)(p) \quad = \quad \alpha_2 + \beta_2 s_i - \gamma_2 p \tag{72}$$

for any realization of the price p for some $\alpha_2, \beta_2, \gamma_2 \in \mathbb{R}$ with $\beta_2, \gamma_2 \neq 0$.

Again, note that we can interpret the quantity $\tilde{x}_2^* = X_2^*(\tilde{s}_i)(p)$ as a random variable that is normally distributed since it is a linear transformation of the normally distributed random variable $\tilde{s}_i$.

Using the market clearing equation (5) and equilibrium strategies of awareness type 1 (equation (11)), we obtain

$$\int_{M_1 \cup M_3} (\beta_1 \tilde{s}_i - \gamma_1 p) di + \int_{M_2 \cup M_4} (\alpha_2 + \beta_2 \tilde{s}_i - \gamma_2 p) di + z \quad = \quad 0 \tag{73}$$

Informed traders of awareness type 2 understand that the information signal of informed traders of awareness 1 should be interpreted as $v + c + \varepsilon_i$ rather than $v + \varepsilon_i$.[6] Thus, we rewrite previous market clearing equation as

$$\int_{M_1 \cup M_3} (\beta_1 (v + c + \tilde{\varepsilon}_i) - \gamma_1 p) di + \int_{M_2 \cup M_4} (\alpha_2 + \beta_2 (v + c + \tilde{\varepsilon}_i) - \gamma_2 p) di + z \quad = \quad 0 \tag{74}$$

$$\begin{aligned} & \beta_1 \int_{M_1 \cup M_3} (v + c + \tilde{\varepsilon}_i) di - (m_1 + m_3) \gamma_1 p \\ + & \quad (m_2 + m_4) \alpha_2 + \beta_2 \int_{M_2 \cup M_4} (v + c + \tilde{\varepsilon}_i) di - (m_2 + m_4) \gamma_2 p + z = 0 \end{aligned} \tag{75}$$

Invoking again the implication of the Strong Law of Large Numbers (equation (1)), we obtain

$$\begin{aligned} & (m_1 + m_3) \beta_1 (v + c) - (m_1 + m_3) \gamma_1 p \\ + & \quad (m_2 + m_4) \alpha_2 + (m_2 + m_4) \beta_2 (v + c) - (m_2 + m_4) \gamma_2 p + z = 0 \end{aligned} \tag{76}$$

We simplify notation by defining $\gamma_{12} := (m_1 + m_3)\gamma_1 + (m_2 + m_4)\gamma_2$ and $\beta_{12} := (m_1 + m_3)\beta_1 + (m_2 + m_4)\beta_2$ and solve for the price p:

$$p \quad = \quad \frac{(m_2 + m_4)\alpha_2}{\gamma_{12}} + \frac{\beta_{12}}{\gamma_{12}} c + \frac{\beta_{12}}{\gamma_{12}} v + \frac{1}{\gamma_{12}} z. \tag{77}$$

As in the previous step, we can interpret the price as a random variable that is normally distributed because it is a linear combination of normally distributed variables. This is the price random variable that emerges from equilibrium strategies of informed traders of awareness types 1 and 2 (in addition to noise traders). Thus, we write

$$\tilde{p}(X_2^*, X_1^*) \quad = \quad \frac{(m_2 + m_4)\alpha_2}{\gamma_{12}} + \frac{\beta_{12}}{\gamma_{12}} c + \frac{\beta_{12}}{\gamma_{12}} \tilde{v} + \frac{1}{\gamma_{12}} \tilde{z}. \tag{78}$$

[6] Note that it does not mean that awareness type 2 believes awareness type 1 received s_i different from the one they actually did. Awareness type 2 just interprets s_i received by awareness type 2 differently by considering the constant c as well.

Since strategies are mutually known among traders with awareness level 2 in equilibrium, they can deduce information contained in the realization of the price about the fundamental and noise trading. Applying a linear transformation to $\tilde{p}(X_2^*, X_1^*)$ and denoting by $\tilde{z}_2 := \frac{1}{\beta_{12}}\tilde{z}$, we isolate the information contained in the price by defining

$$\tilde{h}_2 := \left(\tilde{p}(X_2^*, X_1^*) - \frac{m_2 + m_4}{\gamma_{12}}\alpha_2 - \frac{\beta_{12}}{\gamma_{12}}c\right)\frac{\gamma_{12}}{\beta_{12}} = \tilde{v} + \frac{1}{\beta_{12}}\tilde{z} = \tilde{v} + \tilde{z}_2 \tag{79}$$

Denote by $\tau_2 := \frac{1}{Var[\tilde{v}|s_i, p=\tilde{p}(X_2^*, X_1^*)]}$ the precision of the fundamental perceived by trader i of awareness type 2 conditional on signal realization s_i and the price $p = \tilde{p}(X_2^*, X_1^*)$. Clearly, $\tau_2 = \frac{1}{Var[\tilde{v}|s_i - c, h_2]}$. From the projection theorem applied to normally distributed variables (e.g., DeGroot (1970, Chapter 5), Vives (2008, 10.2.1)) follows that

$$\tau_2 = \tau_v + \tau_{\varepsilon_i} + \tau_{z_2} \tag{80}$$

and

$$\begin{aligned}
\mathbb{E}[\tilde{v} \mid s_i, p = \tilde{p}(X_2^*, X_1^*)] &= \mathbb{E}[\tilde{v} \mid s_i - c, h_2] \\
&= \frac{\tau_{\varepsilon_i}}{\tau_2}(s_i - c) + \frac{\tau_{z_2}}{\tau_2}h_2 \\
&= \frac{\tau_{\varepsilon_i}}{\tau_2}(s_i - c) + \frac{\tau_{z_2}}{\tau_2}\left(p - \frac{m_2 + m_4}{\gamma_{12}}\alpha_2 - \frac{\beta_{12}}{\gamma_{12}}c\right)\frac{\gamma_{12}}{\beta_{12}} \\
&= \frac{\tau_{\varepsilon_i}}{\tau_2}s_i + \frac{\tau_{z_2}}{\tau_2}\frac{\gamma_{12}}{\beta_{12}}p - \frac{\tau_{z_2}}{\tau_2}\frac{m_2 + m_4}{\beta_{12}}\alpha_2 - \frac{\tau_{\varepsilon_i} + \tau_{z_2}}{\tau_2}c
\end{aligned} \tag{81}$$

If all informed traders of awareness type 1 use equilibrium strategies X_1^* and all but i of awareness type 2 use equilibrium strategies X_2^*, then conditional on information signal s_i and price $p = \tilde{p}(X_2^*, X_1^*)$, trading returns $(\tilde{v} + c - p)x_i$ of informed trader i from quantity x_i is a random variable that is distributed normally. Thus, as it is well-known in this case (e.g., Danthine and Moresi (1993, pp. 979–980), Vives (2008, 10.2.4)), CARA expected utility conditional on information signal s_i and price $p = \tilde{p}(X_2^*, X_1^*)$ takes on the "mean-variance" form

$$\begin{aligned}
&\mathbb{E}[U((\tilde{v} + c - p)x_i) \mid s_i, p = \tilde{p}(X_2^*, X_1^*)] \\
&= \mathbb{E}[(\tilde{v} + c - p)x_i \mid s_i, p = \tilde{p}(X_2^*, X_1^*)] - \frac{\rho}{2}Var[(\tilde{v} + c - p)x_i \mid s_i, p = \tilde{p}(X_2^*, X_1^*)] \\
&= x_i\mathbb{E}[\tilde{v} \mid s_i, p = \tilde{p}(X_2^*, X_1^*)] + (c - p)x_i - \frac{\rho}{2}x_i^2 Var[\tilde{v} \mid s_i, p = \tilde{p}(X_2^*, X_1^*)]
\end{aligned} \tag{82}$$

where the last line follows from well-known properties of the expected value and the variance.

Any strategy of informed trader i such that it's realized quantity is (ex post) optimal conditional on the price p and information signal s_i for every price p and information signal s_i is (when understood as a demand function) also interim optimal for i given just information signal s_i when other informed traders of awareness type 2 follow strategy X_2^* and informed traders of awareness type 1 follow strategy X_1^*.

For every p and s_i, maximizing expected utility given by equation (82) w.r.t. x_i yields the first-order condition

$$x_i^* \;=\; \frac{\mathbb{E}[v \mid s_i, p = \tilde{p}(X_2^*, X_1^*)] + c - p}{\rho Var[\tilde{v} \mid s_1, p = \tilde{p}(X_2^*, X_1^*)]} \tag{83}$$

Since the objective function (82) is strictly concave, the first-order condition is also sufficient for the maximum. Using equations (80) and (81), we rewrite equation (83)

$$\begin{aligned}
x_i^* &= \left(\frac{\tau_{\varepsilon_i}}{\tau_2}s_i + \frac{\tau_{z_2}}{\tau_2}\frac{\gamma_{12}}{\beta_{12}}p - \frac{\tau_{z_2}}{\tau_2}\frac{m_2+m_4}{\beta_{12}}\alpha_2 - \frac{\tau_{\varepsilon_i}+\tau_{z_2}}{\tau_2}c + c - p\right)\frac{\tau_2}{\rho} \\
&= \frac{\tau_2 - \tau_{\varepsilon_i} - \tau_{z_2}}{\rho}c - \frac{\tau_{z_2}}{\rho}\frac{m_2+m_4}{\beta_{12}}\alpha_2 + \frac{\tau_{\varepsilon_i}}{\rho}s_i - \frac{\beta_{12}\tau_2 - \gamma_{12}\tau_{z_2}}{\beta_{12}\rho}p & (84)\\
&= \alpha_2 + \beta_2 s_i - \gamma_2 p & (85)
\end{aligned}$$

where last line follows from identifying coefficients with

$$\begin{aligned}
\alpha_2 &= c\frac{\tau_v\rho}{(m_2+m_4)\tau_{\varepsilon_i}\tau_z + \rho^2} & (86)\\
\beta_2 &= \frac{\tau_{\varepsilon_i}}{\rho} \\
\gamma_2 &= \frac{\tau_2\rho}{\tau_{\varepsilon_i}\tau_z + \rho^2}
\end{aligned}$$

using the fact that $\tau_{z_2} = \beta_{12}^2\tau_z$. Note also that $\tau_2 = \tau_1$ follows from $\beta_2 = \beta_1$. Recall that $\tilde{\varepsilon}_i$ is i.i.d. across informed traders i. Thus, α_2, β_2 and γ_2 are identical for all traders with awareness level 2. From Step 1 we concluded that also equilibrium strategies are identical for all traders with awareness level 1. Thus, the equilibrium of the partial game with awareness types 1 and 2 only is symmetric. This proves the characterization of equilibrium strategies of awareness types 2.

Awareness types 3: The step is similar the previous two steps except that all informed traders of awareness type 3 take into account equilibrium strategies of informed traders of awareness type 1. Suppose all traders of awareness type 3 use the linear equilibrium strategy for all s_i

$$X_3^*(s_i)(p) \;=\; \beta_3 s_i - \gamma_3 p \tag{87}$$

for any realization of the price p for some $\beta_3, \gamma_3 \in \mathbb{R}$ with $\beta_3, \gamma_3 \neq 0$.

Again, note that we can interpret the quantity $\tilde{x}_3^* = X_3^*(\tilde{s}_i)(p)$ as a random variable that is normally distributed since it is a linear transformation of the normally distributed random variable $\tilde{s}_i$.

Using the market clearing equation (6) and equilibrium strategies of awareness type 1 (equation (11)), we obtain

$$\int_{M_1 \cup M_2} (\beta_1\tilde{s}_i - \gamma_1 p)di + \int_{M_3 \cup M_4} (\beta_3\tilde{s}_i - \gamma_3 p)di + z \;=\; 0 \tag{88}$$

Informed traders of awareness type 3 understand that the information signal of informed traders of awareness 1 should be interpreted as $v + w + \varepsilon_i$ rather than $v + \varepsilon_i$. Thus, we rewrite previous market clearing equation as

$$\int_{M_1 \cup M_2} (\beta_1(v + w + \tilde{\varepsilon}_i) - \gamma_1 p) di + \int_{M_3 \cup M_4} (\beta_3(v + w + \tilde{\varepsilon}_i) - \gamma_3 p) di + z = 0 \quad (89)$$

$$\begin{aligned} & \beta_1 \int_{M_1 \cup M_2} (v + w + \tilde{\varepsilon}_i) di - (m_1 + m_2)\gamma_1 p \\ + \quad & \beta_3 \int_{M_3 \cup M_4} (v + w + \tilde{\varepsilon}_i) di - (m_3 + m_4)\gamma_3 p + z = 0 \end{aligned} \quad (90)$$

Invoking again the implication of the Strong Law of Large Numbers (equation (1)), we obtain

$$\begin{aligned} & (m_1 + m_2)\beta_1(v + w) - (m_1 + m_2)\gamma_1 p \\ + \quad & (m_3 + m_4)\beta_3(v + w) - (m_3 + m_4)\gamma_3 p + z = 0 \end{aligned} \quad (91)$$

We simplify notation by defining $\gamma_{13} := (m_1 + m_2)\gamma_1 + (m_3 + m_4)\gamma_3$ and $\beta_{13} := (m_1 + m_2)\beta_1 + (m_3 + m_4)\beta_3$ and solve for the price p:

$$p = \frac{\beta_{13}}{\gamma_{13}}(v + w) + \frac{1}{\gamma_{13}} z. \quad (92)$$

As in the previous step, we can interpret the price as a random variable that is normally distributed because it is a linear combination of normally distributed variables. This is the price random variable that emerges from equilibrium strategies of informed traders of awareness types 1 and 3 (in addition to noise traders). Thus, we write

$$\tilde{p}(X_3^*, X_1^*) = \frac{\beta_{13}}{\gamma_{13}}(\tilde{v} + \tilde{w}) + \frac{1}{\gamma_{13}} \tilde{z}. \quad (93)$$

Since strategies are mutually known among traders with awareness level 3 in equilibrium, they can deduce information contained in the realization of the price about the fundamental and noise trading. Applying a linear transformation to $\tilde{p}(X_3^*, X_1^*)$ and denoting by $\tilde{z}_3 := \frac{1}{\beta_{13}}\tilde{z}$, we isolate the information contained in the price by defining

$$\tilde{h}_3 := \frac{\gamma_{13}}{\beta_{13}} \tilde{p}(X_3^*, X_1^*) = \tilde{v} + \tilde{w} + \frac{1}{\beta_{13}} \tilde{z} = \tilde{v} + \tilde{w} + \tilde{z}_3 \quad (94)$$

Denote by $\tau_3 := \frac{1}{Var[\tilde{v}+\tilde{w}|s_i, p=\tilde{p}(X_3^*, X_1^*)]}$ the precision of the fundamental perceived by trader i of awareness type 2 conditional on signal realization s_i and the price $p = \tilde{p}(X_3^*, X_1^*)$. Clearly, $\tau_3 = \frac{1}{Var[\tilde{v}+\tilde{w}|s_i, h_3]}$. From the projection theorem applied to normally distributed variables (e.g., DeGroot (1970, Chapter 5), Vives (2008, 10.2.1)) follows that

$$\tau_3 = \tau_{v+w} + \tau_{\varepsilon_i} + \tau_{z_3} \quad (95)$$

and

$$
\begin{aligned}
\mathbb{E}[\tilde{v}+\tilde{w} \mid s_i, p=\tilde{p}(X_3^*, X_1^*)] &= \mathbb{E}[\tilde{v}+\tilde{w} \mid s_i, h_3] \\
&= \frac{\tau_{\varepsilon_i}}{\tau_3} s_i + \frac{\tau_{z_3}}{\tau_3} h_3 \\
&= \frac{\tau_{\varepsilon_i}}{\tau_3} s_i + \frac{\tau_{z_3}}{\tau_3}\frac{\gamma_{13}}{\beta_{13}} p \qquad (96)
\end{aligned}
$$

If all informed traders of awareness type 1 use equilibrium strategies X_1^* and all but i of awareness type 3 use equilibrium strategies X_3^*, then conditional on information signal s_i and price $p=\tilde{p}(X_3^*, X_1^*)$, trading returns $(\tilde{v}+\tilde{w}-p)x_i$ of informed trader i from quantity x_i is a random variable that is distributed normally since it is a linear combination of normally distributed random variables. Thus, as it is well-known in this case (e.g., Danthine and Moresi (1993, pp. 979–980), Vives (2008, 10.2.4)), CARA expected utility conditional on information signal s_i and price $p=\tilde{p}(X_3^*, X_1^*)$ takes on the "mean-variance" form

$$
\begin{aligned}
&\mathbb{E}[U((\tilde{v}+\tilde{w}-p)x_i) \mid s_i, p=\tilde{p}(X_3^*, X_1^*)] \\
&= \mathbb{E}[(\tilde{v}+\tilde{w}-p)x_i \mid s_i, p=\tilde{p}(X_3^*, X_1^*)] - \frac{\rho}{2} Var[(\tilde{v}+\tilde{w}-p)x_i \mid s_i, p=\tilde{p}(X_3^*, X_1^*)] \\
&= x_i \mathbb{E}[\tilde{v}+\tilde{w} \mid s_i, p=\tilde{p}(X_3^*, X_1^*)] + p x_i - \frac{\rho}{2} x_i^2 Var[\tilde{v}+\tilde{w} \mid s_i, p=\tilde{p}(X_3^*, X_1^*)] \qquad (97)
\end{aligned}
$$

where the last line follows from well-known properties of the expected value and the variance.

Any strategy of informed trader i such that it's realized quantity is (ex post) optimal conditional on the price p and information signal s_i for every price p and information signal s_i is (when understood as a demand function) also interim optimal for i given just information signal s_i when other informed traders of awareness type 3 follow strategy X_3^* and informed traders of awareness type 1 follow strategy X_1^*.

For every p and s_i, maximizing expected utility given by equation (97) w.r.t. x_i yields the first-order condition

$$
x_i^* = \frac{\mathbb{E}[\tilde{v}+\tilde{w} \mid s_i, p=\tilde{p}(X_3^*, X_1^*)] - p}{\rho Var[\tilde{v}+\tilde{w} \mid s_1, p=\tilde{p}(X_3^*, X_1^*)]} \qquad (98)
$$

Since the objective function (97) is strictly concave in x_i, the first-order condition is also sufficient for the maximum. Using equations (95) and (96), we rewrite equation (98)

$$
\begin{aligned}
x_i^* &= \left(\frac{\tau_{\varepsilon_i}}{\tau_3} s_i + \frac{\tau_{z_3}}{\tau_3}\frac{\gamma_{13}}{\beta_{13}} p - p\right)\frac{\tau_3}{\rho} \\
&= \frac{\tau_{\varepsilon_i}}{\rho} s_i - \frac{\beta_{13}\tau_3 - \gamma_{13}\tau_{z_3}}{\beta_{13}\rho} p \qquad (99) \\
&= \beta_3 s_i - \gamma_3 p \qquad (100)
\end{aligned}
$$

where last line follows from identifying coefficients with

$$
\begin{aligned}
\beta_3 &= \frac{\tau_{\varepsilon_i}}{\rho} \\
\gamma_3 &= \frac{\left(\tau_{v+w} + \tau_{\varepsilon_i} + \left(\frac{\tau_{\varepsilon_i}}{\rho}\right)^2 \tau_z\right)\rho}{(m_3+m_4)\tau_{\varepsilon_i}\tau_z + \rho^2} - \frac{(m_1+m_2)\tau_{\varepsilon_i}\tau_z}{(m_3+m_4)\tau_{\varepsilon_i}\tau_z + \rho^2}\gamma_1 \qquad (101)
\end{aligned}
$$

using the fact that $\tau_{z_3} = \beta_{13}^2 \tau_z$. Recall that $\tilde{\varepsilon}_i$ is i.i.d. across informed traders i. Thus, β_3 and γ_3 are identical for all traders with awareness level 3. From Step 1 we concluded that also equilibrium strategies are identical for all traders with awareness level 1. Thus, the equilibrium of the partial game with awareness types 1 and 3 only is symmetric. This proves the characterization of equilibrium strategies of awareness types 3.

Awareness types 4: The step is similar to the previous steps except that all informed traders of awareness type 4 take into account equilibrium strategies of informed traders with lower awareness levels. Suppose all traders of awareness type 4 use the linear equilibrium strategy for all s_i

$$X_4^*(s_i)(p) \quad = \quad \alpha_4 + \beta_4 s_i - \gamma_4 p \tag{102}$$

for any realization of the price p for some $\alpha_4, \beta_4, \gamma_4 \in \mathbb{R}$ with $\beta_4, \gamma_4 \neq 0$.

Again, note that we can interpret the quantity $\tilde{x}_4^* = X_4^*(\tilde{s}_i)(p)$ as a random variable that is normally distributed since it is a linear transformation of the normally distributed random variable $\tilde{s}_i$.

Using the market clearing equation (5) and equilibrium strategies of awareness type 1 (equation (11)), we obtain

$$\begin{aligned} &\int_{M_1} (\beta_1 \tilde{s}_i - \gamma_1 p) di + \int_{M_2} (\alpha_2 + \beta_2 \tilde{s}_i - \gamma_2 p) di \\ &+ \int_{M_3} (\beta_3 \tilde{s}_i - \gamma_3 p) di + \int_{M_4} (\alpha_4 + \beta_4 \tilde{s}_i - \gamma_4 p) di + z = 0 \end{aligned} \tag{103}$$

Informed traders of awareness type 4 understand that the information signal of informed traders of lower awareness levels should be interpreted as $v + w + c + \varepsilon_i$. Thus, we rewrite previous market clearing equation as

$$\begin{aligned} &\int_{M_1} (\beta_1 (v + w + c + \tilde{\varepsilon}_i) - \gamma_1 p) di + \int_{M_2} (\alpha_2 + \beta_2 (v + w + c + \tilde{\varepsilon}_i) - \gamma_2 p) di \\ &+ \int_{M_3} (\beta_3 (v + w + c + \tilde{\varepsilon}_i) - \gamma_3 p) di + \int_{M_4} (\alpha_4 + \beta_4 (v + w + c + \tilde{\varepsilon}_i) - \gamma_4 p) di + z = 0 \end{aligned} \tag{104}$$

$$\begin{aligned} &\beta_1 \int_{M_1} (v + w + c + \tilde{\varepsilon}_i) di + m_2 \alpha_2 + \beta_2 \int_{M_2} (v + w + c + \tilde{\varepsilon}_i) di + \beta_3 \int_{M_3} (v + w + c + \tilde{\varepsilon}_i) di \\ &+ \ m_4 \alpha_4 + \beta_4 \int_{M_4} (v + w + c + \tilde{\varepsilon}_i) di - p \sum_{\ell=1}^{4} m_\ell \gamma_\ell + z = 0 \end{aligned} \tag{105}$$

Invoking again the implication of the Strong Law of Large Numbers (equation (1)), we obtain that aggregate excess demand is equal to zero:

$$m_2 \alpha_2 + m_4 \alpha_4 + (v + w + c) \sum_{\ell=1}^{4} m_\ell \beta_\ell - p \sum_{\ell=1}^{4} m_\ell \gamma_\ell + z = 0 \tag{106}$$

We simplify notation by defining $\alpha_{14} := m_2\alpha_2 + m_4\alpha_4$, $\beta_{14} := \sum_{\ell=1}^{4} m_\ell \beta_\ell$, and[7] $\gamma_{14} := \sum_{\ell=1}^{4} = m_\ell \gamma_\ell$ and solve for the price p:

$$p \quad = \quad \frac{\alpha_{14}}{\gamma_{14}} + \frac{\beta_{14}}{\gamma_{14}} c + \frac{\beta_{14}}{\gamma_{14}} v + \frac{\beta_{14}}{\gamma_{14}} w + \frac{1}{\gamma_{14}} z. \tag{107}$$

As in the previous steps, we can interpret the price as a random variable that is normally distributed because it is a linear combination of normally distributed variables. This is the price random variable that emerges from equilibrium strategies of informed traders (in addition to noise traders). Thus, we write

$$\tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*) \quad = \quad \frac{\alpha_{14}}{\gamma_{14}} + \frac{\beta_{14}}{\gamma_{14}} c + \frac{\beta_{14}}{\gamma_{14}} \tilde{v} + \frac{\beta_{14}}{\gamma_{14}} \tilde{w} + \frac{1}{\gamma_{14}} \tilde{z}. \tag{108}$$

Since strategies are mutually known among traders with awareness level 4 in equilibrium, they can deduce information contained in the realization of the price about the fundamental and noise trading. Applying a linear transformation to $\tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*)$ and denoting by $\tilde{z}_4 := \frac{1}{\beta_{14}} \tilde{z}$, we isolate the information contained in the price by defining

$$\tilde{h}_4 := \left(\tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*) - \frac{\alpha_{14}}{\gamma_{14}} - \frac{\beta_{14}}{\gamma_{14}} c \right) \frac{\gamma_{14}}{\beta_{14}} = \tilde{v} + \tilde{w} + \frac{1}{\beta_{14}} \tilde{z} = \tilde{v} + \tilde{w} + \tilde{z}_4. \tag{109}$$

Denote by $\tau_4 := \frac{1}{Var[\tilde{v}+\tilde{w} | s_i, p = \tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*)]}$ the precision of the fundamental perceived by trader i of awareness type 4 conditional on signal realization s_i and the price $p = \tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*)$. Clearly, $\tau_4 = \frac{1}{Var[\tilde{v}+\tilde{w} | s_i - c, h_4]}$. From the projection theorem applied to normally distributed variables (e.g., DeGroot (1970, Chapter 5), Vives (2008, 10.2.1)) follows that

$$\tau_4 \quad = \quad \tau_{v+w} + \tau_{\varepsilon_i} + \tau_{z_4} \tag{110}$$

and

$$\begin{aligned} & \mathbb{E}[\tilde{v} + \tilde{w} \mid s_i, p = \tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*)] \\ & = \mathbb{E}[\tilde{v} + \tilde{w} \mid s_i - c, h_4] \\ & = \frac{\tau_{\varepsilon_i}}{\tau_4}(s_i - c) + \frac{\tau_{z_4}}{\tau_4} h_4 \\ & = \frac{\tau_{\varepsilon_i}}{\tau_4}(s_i - c) + \frac{\tau_{z_4}}{\tau_4} \left(p - \frac{\alpha_{14}}{\gamma_{14}} - \frac{\beta_{14}}{\gamma_{14}} c \right) \frac{\gamma_{14}}{\beta_{14}} \\ & = \frac{\tau_{\varepsilon_i}}{\tau_4} s_i + \frac{\tau_{z_4}}{\tau_4} \frac{\gamma_{14}}{\beta_{14}} p - \frac{\tau_{z_4}}{\tau_4} \frac{\alpha_{14}}{\beta_{14}} - \frac{\tau_{\varepsilon_i} + \tau_{z_4}}{\tau_4} c \end{aligned} \tag{111}$$

If all informed traders of lower awareness types use equilibrium strategies X_1^*, X_2^*, and X_3^*, respectively, and all informed traders but i of awareness type 4 use equilibrium strategies X_4^*, then conditional on information signal s_i and price $p = \tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*)$, trading returns $(\tilde{v} + \tilde{w} + c - p)x_i$ of informed trader i from quantity x_i is a random variable that is distributed

[7] The subscript "14" stands for awareness levels 1 to 4.

normally. Thus, as it is well-known in this case (e.g., Danthine and Moresi (1993, pp. 979–980), Vives (2008, 10.2.4)), CARA expected utility conditional on information signal s_i and price $p = \tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*)$ takes on the "mean-variance" form

$$\begin{aligned}
&\mathbb{E}[U((\tilde{v} + \tilde{w} + c - p)x_i) \mid s_i, p = \tilde{p}(X_4^*, X_4^*, X_2^*, X_1^*)] \\
&= \mathbb{E}[(\tilde{v} + \tilde{w} + c - p)x_i \mid s_i, p = \tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*)] \\
&\quad -\frac{\rho}{2} Var[(\tilde{v} + \tilde{w} + c - p)x_i \mid s_i, p = \tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*)] \\
&= x_i \mathbb{E}[\tilde{v} + \tilde{w} \mid s_i, p = \tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*)] + (c - p)x_i \\
&\quad -\frac{\rho}{2} x_i^2 Var[\tilde{v} + \tilde{w} \mid s_i, p = \tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*)] \qquad (112)
\end{aligned}$$

where the last line follows from well-known properties of the expected value and the variance.

Any strategy of informed trader i such that it's realized quantity is (ex post) optimal conditional on the price p and information signal s_i for every price p and information signal s_i is (when understood as a demand function) also interim optimal for i given just information signal s_i when other informed traders of awareness type 4 follow strategy X_4^* and informed traders of lower awareness types follow strategies X_1^*, X_2^*, and X_3^*, respectively.

For every p and s_i, maximizing expected utility given by equation (112) w.r.t. x_i yields the first-order condition

$$x_i^* = \frac{\mathbb{E}[\tilde{v} + \tilde{w} \mid s_i, p = \tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*)] + c - p}{\rho Var[\tilde{v} + \tilde{w} \mid s_1, p = \tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*)]} \qquad (113)$$

Since the objective function (112) is strictly concave in x_i, the first-order condition is also sufficient for the maximum. Using equations (110) and (111), we rewrite equation (113)

$$\begin{aligned}
x_i^* &= \left(\frac{\tau_{\varepsilon_i}}{\tau_4} s_i + \frac{\tau_{z_4}}{\tau_4} \frac{\gamma_{14}}{\beta_{14}} p - \frac{\tau_{z_4}}{\tau_4} \frac{\alpha_{14}}{\beta_{12}} - \frac{\tau_{\varepsilon_i} + \tau_{z_4}}{\tau_4} c + c - p \right) \frac{\tau_4}{\rho} \\
&= \frac{\tau_4 - \tau_{\varepsilon_i} - \tau_{z_4}}{\rho} c - \frac{\tau_{z_4}}{\rho} \frac{\alpha_{14}}{\beta_{14}} + \frac{\tau_{\varepsilon_i}}{\rho} s_i - \frac{\beta_{14}\tau_4 - \gamma_{14}\tau_{z_4}}{\beta_{14}\rho} p \qquad (114) \\
&= \alpha_4 + \beta_4 s_i - \gamma_4 p \qquad (115)
\end{aligned}$$

where last line follows from identifying coefficients with

$$\begin{aligned}
\alpha_4 &= c\frac{(m_2 + m_4)\tau_{v+w}\tau_{\varepsilon_i}\tau_z\rho + \tau_{v+w}\rho^3}{((m_2 + m_4)\tau_{\varepsilon_i}\tau_z + \rho^2)(m_4\tau_{\varepsilon_i}\tau_z + \rho^2)} - \left(c\frac{\tau_v \rho}{(m_2 + m_4)\tau_{\varepsilon_i}\tau_z + \rho^2} \right) \left(\frac{\tau_{v+w}\tau_{\varepsilon_i}\tau_z}{m_4\tau_{\varepsilon_i}\tau_z + \rho^2} \right) \qquad (116) \\
&= c\frac{(m_2 + m_4)\tau_{v+w}\tau_{\varepsilon_i}\tau_z\rho + \tau_{v+w}\rho^3}{((m_2 + m_4)\tau_{\varepsilon_i}\tau_z + \rho^2)(m_4\tau_{\varepsilon_i}\tau_z + \rho^2)} - \alpha_2 \frac{\tau_{v+w}\tau_{\varepsilon_i}\tau_z}{m_4\tau_{\varepsilon_i}\tau_z + \rho^2} \qquad (117) \\
\beta_4 &= \frac{\tau_{\varepsilon_i}}{\rho} \\
\gamma_4 &= \frac{\left(\tau_{v+w} + \tau_{\varepsilon_i} + \left(\frac{\tau_{\varepsilon_i}}{\rho} \right)^2 \tau_z \right) \rho}{m_4\tau_{\varepsilon_i}\tau_z + \rho^2} - \frac{\tau_{\varepsilon_i}\tau_z}{m_4\tau_{\varepsilon_i}\tau_z + \rho^2} \sum_{\ell=1}^{3} m_\ell \gamma_\ell
\end{aligned}$$

using the fact that $\tau_{z_4} = \beta_{14}^2 \tau_z$. Recall that $\tilde{\varepsilon}_i$ is i.i.d. across informed traders i. Thus, α_4, β_4 and γ_4 are identical for all traders with awareness level 4.

What remains to be shown is that $\gamma_4 = \gamma_3$. First, note that $\tau_4 = \tau_3$. Rewriting above equations for γ_3 and γ_4

$$\begin{aligned} ((m_3+m_4)\tau_{\varepsilon_i}\tau_z+\rho^2)\gamma_3 &= \tau_3\rho-\tau_{\varepsilon_i}\tau_z(m_1+m_2)\gamma_1 && (118)\\ (m_4\tau_{\varepsilon_i}\tau_z+\rho^2)\gamma_4 &= \tau_3\rho-\tau_{\varepsilon_i}\tau_z(m_1+m_2)\gamma_1-\tau_{\varepsilon_i}\tau_z m_3\gamma_3 && (119)\end{aligned}$$

Subtracting equation (118) from equation (119) yields

$$\begin{aligned} (m_4\tau_{\varepsilon_i}\tau_z+\rho^2)\gamma_4-((m_3+m_4)\tau_{\varepsilon_i}\tau_z+\rho^2)\gamma_3 &= -\tau_{\varepsilon_i}\tau_z m_3\gamma_3 && (120)\\ (m_4\tau_{\varepsilon_i}\tau_z+\rho^2)\gamma_4 &= (m_4\tau_{\varepsilon_i}\tau_z+\rho^2)\gamma_3 && (121)\\ \gamma_4 &= \gamma_3 && (122)\end{aligned}$$

From the previous steps we concluded that for awareness level, equilibrium strategies are identical for all traders with that awareness level. Thus, the equilibrium is symmetric. This proves the characterization of equilibrium strategies of awareness types 4 and completes the characterization of linear symmetric Bayes-Nash equilibrium of the market game with unawareness. □

Proof of Proposition 4

Define

$$\begin{aligned} A &:= c\frac{(m_2+m_4)\tau_{v+w}\tau_{\varepsilon_i}\tau_z\rho+\tau_{v+w}\rho^3}{(m_2+m_4)\tau_{\varepsilon_i}\tau_z+\rho^2}\\ D &:= m_4\tau_{\varepsilon_i}\tau_z+\rho^2\\ B &:= \tau_{v+w}\tau_{\varepsilon_i}\tau_z\end{aligned}$$

Then

$$\alpha_4 = \frac{A}{D}-\frac{B}{D}\alpha_2.$$

It follows

$$\begin{aligned} \alpha_4 &< \alpha_2\\ A &< (B+D)\alpha_2\\ c\frac{((m_2+m_4)\tau_{v+w}\tau_{\varepsilon_i}\tau_z+\tau_{v+w}\rho^2)\rho}{(m_2+m_4)\tau_{\varepsilon_i}\tau_z+\rho^2} &< (\tau_{v+w}\tau_{\varepsilon_i}\tau_z+m_4\tau_{\varepsilon_i}\tau_z+\rho^2)\alpha_2\\ c\frac{((m_2+m_4)\tau_{v+w}\tau_{\varepsilon_i}\tau_z+\tau_{v+w}\rho^2)\rho}{(m_2+m_4)\tau_{\varepsilon_i}\tau_z+\rho^2} &< (\tau_{v+w}\tau_{\varepsilon_i}\tau_z+m_4\tau_{\varepsilon_i}\tau_z+\rho^2)c\frac{\tau_v\rho}{(m_2+m_4)\tau_{\varepsilon_i}\tau_z+\rho^2}\\ (m_2+m_4)\tau_{v+w}\tau_{\varepsilon_i}\tau_z+\tau_{v+w}\rho^2 &< (\tau_{v+w}\tau_{\varepsilon_i}\tau_z+m_4\tau_{\varepsilon_i}\tau_z+\rho^2)\tau_v\\ m_4\tau_{v+w}\tau_{\varepsilon_i}\tau_z+\tau_{v+w}\rho^2+m_2\tau_{v+w}\tau_{\varepsilon_i}\tau_z &< m_4\tau_v\tau_{\varepsilon_i}\tau_z+\tau_v\rho^2+\tau_v\tau_{v+w}\tau_\varepsilon\tau_z\end{aligned}$$

Observe that $\tau_{v+w} = \frac{1}{\sigma_v^2+\sigma_w^2} < \tau_v = \frac{1}{\sigma_v^2}$. Thus, a sufficient condition for the inequality to hold is $m_2 \leq \tau_v$. □

Proof of Proposition 5

From Proposition 1 we know $\gamma_3 = \gamma_4$ and $\gamma_1 = \gamma_2$. We claim $\gamma_3 < \gamma_1$. This is equivalent to

$$\frac{\tau_3\rho}{(m_3+m_4)\tau_{\varepsilon_i}\tau_z+\rho^2} - \frac{\tau_{\varepsilon_i}\tau_z}{(m_3+m_4)\tau_{\varepsilon_i}\tau_z+\rho^2}(m_1+m_2)\gamma_1 \quad < \quad \gamma_1$$

$$\begin{aligned}
\tau_3\rho - \tau_{\varepsilon_i}\tau_z(m_1+m_2)\gamma_1 &< ((m_3+m_4)\tau_{\varepsilon_i}\tau_z+\rho^2)\gamma_1 \\
\tau_3\rho &< (\tau_{\varepsilon_i}\tau_z+\rho^2)\gamma_1 \\
\tau_3\rho &< (\tau_{\varepsilon_i}\tau_z+\rho^2)\frac{\tau_1\rho}{(\tau_{\varepsilon_i}\tau_z+\rho^2)} \\
\tau_{v+w}+\tau_{\varepsilon_i}+\left(\frac{\tau_{\varepsilon_i}}{\rho}\right)^2\tau_z &< \tau_v+\tau_{\varepsilon_i}+\left(\frac{\tau_{\varepsilon_i}}{\rho}\right)^2\tau_z \\
\tau_{v+w} &< \tau_v \\
\frac{1}{\sigma_v^2+\sigma_w^2} &< \frac{1}{\sigma_v^2}
\end{aligned}$$

□

Proof of Proposition 6

Using the characterization of Proposition 1,

$$\begin{aligned}
\gamma_3 &< 0 \\
\frac{\tau_3\rho}{(m_3+m_4)\tau_{\varepsilon_i}\tau_z+\rho^2} &< \frac{\tau_{\varepsilon_i}\tau_z}{(m_3+m_4)\tau_{\varepsilon_i}\tau_z+\rho^2}(m_1+m_2)\gamma_1 \\
\tau_3\rho &< (m_1+m_2)\tau_{\varepsilon_i}\tau_z\frac{\tau_1\rho}{\tau_{\varepsilon_i}\tau_z+\rho^2} \\
\frac{\tau_3}{\tau_1} &< \frac{(m_1+m_2)\tau_{\varepsilon_i}\tau_z}{\tau_{\varepsilon_i}\tau_z+\rho^2}
\end{aligned}$$

Precisions τ_3 and τ_1 differ just by τ_{v+w} and τ_v, respectively. Note that by definition $\tau_{v+w} = \frac{1}{\sigma_v^2+\sigma_w^2} < \tau_v = \frac{1}{\sigma_v^2}$. If σ_w^2 is sufficiently larger than σ_v^2, then the inequality is satisfied.

The argument is analogous for γ_4. □

Proof of Proposition 9

Awareness level 1: From the proof of Proposition 1 follows

$$\begin{aligned}
\mathbb{E}\left[\int_{M_1} X_1^*(\tilde{s}_i)(\tilde{p}(X_1^*))di\right] &= \mathbb{E}[m_1(\beta_1\tilde{v}-\gamma_1\tilde{p}(X_1^*))] \\
&= m_1\beta_1\mathbb{E}[\tilde{v}] - m_1\gamma_1\frac{\beta_1}{\gamma_1}\mathbb{E}[\tilde{v}] + m_1\frac{1}{\gamma_1}\mathbb{E}[\tilde{z}] = 0.
\end{aligned}$$

We observe that $\int_{M_1} X_1^*(\tilde{s}_i)(\tilde{p}(X_1^*))di$ follows a normal distribution, which, as we just computed, has mean zero. $\left|\int_{M_1} X_1^*(\tilde{s}_i)(\tilde{p}(X_1^*))di\right|$ follows a folded normal distribution. Thus, by standard properties of the folded normal distribution,

$$\begin{aligned}
\nu_1 &= \mathbb{E}\left[\left|\int_{M_1} X_1^*(\tilde{s}_i)(\tilde{p}(X_1^*))di\right|\right] \\
&= \mathbb{E}[|m_1(\beta_1\tilde{v} - \gamma_1\tilde{p}(X_1^*))|] \\
&= \sqrt{\frac{2}{\pi}}\sqrt{Var(m_1(\beta_1\tilde{v} - \gamma_1\tilde{p}(X_1^*)))} \\
&= \sqrt{\frac{2}{\pi}}\sqrt{Var(m_1\tilde{z})} \\
&= \sqrt{\frac{2}{\pi}}m_1\sigma_z
\end{aligned}$$

Awareness level 2: From the proof of Proposition 1 follows

$$\begin{aligned}
\mathbb{E}\left[\int_{M_2} X_2^*(\tilde{s}_i)(\tilde{p}(X_2^*, X_1^*))di\right] &= \mathbb{E}[m_2(\alpha_2 + \beta_1(\tilde{v} + c) - \gamma_1\tilde{p}(X_2^*, X_1^*))] \\
&= m_2\alpha_2 + m_2\beta_1 c - m_2(m_2 + m_4)\alpha_2 - m_2\beta_1 c \\
&= m_2(1 - m_2 - m_3)\alpha_2 := \mu
\end{aligned}$$

Observe $\mu > 0$. We also observe that $\int_{M_2} X_2^*(\tilde{s}_i)(\tilde{p}(X_2^*, X_1^*))di$ is normally distributed but not with mean zero. Define also

$$\sigma^2 := Var\left[\int_{M_2} X_2^*(\tilde{s}_i)(\tilde{p}(X_2^*, X_1^*))di\right] \tag{123}$$

$$A := e^{-\frac{\mu^2}{2\sigma^2}} \tag{124}$$

$$B := \mu\left(1 - 2\Phi\left(-\frac{\mu}{\sigma}\right)\right) \tag{125}$$

Observe that $A > 1$ and $B > 0$. By standard properties of the folded normal distribution,

$$\begin{aligned}
v_2 &= \mathbb{E}\left[\left|\int_{M_2} X_2^*(\tilde{s}_i)(\tilde{p}(X_2^*, X_1^*))di\right|\right] \\
&= \mathbb{E}[|m_2(\alpha_2 + \beta_2\tilde{v} - \gamma_2\tilde{p}(X_2^*, X_1^*))|] \\
&= \sqrt{\frac{2}{\pi}}\sqrt{Var(m_2(\alpha_2 + \beta_2\tilde{v} - \gamma_2\tilde{p}(X_2^*, X_1^*)))}A + B \\
&= \sqrt{\frac{2}{\pi}}\sqrt{Var(m_2(\alpha_2 - (m_2 + m_4)\alpha_2 - \beta_{12}c - \tilde{z}))}A + B \\
&= \sqrt{\frac{2}{\pi}}m_2\sigma_z A + B
\end{aligned}$$

Since $m_1 = m_2$, $A > 1$ and $B > 0$, $\nu_1 < \nu_2$.

Awareness level 3: From the proof of Proposition 1 follows

$$\begin{aligned}\mathbb{E}\left[\int_{M_3} X_3^*(\tilde{s}_i)(\tilde{p}(X_3^*, X_1^*))di\right] &= \mathbb{E}[m_3(\beta_1(\tilde{v}+\tilde{w}) - \gamma_1\tilde{p}(X_3^*, X_1^*))] \\ &= \mathbb{E}[m_3(\beta_3(\tilde{w}+\tilde{w}) - \gamma_3\frac{\beta_{13}}{\gamma_{13}}(\tilde{v}+\tilde{w}) - \frac{\gamma_3}{\gamma_{13}}\tilde{z})] = 0\end{aligned}$$

By standard properties of the folded normal distribution,

$$\begin{aligned}v_3 &= \mathbb{E}\left[\left|\int_{M_3} X_2^*(\tilde{s}_i)(\tilde{p}(X_3^*, X_1^*))di\right|\right] \\ &= \mathbb{E}[|m_3(\beta_1(\tilde{v}+\tilde{w}) - \gamma_1\tilde{p}(X_3^*, X_1^*))|] \\ &= \sqrt{\frac{2}{\pi}}\sqrt{Var\left[m_3\beta_3\tilde{v} + m_3\beta_3\tilde{w} - m_3\gamma_3\frac{\beta_3}{\gamma_{13}}\tilde{v} - m_3\gamma_3\frac{\beta_3}{\gamma_{13}}\tilde{w} - m_3\gamma_3\frac{1}{\gamma_{13}}\tilde{z}\right]} \\ &= \sqrt{\frac{2}{\pi}}\sqrt{m_3^2\beta_3^2\left(1-\frac{\gamma_3}{\gamma_{13}}\right)^2\sigma_v^2 + m_3^2\beta_3^2\left(1-\frac{\gamma_3}{\gamma_{13}}\right)^2\sigma_w^2 + m_3^2\left(\frac{\gamma_3}{\gamma_{13}}\right)^2\sigma_z^2}\end{aligned}$$

Since $\gamma_3 < \gamma_1$ by Proposition 5 and thus $\gamma_3 < \gamma_{13}$, we are unable to compare v_3 with v_1 and v_2 without further assumptions.

Awareness level 4: From the proof of Proposition 1 follows

$$\begin{aligned}\mathbb{E}\left[\int_{M_4} X_4^*(\tilde{s}_i)(\tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*))di\right] &= \mathbb{E}[m_4(\alpha_4 + \beta_4(\tilde{v}+\tilde{w}+c) - \gamma_4\tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*))] \\ &= m_4\alpha_4 + m_4\beta_1 c - m_4\frac{\gamma_4}{\gamma_{14}}\alpha_{14} - m_4\frac{\gamma_4}{\gamma_{14}}\beta_1 c \\ &= m_4\left(1 - m_4\frac{\gamma_4}{\gamma_{14}}\right)\alpha_4 + m_4\beta_1\left(1-\frac{\gamma_4}{\gamma_{14}}\right)c - m_2 m_4\frac{\gamma_4}{\gamma_{14}}\alpha_2 := \zeta\end{aligned}$$

In general, we may have $\zeta \neq 0$. Define also

$$\varsigma^2 := Var\left[\int_{M_4} X_4^*(\tilde{s}_i)(\tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*))di\right] \tag{126}$$

$$C := e^{-\frac{\zeta^2}{2\varsigma^2}} \tag{127}$$

$$D := \zeta\left(1 - 2\Phi\left(-\frac{\zeta}{\varsigma}\right)\right) \tag{128}$$

By standard properties of the folded normal distribution,

$$\begin{aligned}v_4 &= \mathbb{E}\left[\left|\int_{M_4} X_4^*(\tilde{s}_i)(\tilde{p}(X_4^*, X_3^*, X_2^*, X_1^*))di\right|\right] \\ &= \mathbb{E}[|m_4(\alpha_4 + \beta_4\tilde{v} + \beta_4\tilde{w} + \beta_4 c - \gamma_4\tilde{p}(X_2^*, X_1^*))|] \\ &= \sqrt{\frac{2}{\pi}}\sqrt{Var\left[m_4\beta_1\left(1-\frac{\gamma_4}{\gamma_{14}}\right)\tilde{v} + m_4\beta_1\left(1-\frac{\gamma_4}{\gamma_{14}}\right)\tilde{w} - m_4\frac{\gamma_4}{\gamma_{14}}\tilde{z}\right]} \\ &= \sqrt{\frac{2}{\pi}}\sqrt{m_4^2\beta_1^2\left(1-\frac{\gamma_4}{\gamma_{14}}\right)^2\sigma_v^2 + m_4^2\beta_1^2\left(1-\frac{\gamma_4}{\gamma_{14}}\right)^2\sigma_w^2 - m_4^2\frac{\gamma_4^2}{\gamma_{14}^2}\sigma_z^2 C + D}\end{aligned}$$

We are unable to compare ν_4 with ν_1, ν_2, and ν_3 without further assumptions. $\square$

Proof of Proposition 10

Market game as perceived by traders of type B before disclosure: In the market game perceived by traders of type B before disclosure, all traders are of awareness type B. Suppose they use the linear equilibrium strategy defined by for all s

$$X^*(s)(p) = \beta_B s - \gamma_B p \tag{129}$$

for any realization of the price p and some $\beta_B, \gamma_B \in \mathbb{R}$ with $\beta_B, \gamma_B \neq 0$.

We can interpret the quantity $\tilde{x}^* = X^*(\tilde{s})(p)$ as a random variable that is normally distributed since it is a linear transformation of the normally distributed random variables.

Using the market clearing condition in equation (42), we obtain

$$\beta_B s - \gamma_B p + z = 0.$$

Solving for p, we obtain

$$p = \frac{\beta_B}{\gamma_B} s + \frac{1}{\gamma_B} z. \tag{130}$$

Since this holds for all s and z, we interpret the price as a random variable that is normally distributed since it is a linear combination of normally distributed random variables. Since the price random variable emerges from equilibrium demand functions X^*, we write

$$\tilde{p}(X^*) = \frac{\beta_B}{\gamma_B} \tilde{s} + \frac{1}{\gamma_B} \tilde{z}. \tag{131}$$

Observe that the price does not contain more information about v than the common signal. Thus, $\mathbb{E}[v \mid s, p = \tilde{p}(X^*)] = \mathbb{E}[v \mid s]$ and $Var[v \mid s, p = \tilde{p}(X^*)] = Var[v \mid s]$.

From the projection theorem applied to normally distributed variables (e.g., DeGroot (1970, Chapter 5), Vives (2008, 10.2.1)) follows that

$$\frac{1}{Var[v \mid s, p = \tilde{p}(X^*)]} = \frac{1}{Var[\tilde{v} \mid s]} =: \tau_B = \tau_v + \tau_\varepsilon \tag{132}$$

and

$$\mathbb{E}[v \mid s, p = \tilde{p}(X^*)] = \mathbb{E}[\tilde{v} \mid s] = \frac{\tau_\varepsilon}{\tau_B} s \tag{133}$$

If all non-noise traders but i use equilibrium demand schedules X^*, then conditional on the common signal s and price $p = \tilde{p}(X^*)$, trading returns $(\tilde{v} - p)x_i$ of trader i from quantity x_i is a random variable that is distributed normally. It is well-known that in such a case (e.g., Danthine and Moresi (1993, pp. 979–980), Vives (2008, 10.2.4)) that the expected CARA utility conditional on the common signal s and price $p = \tilde{p}(X^*)$ takes the "mean-variance" form

$$\begin{aligned}
&\mathbb{E}[U((\tilde{v} - p)x_i) \mid s, p = \tilde{p}(X^*)] \\
&= \mathbb{E}[(\tilde{v} - p)x_i \mid s, p = \tilde{p}(X^*)] - \frac{\rho}{2} Var[(\tilde{v} - p)x_i \mid s, p = \tilde{p}(X^*)] \\
&= x_i \mathbb{E}[\tilde{v} \mid s, p = \tilde{p}(X^*)] - p x_i - \frac{\rho}{2} x_i^2 Var[\tilde{v} \mid s, p = \tilde{p}(X^*)]
\end{aligned} \tag{134}$$

where the last line follows from well-known properties of the expected value and the variance.

Any demand schedule of trader i such that it's realized quantity is (ex post) optimal conditional on the price p and common signal s for every p and s, is when understood as a demand function also interim optimal for i given just the common signal s when other non-noise traders use the equilibrium demand schedule X^*. For every p and s, maximizing expected utility given in equation (134) w.r.t. x_i yields the first-order condition

$$x_i^* = \frac{\mathbb{E}[\tilde{v} \mid s, p = \tilde{p}(X^*)] - p}{\rho Var[\tilde{v} \mid s, p = \tilde{p}(X^*)]}. \tag{135}$$

Since the objective function is strictly concave in x_i, the first-order condition is also sufficient for the maximum. Substituting equations (132) and (133) into equation (135) yields

$$x_i^* = \beta_B s - \gamma_B p \tag{136}$$

with coefficients

$$\beta_B = \frac{\tau_\varepsilon}{\rho} \tag{137}$$

$$\gamma_B = \frac{\tau_v + \tau_\varepsilon}{\rho}. \tag{138}$$

This proves the characterization of the equilibrium demand functions in the market game of the less expressive tree.

Market game perceived by traders of type A under non-disclosure: Under non-disclosure, all traders of type A in M_A take into account that all traders in M_B remain unaware. Thus, they take into account that traders in M_B use equilibrium demand functions derived above. Suppose all traders of type A use linear equilibrium demand functions for all s

$$X_A^{N*}(s)(p) = \alpha_A^N + \beta_A^N s - \gamma_A^N p \tag{139}$$

for any realization of the price p and some $\alpha_A^N, \beta_A^N, \gamma_A^N \in \mathbb{R}$ with $\beta_A^N, \gamma_A^N \neq 0$.

We can interpret the quantity $\tilde{x}_A^{N*} = X_A^{N*}(\tilde{s})(p)$ as a random variable that is normally distributed since it is a linear transformation of the normally distributed random variables.

Using the market clearing condition of equation (43) and equilibrium demand functions of traders of type B (equation (136)), we obtain

$$\int_{M_B} (\beta_B s - \gamma_B p) di + \int_{M_A} (\alpha_A^N + \beta_A^N s - \gamma_A^N p) di + z = 0 \tag{140}$$

$$m_B \beta_B s - m_B \gamma_B p + m_A \alpha_A^N + m_A \beta_A^N s - m_A \gamma_A^N p + z = 0 \tag{141}$$

We simplify notation by defining $\gamma^N := m_A \gamma_A^N + m_B \gamma_B$ and $\beta^N := m_A \beta_A^N + m_B \beta_B$. We solve for the equilibrium price

$$p = \frac{m_A \alpha_A^N}{\gamma^N} + \frac{\beta^N}{\gamma^N} s + \frac{1}{\gamma^N} z. \tag{142}$$

We can interpret the price as a random variable that is normally distributed because it is a linear combination of normally distributed variables. Since it is the random price that emerges from equilibrium demand functions in this market game, we write

$$\tilde{p}(X_A^{N*}, X^*) = \frac{m_A \alpha_A^N}{\gamma^N} + \frac{\beta^N}{\gamma^N}\tilde{s} + \frac{1}{\gamma^N}\tilde{z}. \tag{143}$$

Observe that the price does not contain more information about $v + w$ than the common signal. Thus, $\mathbb{E}[v + w \mid s, p = \tilde{p}(X^*)] = \mathbb{E}[v + w \mid s]$ and $Var[v + w \mid s, p = \tilde{p}(X^*)] = Var[v + w \mid s]$.

From the projection theorem applied to normally distributed variables (e.g., DeGroot (1970, Chapter 5), Vives (2008, 10.2.1)) follows that

$$\frac{1}{Var[\tilde{v} + \tilde{w} \mid s, p = \tilde{p}(X_A^{N*}, X^*)]} = \frac{1}{Var[\tilde{v} + \tilde{w} \mid s]} := \tau^N \quad = \quad \tau_{v+w} + \tau_\varepsilon \tag{144}$$

and

$$\mathbb{E}[\tilde{v} + \tilde{w} \mid s, p = \tilde{p}(X_A^{N*}, X^*)] = \mathbb{E}[\tilde{v} + \tilde{w} \mid s] \quad = \quad \bar{w} + \frac{\tau_\varepsilon}{\tau^N}(s - \bar{w}). \tag{145}$$

If all traders of type B use equilibrium demand X^* and all traders of type A but i use equilibrium demand X_A^{N*}, then conditional on the common signal s and price $p = \tilde{p}(X_A^{N*}, X^*)$, trading returns $(\tilde{v} + \tilde{w} - p)x_i$ of trader $i \in M_A$ from quantity x_i is a random variable that is distributed normally. Thus, as it is well-known in this case (e.g., Danthine and Moresi (1993, pp. 979–980), Vives (2008, 10.2.4)), CARA expected utility conditional on s and price $p = \tilde{p}(X_A^{N*}, X^*)$ takes the "mean-variance" form

$$\begin{aligned} &\mathbb{E}[U((\tilde{v} + \tilde{w} - p)x_i) \mid s, p = \tilde{p}(X_A^{N*}, X^*)] \\ &= \mathbb{E}[(\tilde{v} + \tilde{w} - p)x_i \mid s, p = \tilde{p}(X_A^{N*}, X^*)] - \frac{\rho}{2}Var[(\tilde{v} + \tilde{w} - p)x_i \mid s, p = \tilde{p}(X_A^{N*}, X^*)] \end{aligned} \tag{146}$$

Any demand schedule of trader i such that its realized quantity is (ex post) optimal conditional on the price p and signal s for every price p and s is (when understood as a demand function) also interim optimal for i given just s when other traders of type A follow X_A^{N*} and traders of type B use X^*.

For every p and s, maximizing expected utility given in equation (146) w.r.t. x_i yields the first-order condition

$$x_i^* \quad = \quad \frac{\mathbb{E}[\tilde{v} + \tilde{w} \mid s, p = \tilde{p}(X_A^{N*}, X^*)] - p}{\rho Var[\tilde{v} + \tilde{w} \mid s, p = \tilde{p}(X_A^{N*}, X^*)]}. \tag{147}$$

Since the objective function (146) is strictly concave in x_i, the first-order condition is also sufficient for the maximum. Substituting equations (144) and (145) into equation (147), we obtain

$$x_i^* \quad = \quad \alpha_A^N + \beta_A^N s - \gamma_A^N p \tag{148}$$

with the identification of coefficients

$$\alpha_A^N = \frac{\tau_{v+w}}{\rho}\bar{w} \tag{149}$$

$$\beta_A^N = \frac{\tau_\varepsilon}{\rho} \tag{150}$$

$$\gamma_A^N = \frac{\tau_{v+w} + \tau_\varepsilon}{\rho} \tag{151}$$

This proofs the characterization of demand functions in the market game under non-disclosure.

Market game after disclosure: Under disclosure by some trader of type A, all traders of type B have the same awareness as traders of type A and this is commonly known among all traders. Suppose all traders use linear equilibrium demand functions for all s

$$X^{D*}(s)(p) = \alpha^D + \beta^D s - \gamma^D p \tag{152}$$

for any realization of the price p and some $\alpha^D, \beta^D, \gamma^D \in \mathbb{R}$ with $\beta^D, \gamma^D \neq 0$.

We can interpret the quantity $\tilde{x}^{D*} = X^{D*}(\tilde{s})(p)$ as a random variable that is normally distributed since it is a linear transformation of the normally distributed random variables.

Using the market clearing condition of equation (44), we obtain

$$\alpha^D + \beta^D s - \gamma^D p + z = 0 \tag{153}$$

We solve for the equilibrium price

$$p = \frac{\alpha^D}{\gamma^D} + \frac{\beta^D}{\gamma^D}s + \frac{1}{\gamma^D}z. \tag{154}$$

We can interpret the price as a random variable that is normally distributed because it is a linear combination of normally distributed variables. Since it is the random price that emerges from equilibrium demand functions in this market game, we write

$$\tilde{p}(X^{D*}) = \frac{\alpha^D}{\gamma^D} + \frac{\beta^D}{\gamma^D}\tilde{s} + \frac{1}{\gamma^D}\tilde{z}. \tag{155}$$

Observe that the price does not contain more information about $v + w$ than the common signal. Thus, $\mathbb{E}[v + w \mid s, p = \tilde{p}(X^{D*})] = \mathbb{E}[v + w \mid s]$ and $Var[v + w \mid s, p = \tilde{p}(X^{D*})] = Var[v + w \mid s]$.

From the projection theorem applied to normally distributed variables (e.g., DeGroot (1970, Chapter 5), Vives (2008, 10.2.1)) follows that

$$\frac{1}{Var[\tilde{v} + \tilde{w} \mid s, p = \tilde{p}(X^{D*})]} = \frac{1}{Var[\tilde{v} + \tilde{w} \mid s]} := \tau^D = \tau_{v+w} + \tau_\varepsilon$$

and

$$\mathbb{E}[\tilde{v} + \tilde{w} \mid s, p = \tilde{p}(X^{D*})] = \mathbb{E}[\tilde{v} + \tilde{w} \mid s] = \bar{w} + \frac{\tau_\varepsilon}{\tau^D}(s - \bar{w}).$$

We observe that $\tau^D = \tau^N$ and $\mathbb{E}[\tilde{v} + \tilde{w} \mid s, p = \tilde{p}(X^{D*})] = \mathbb{E}[\tilde{v} + \tilde{w} \mid s, p = \tilde{p}(X_A^{N*}, X^*)]$. Thus, we can conclude that $X^{D*} = X_A^{N*} = X_B^{D*}$. This completes the characterization of equilibrium demand functions of all non-noise traders after disclosure. □

Proof of Proposition 11

$$\begin{aligned} \gamma_A &< \gamma_B && (156) \\ \frac{\tau_{v+w}+\tau_\varepsilon}{\rho} &< \frac{\tau_v+\tau_\varepsilon}{\rho} \\ \tau_{v+w} &< \tau_v \\ \frac{1}{\sigma_v^2+\sigma_w^2} &< \frac{1}{\sigma_v^2} && (157) \end{aligned}$$

□

Proof of Proposition 12

Expected CARA equilibrium utility of a trader of type A conditional on common signal s and price p takes the familiar "mean-variance" form

$$\begin{aligned} & \mathbb{E}[U((\tilde{f}_A-p)X_A^{N*}(i,s)(p)) \mid s,p] && (158) \\ = & \ (\mathbb{E}[f_A \mid s,p]-p)X_A^{N*}(i,s)(p)-\frac{1}{2}\rho(X_A^{N*}(i,s)(p))^2 Var[f_A \mid s,p] && (159) \\ = & \ (\mathbb{E}[f_A \mid s]-p)(\alpha+\beta s-\gamma_A p)-\frac{1}{2}\frac{(\alpha+\beta s-\gamma_A p)^2}{\gamma_A} \\ = & \ \frac{1}{2}\gamma_A p^2-\mathbb{E}[f_A \mid s]\gamma_A p+k && (160) \end{aligned}$$

with

$$k \ := \ \beta\mathbb{E}[f_A \mid s]s+\alpha\mathbb{E}[f_A \mid s]-\frac{1}{2}\frac{\beta^2 s^2+2\alpha\beta s+\alpha^2}{\gamma_A} \qquad (161)$$

where the second equation uses the equilibrium demand schedules of traders of type A characterized in Proposition 10, the fact that signals are common and hence price does not contain further information about the fundamental f_B, as well as the observation that

$$Var[f_A \mid s,p] \ = \ \frac{1}{\rho\gamma_A}. \qquad (162)$$

Note that equation (160) is a function convex in prices p. We differentiate equation (160) w.r.t. p to obtain the necessary and sufficient condition for a minimum,

$$\frac{\partial\mathbb{E}[U((\tilde{f}_A-p)X_A^{N*}(i,s)(p)) \mid s,p]}{\partial p} \ = \ \gamma_A p-\gamma_A\mathbb{E}[f_A \mid s]=0. \qquad (163)$$

Thus, the price minimizing expected CARA equilibrium utility of a trader of type A conditional on common signal s and price p is

$$p_{\min} \ = \ \mathbb{E}[f_A \mid s]. \qquad (164)$$

From Proposition 10 follows that conditional on s and p,

$$\mathbb{E}[U((\tilde{f}_A-p)X_A^{N*}(i,s)(p)) \mid s,p] \ = \ \mathbb{E}[U((\tilde{f}_A-p)X_A^{D*}(i,s)(p)) \mid s,p]. \qquad (165)$$

That is, any effect of disclosure on expected utility of trader A comes only through the change in the market clearing price.

Recall the market clearing price under nondisclosure,

$$\tilde{p}(X_A^{N*}, X^*) \quad = \quad m_A \frac{\alpha}{m_A\gamma_A + (1-m_A)\gamma_B} + \frac{\beta}{m_A\gamma_A + (1-m_A)\gamma_B} s + \frac{\tilde{z}}{m_A\gamma_A + (1-m_A)\gamma_B}$$

Note that given s and $p = \tilde{p}(X_A^{N*}, X^*)$, noise-trading can be inferred via this formula. By solving for z, we define a function that maps the signal and the equilibrium market clearing price in the market game under non-disclosure to the demand of noise-traders:

$$z(s,p) \quad := \quad (m_A\gamma_A + (1-m_A)\gamma_B)p - \beta s - m_A\alpha \tag{166}$$

Next, note that (with some abuse of notation) we can rewrite the market clearing price after disclosure, equation (155), as

$$\tilde{p}(X_A^{D*}) \quad = \quad \mathbb{E}[f_A \mid s] + \frac{\tilde{z}}{\gamma_A} \tag{167}$$

Note further, we can write the conditional expectation of the fundamental as (Vives (2008, p. 378))

$$\mathbb{E}[f_A \mid s] \quad = \quad (1-\xi)\bar{w} + \xi s \tag{168}$$

with

$$\xi \quad := \quad \frac{\tau_\varepsilon}{\tau_{v+w} + \tau_\varepsilon}. \tag{169}$$

Thus, we can write the market clearing price after disclosure as

$$\tilde{p}(X_A^{D*}) \quad = \quad (1-\xi)\bar{w} + \xi s + \frac{\tilde{z}}{\gamma_A}. \tag{170}$$

Given s and $p = \tilde{p}(X_A^{N^*}, X^*)$, a trader of type A can anticipate the market clearing price after disclosure using function $z(s,p)$

$$p^D(s,p) \quad = \quad (1-\xi)\bar{w} + \xi s + \frac{z(s,p)}{\gamma_A}. \tag{171}$$

Hence, conditional on s and $p = \tilde{p}(X_A^{N^*}, X^*)$, she can also anticipate her expected utility after disclosure, namely $\mathbb{E}[U((\tilde{f}_A - p)X_A^{D*}(i,s)(p^D(s,p))) \mid s, p]$. Her equilibrium disclosure strategy must satisfy $d^*(i,s)(p = \tilde{p}(X_A^{N^*}, X^*)) = D$ if and only if

$$\mathbb{E}[U((\tilde{f}_A - p^D(s,p))X_A^{D*}(i,s)(p^D(s,p))) \mid s,p] \quad \geq \quad \mathbb{E}[U((\tilde{f}_A - p)X_A^{N*}(i,s)(p)) \mid s,p]. \tag{172}$$

Since the conditional expected CARA utility is quadratic in p, we have that

$$\mathbb{E}[U((\tilde{f}_A - p^D(s,p))X_A^{D*}(i,s)(p^D(s,p))) \mid s,p] \quad = \quad \mathbb{E}[U((\tilde{f}_A - q^D(s,p))X_A^{D*}(i,s)(q^D(s,p))) \mid s,p] \tag{173}$$

with $q^D(s,p) = (1-\xi)\bar{w} + \xi s - \frac{z(s,p)}{\gamma_A}$. This is the price that has the same distance to $p_{\min}$ as $p^D(s,p)$ but lies opposite to $p^D(s,p)$ (see Figure (4)). It becomes clear that since the conditional expected CARA utility is quadratic in price, above inequality (172) is satisfied if and only if for $p = \tilde{p}(X_A^{D*}, X^*)$

$$\begin{aligned}
&\min\left\{(1-\xi)\bar{w} + \xi s - \frac{z(s,p)}{\gamma_A}, (1-\xi)\bar{w} + \xi s + \frac{z(s,p)}{\gamma_A}\right\} \\
\leq\ & p \leq \max\left\{(1-\xi)\bar{w} + \xi s - \frac{z(s,p)}{\gamma_A}, (1-\xi)\bar{w} + \xi s + \frac{z(s,p)}{\gamma_A}\right\}. \qquad (174)
\end{aligned}$$

The rest of the proof is to simplify this expression and bring into the form as presented in the proposition. Note that

$$\begin{aligned}
p^D(s,p) &= (1-\xi)\bar{w} + \xi s + \frac{z(s,p)}{\gamma_A} && (175)\\
&= (1-\xi)\bar{w} + \xi s + \frac{(m_A\gamma_A + (1-m_A)\gamma_B)p - \beta s - m_A\alpha}{\gamma_A} \\
&= (1-\xi)\bar{w} + \frac{(m_A\gamma_A + (1-m_A)\gamma_B)p - m_A\alpha}{\gamma_A} && (176)
\end{aligned}$$

Thus, the post-disclosure market clearing price does not depend on s anymore. Using the definitions of the coefficients, we obtain the conditions presented in the proposition.

Define

$$\begin{aligned}
\lambda &:= \frac{m_A\gamma_A + (1-m_A)\gamma_B}{\gamma_A} && (177)\\
\kappa &= (1-\xi)\bar{w} && (178)\\
\chi &= \frac{m_A\alpha}{\gamma_A} && (179)\\
\eta &= 2\xi && (180)
\end{aligned}$$

Note that $\lambda > 1$ and $\eta > 0$. Note further that

$$\begin{aligned}
p^D(p) &= \lambda p + \kappa - \chi && (181)\\
q^D(s,p) &= -\lambda p + \eta s + \kappa + \chi && (182)
\end{aligned}$$

The problem becomes

$$\min\{\lambda p + \kappa - \chi, -\lambda p + \eta s + \kappa + \chi\} \leq p \leq \min\{\lambda p + \kappa - \chi, -\lambda p + \eta s + \kappa + \chi\} \qquad (183)$$

with $\lambda > 1$ and $\eta > 0$.

Case 1: $\lambda p + \kappa - \chi \leq -\lambda p + \eta s + \kappa + \chi$. The solution is

$$K_1 := \left\{(s,p) : p \leq \frac{\kappa - \chi}{1-\lambda}, \frac{1+\lambda}{\eta}p - \frac{\kappa + \chi}{\eta} \leq s\right\} \qquad (184)$$

Case 2: $\lambda p + \kappa - \chi \geq -\lambda p + \eta s + \kappa + \chi$. The solution is

$$K_2 := \left\{(s,p) : p \geq \frac{\kappa - \chi}{1-\lambda}, \frac{1+\lambda}{\eta}p - \frac{\kappa + \chi}{\eta} \geq s\right\} \qquad (185)$$

Using the definitions of the coefficients, replacing them with the primitives, simplifying, and rearranging, yields the expressions stated in Proposition 12 □

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