Submodularity and the Evolution of Walrasian Behavior*

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Abstract

Vega-Redondo (1997) showed that imitation leads to the Walrasian outcome in Cournot Oligopoly. We generalize his result to aggregative quasi-submodular games. Examples are the Cournot Oligopoly, Bertrand games with differentiated complementary products, Common-Pool Resource games, Rent-Seeking games and generalized Nash-Demand games.

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1 Introduction

Vega-Redondo (1997) showed that outputs of imitators in a symmetric finite n-firm Cournot Oligopoly with strictly decreasing inverse demand for a homogeneous good converge to the Walrasian outcome. This result is rather striking since the Cournot Nash equilibrium appears to be very robust. Imitators mimic the action of a most successful player in the previous round. The imitation process exhibits inertia such that in any period not every player adjusts its action. Players are allowed to make mistakes, i.e., with a small probability they randomize with full support. The imitation dynamics is a finite Markov chain that is perturbed by mistakes. Vega-Redondo (1997) showed that as the noise goes to zero, the support of the unique invariant distribution converges to the Walrasian outcome. The key factor to understand this result is that a player adjusting towards the Walrasian outcome may decrease its payoff but decreases the opponents' payoffs even further.

We generalize Vega-Redondo's result to symmetric finite aggregative games that are quasi-submodular in a player's action and the aggregate of all players' actions. E.g., if a player prefers an action to a lower action for a given aggregate of all players' actions, then he must also prefer this action to the lower action for a lower aggregate. In short, we show that Vega-Redondo's result applies to a wider class of games than just Cournot games.

This article is related to Possajennikov (2003). He observed that the Walrasian outcome is an evolutionary stable strategy in some aggregative games. However, his approach focuses on the first and second order conditions of the optimization problem under differentiability, whereas we rely just on the order structure of the game.

As in Vega-Redondo (1997), our analysis makes use of stochastic stability analysis. However, instead of using the basic graph theoretic arguments applied by Kandori, Rob, and Mailath (1993) and Young (1993), we employ as a short cut the concept of a recurrent set (see Nöldeke and Samuelson, 1993, 1997, and Samuelson, 1994, 1997).

2 Submodularity

A *lattice* is a partially order set $\langle X, \succcurlyeq \rangle$ whose least upper bound and greatest lower bound are defined by $x' \vee x'' = \sup\{x', x''\}$ and $x' \wedge x'' = \inf\{x', x''\}$, for all $x', x'' \in X$ respectively. For example, if X is the product of several ordered sets, one may define $x' \vee x''$ (likewise $x' \wedge x''$) as the component-wise max (min) to define a lattice. Note that the direct product of a lattice is a lattice, i.e., if X is a lattice then so is $X^2 = X \times X$. A real-valued function $f : X \longrightarrow \mathbb{R}$ on a lattice X is *submodular* on X if for all $x', x'' \in X$,

$$f(x' \wedge x'') + f(x' \vee x'') \leq f(x') + f(x''). \tag{1}$$

The function f is *strictly submodular* if the inequality holds strictly for all unordered $x', x'' \in X$. The function f is *quasi-submodular* on X if for all $x', x'' \in X$,

$$f(x' \vee x'') \geq (>) f(x'') \implies f(x') \geq (>) f(x' \wedge x''), \tag{2}$$

$$f(x' \wedge x'') \geq (>) f(x'') \implies f(x') \geq (>) f(x' \vee x''). \tag{3}$$

Note that submodularity implies quasi-submodularity but not vice versa (see Topkis, 1998).

A function g from a partially ordered set X to a partially ordered set Y is *(strictly) isotone* if $x' \preceq (\prec) x''$ in X implies $g(x') \preceq (\prec) g(x'')$.

Definition 1 (Aggregative Quasi-Submodular game). *A symmetric (finite) strategic game* $\Gamma = \langle N, S, a, \pi \rangle$ *is aggregative*[1] *quasi-submodular if*

(i) $N = \{1, ..., n\}$ *is the finite set of players,*

(ii) the set of actions S_i, $i \in N$, *is a totally ordered (finite) lattice,*

(iii) the aggregator $a_i : \times_{j \in N} S_j \longrightarrow T$, T *being a totally ordered (finite) lattice, is strictly isotone and invariant to permutations of its arguments*[2]*,*

(iv) the payoff function $\pi_i : S_i \times T \longrightarrow \mathbb{R}$ *is quasi-submodular in* (s, t) *for all* $i \in N$*, i.e., for all* $(s', t'), (s'', t'') \in S_i \times T$*,*

$$\pi_i((s', t') \vee (s'', t'')) \geq (>) \pi_i(s'', t'') \Longrightarrow$$
$$\pi_i(s', t') \geq (>) \pi_i((s', t') \wedge (s'', t'')), \tag{4}$$
$$\pi_i((s', t') \wedge (s'', t'')) \geq (>) \pi_i(s'', t'') \Longrightarrow$$
$$\pi_i(s', t') \geq (>) \pi_i((s', t') \vee (s'', t'')). \tag{5}$$

(v) the action sets and payoff functions are symmetric, i.e., $S_i = S$ *and* $\pi_i = \pi$*, for all* $i \in N$*.*

Examples of the class of aggregative quasi-submodular games are as follows:

Example 1: Cournot Oligopoly with a homogeneous good. Let $N = \{1, ..., n\}$ be a finite set of firms. Every firm $i \in N$ chooses an output $s_i \in S \subseteq \mathbb{R}_+$. It faces the same demand function p and cost function c,

[1] See Corchón (1994) for a definition and analysis of aggregative games.

[2] The function $a_i : \times_{j \in N} S_j \longrightarrow T$ is invariant to permutations of its arguments if $a_i(s_1, ..., s_n) = a_i(s_{b(1)}, ..., s_{b(n)})$ for all bijections $b : N \longrightarrow N$.

whereby p is strictly decreasing in the total quantity over all firms $t = \Sigma_{i\in N} s_i$. A firm's payoff function is $\pi(s_i, t) = p(t)s_i - c(s)$. The aggregator is the total output of all firms $a(s_1, ..., s_n) = \Sigma_{i\in N} s_i$.

Example 2: Cournot Oligopoly with differentiated substitute products. The game is similar to Example 1. Goods are substitutes if for example the price function that firm i faces is $p_i(s_i, t) = \beta\theta(\Sigma_{j=1}^n s_j^\beta)^{\theta-1} s_i^{\beta-1}$ with $0 < \beta\theta < 1$, $\theta < 1$ and $1 \geq \beta > 0$ (Vives, 2000). The aggregator is $a(s_1, ..., s_n) = \Sigma_{j=1}^n s_j^\beta = t$. Firm i's payoff function is $\pi_i(s_i, t) = p_i(s_i, t)s_i - c(s_i)$.

Example 3: Bertrand Oligopoly with differentiated complementary products. By contrast to the previous examples, each firm $i \in N$ chooses a price $s_i \in S \subseteq \mathbb{R}_+$ for its good. Let d_i be the demand function for the good of firm i. Goods are complements if for example

$$d_i(s_i, t) = (\beta\theta)^{\frac{1}{1-\beta\theta}} \frac{s_i^{\frac{1}{\beta-1}}}{(\Sigma_{j=1}^n s_j^{\frac{\beta}{\beta-1}})^{\frac{1-\theta}{1-\beta\theta}}},$$

with $0 < \beta\theta < 1$, $\theta < 1$ and $\beta < 0$ (Vives, 2000). The aggregator is $a(s_1, ..., s_n) = \Sigma_{j=1}^n s_j^{\frac{\beta}{\beta-1}} = t$. The payoff function is $\pi_i(s_i, t) = d_i(s_i, t)s_i - c(d_i(s_i, t))$.

Example 4: Common-Pool Resource game. Let $N = \{1, ..., n\}$ be a finite set of appropriators of a common-pool resource. Each appropriator has an endowment $e \in \mathbb{R}_{++}$ that it can invest in an outside activity with marginal payoff $c \in \mathbb{R}_{++}$ or in the common-pool resource. Let $s_i \in [0, e]$ denote appropriator i's investment into the common-pool resource. The return from such investment is $\frac{s_i}{\Sigma_{j=1}^n s_j}[\alpha\Sigma_{j=1}^n s_j - \beta(\Sigma_{j=1}^n s_j)^2]$, with constants

$\alpha, \beta \in \mathbb{R}_{++}$. The aggregator is $a(s_1, ..., s_n) = \Sigma_{j=1}^n s_j = t$. The payoff function is $\pi(s_i, t) = c(e - s_i) + \frac{s_i}{\Sigma_{j=1}^n s_j}[\alpha \Sigma_{j=1}^n s_j - \beta(\Sigma_{j=1}^n s_j)^2]$ if $s_i > 0$, and $\pi(0, 0) = ce$ otherwise (Walker, Gardner, and Ostrom, 1990).

Example 5: Generalized Nash-Demand game. Let $N = \{1, ..., n\}$ be a finite set of players. Every player $i \in N$ demands $s_i \in S \subseteq \mathbb{R}_+$. The probability of getting the demand is $p(t)$, which is strictly decreasing in the total of demands of all players $t = \Sigma_{j=1}^n s_j$. The payoff function is $\pi(s_i, t) = p(t)s_i$. The aggregator is $a(s_1, ..., s_n) = \Sigma_{j=1}^n s_j$.

Example 6: Rent-Seeking game Let $N = \{1, ..., n\}$ be a finite set of contestants. Every contestant $i \in N$ competes for a rent v by bidding $s_i \in S \subseteq \mathbb{R}_+$. Player i's probability of winning is $\frac{s_i^r}{\Sigma_{j=1}^n s_j^r}$ (or zero if all bid zero), $0 < r < 1$, but the cost of bidding equals to the bid. The aggregator is $a(s_1, ..., s_n) = \Sigma_{j=1}^n s_j^r = t$. The payoff function is $\pi(s_i, t) = \frac{s_i^r}{t} v - s_i$ if $t > 0$ and zero otherwise (Hehenkamp, Leininger, and Possajennikov, 2001).

3 Imitation Dynamics

Time is discrete and indexed by $\tau = 0, 1, 2, ...$

Definition 2 (Imitator). [3] *An imitator $i \in N$ chooses with full support from the set*

$$D_I(\tau - 1) := \{s \in S : \exists j \in N \ s.t. \ s = s_j(\tau - 1) \ and \ \forall k \in N, \pi_j(\tau - 1) \geq \pi_k(\tau - 1)\}. \tag{6}$$

[3] See also Vega-Redondo (1997), p. 378.

An imitator mimics the action of the player(s) with highest payoff in the previous period. At every time $\tau = 1, 2, ...$, each player $i \in N$ is assumed to revise its former action $s_i(\tau - 1)$ with a common i.i.d. probability $\rho \in (0,1)$ according to the imitation rule. Thus the process has *inertia*. In $\tau = 0$ players start with any arbitrary action within the action set S.

The process induced by the imitation dynamics is a discrete time finite Markov chain on the state-space $S^n = \times_{i \in N} S_i$. Each state $\omega(\tau) = (s_1(\tau), s_2(\tau), ..., s_n(\tau))$ induces a profit-profile $(\pi_1(\tau), \pi_2(\tau), ..., \pi_n(\tau))$. The Markov operator is defined in the standard way as transition probability matrix $P = (p_{\omega\omega'})_{\omega,\omega' \in S^n}$ with $p_{\omega\omega'} = prob\{\omega' | \omega\}$, $p_{\omega\omega'} \geq 0$, $\omega, \omega' \in S^n$ and $\sum_{\omega' \in S^n} p_{\omega\omega'} = 1$, for all $\omega \in S^n$.

At every output revision opportunity τ, each player follows the imitation rule with probability $(1 - \varepsilon)$, $\varepsilon \in (0, \eta]$, where η is small, but with probability ε he randomizes ("mutates") with full support S. This *noise* makes the perturbed Markov chain $P(\varepsilon)$ irreducible and ergodic. This implies that there exists a unique invariant distribution $\varphi(\varepsilon)$ on S^n (see for example Masaaki, 1997). We focus on the unique limiting invariant distribution φ^* of P defined by $\varphi(\varepsilon)P(\varepsilon) = \varphi(\varepsilon)$, $\varphi^* := \lim_{\varepsilon \to 0} \varphi(\varepsilon)$ and $\varphi^* P = \varphi^*$. This long-run distribution determines the average proportion of time spent in each state of the state-space in the long run (see Samuelson, 1997, for an introduction).

Consider $\varepsilon = 0$ and define an *absorbing set* $A \subseteq S^n$ by

(i) for all $\omega \in A$ and for all $\omega' \notin A$, $p_{\omega\omega'} = 0$, and

(ii) for all $\omega, \omega' \in A$ there exists a finite $m \in \mathbb{N}$ s.t. $p^{(m)}_{\omega\omega'} > 0$, $p^{(m)}_{\omega\omega'}$ being the m-step transition probability from ω to ω'.

Let Z be the collection of all A in S^n.

We call states ω and ω' *adjacent* if exactly one mutation can change the state from ω to ω' (and vice versa). The set of all states adjacent to the state ω is the *single mutation neighborhood* of ω, denoted by $M(\omega)$. The *basin of attraction* of an absorbing set A is the set $B(A) = \{\omega \in S^n | \exists m \in \mathbb{N}, \exists \omega' \in A \text{ s.t. } p^{(m)}_{\omega\omega'} > 0\}$. A *recurrent set* R is a minimal collection of absorbing sets with the property that there do not exist absorbing sets $A \in R$ and $A' \notin R$ such that for all $\omega \in A$, $M(\omega) \cap B(A') \neq \emptyset$. We will use following lemma.

Lemma 1 (Nöldeke and Samuelson). *Given a perturbed finite Markov chain, then at least one recurrent set exists. Recurrent sets are disjoint. Let the state ω be contained in the support of the unique limiting invariant distribution φ^*. Then $\omega \in R$, R being a recurrent set. Moreover, for all $\omega' \in R$, $\varphi^*(\omega') > 0$.*

For a proof see for example Samuelson (1997), Lemma 7.1 and Proposition 7.7., proof pp. 236-238.

4 Result

Definition 3 (Aggregate-Taking Outcome). $\omega^* = (s_1^*, ..., s_n^*)$ *is an aggregate-taking outcome if for* $t^* = a(\omega^*)$,

$$\pi(s^*, t^*) \geq \pi(s, t^*), \forall s \in S. \tag{7}$$

The aggregate-taking outcome describes a solution in which the player does not perceive the externality of its action and takes the aggregate of all players' actions as given. An example is price-taking behavior. Definition 3 is a generalization of the Walrasian outcome in Vega-Redondo (1997).

Theorem. *Given imitators with inertia and noise in an aggregative quasi-submodular game, suppose the aggregate-taking outcome $\omega^* \in S^n$ exist uniquely. Then $\varphi^*(\omega^*) = 1$.*

The proof follows from the lemmatas below. Recall that Z is the collection of absorbing sets. We write $\omega = (s, ..., s)$ to indicate that all players in N play the same action $s \in S$ in ω.

Lemma 2. *$Z = \{A_\omega = \{\omega\} : \omega = (s, ..., s) \in S^n$ for some $s \in S\}$.*

Proof. By symmetry of Γ, we have by D_I for all $\omega = (s, ..., s) \in S^n$ that $p_{\omega\omega} = 1$ and $p_{\omega\omega'} = 0$, for all $\omega' \neq \omega$. Conversely, since at any τ and i.i.d. probability $\rho > 0$, there is positive probability that all firms adjust towards the same action in $D_I(\tau - 1)$ given any arbitrary $\omega(\tau - 1)$. □

With the next two lemmata we show that the aggregate-taking outcome is the unique recurrent set.

Lemma 3. *$M(\omega) \cap B(\{\omega^*\}) \neq \emptyset$, for all $\{\omega\} \in Z \backslash \{\omega^*\}$.*

Proof. By assumption, ω^* is unique and by Lemma 2 it is an absorbing state $A_{\omega^*} = \{\omega^*\}$. Consider any absorbing set (state) $A \neq A_{\omega^*}$. We claim that starting in any $A \neq \{\omega^*\}$ a single (suitable) mutation can lead the dynamics to the basin of attraction of the aggregate-taking outcome $B(\{\omega^*\})$. It is sufficient to show that for all $s \in S$, $s \neq s^*$, $k \in \mathbb{N}$, $k \leq n$,

$$\pi(s^*, t) > \pi(s, t), \tag{8}$$

with $t = a(s_1^*, ..., s_k^*, s_{k+1}, ..., s_n)$. Clearly, if Inequality (8) holds, then players setting their aggregate-taking action are strictly better off than are players with a different action. Thus the latter will follow the former by the imitation dynamics.

The function a is strictly isotone and invariant to permutations of it's arguments. The payoff function π is quasi-submodular. Set $s^* \equiv s'$, $s \equiv s''$, $t \equiv t'$ and $t^* \equiv t''$. By quasi-submodularity, Inequality (7) in Definition 3 implies above Inequality (8). I.e., if $s \prec s^*$ then we apply the upper Formula (4), if $s \succ s^*$, then we use the lower Formula (5). Setting $k = 1$ yields the desired claim and completes the proof of the lemma. □

Lemma 4. *$M(\omega^*) \cap B(A) = \emptyset$, for all $A \in Z$, $A \neq \{\omega^*\}$.*

Proof. By setting $k = n - 1$ in Inequality (8), it follows that more than one mutation is needed to escape A_{ω^*} since players setting s^* are still better off after just one mutation. □

From the previous lemmatas follows that $R = \{\omega^*\}$. Moreover, it follows by Lemma 3 that there does not exist any other recurrent set. Thus by Lemma 1, $\varphi^*(\omega^*) = 1$. This completes the proof of the Theorem.

Note that just a single suitable mutation is required to trigger the convergence to the long run outcome. Hence, the convergence is rather fast compared to many results in the literature obtained by the same method.

5 Conclusions

We generalize Vega-Redondo's (1997) result to a class of aggregative quasi-submodular games. Examples of this class are many games with strategic substitutes. The result provides an evolutionary foundation for Walrasian or aggregate-taking behavior in an important class of non-cooperative games. Schipper (2001) also uses quasi-submodularity to prove that imitators are strictly better off than are best-response-players in Cournot oligopoly. This result too applies to aggregative quasi-submodular games.

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