Preference for Flexibility and Freedom of Choice in a Savage Framework

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December 1996

Abstract

This is the working-paper version of the published article, "Preference for Flexibility in a Savage Framework", 1999, Econometrica 67, 121-146.

The last three sections consist of additional material on the uniqueness of the representation (s. 5), interpretation in terms of freedom of choice (s. 6), and an alternative motivation for departing from the Savage framework based on "intra-agent incentive-compatibility".

1. INTRODUCTION

Flexia plans to undertake a plane trip; she has to decide whether to purchase an advance-reservation ticket now at a price p, or whether to wait until right before her intended date of departure and then to finally decide between staying at home and purchasing a ticket at a higher price q. Flexia's "present choice" can be thought of as one among "opportunity sets", here {fly @p, stay @p} and {fly @q, stay @0}, from which her "future choice" is then made.

Flexia has fairly common present preferences over opportunity sets; if required to make a final choice now among basic alternatives (singleton opportunity sets), she would most prefer to purchase a ticket in advance (fly @p $\succ$ stay @0). On the other hand, if possible, she prefers to "wait-and-see" ({fly @p, stay @p} $\prec$ {fly @q, stay @0}). Note that these preferences are not compatible with a ranking of opportunity sets according to their indirect utility (i.e. by ranking the sets as equivalent to their currently best element)[1]. They are naturally explained, however, as due to an uncertainty about her own future preferences between making the trip and staying at home.

Such *preference for flexibility* received its first axiomatic study in a classic paper by Kreps (1979) which characterized the class of preferences that rank opportunity sets in terms of their *expected indirect* (=maximal attainable) *utility* (EIU), the expectation being taken with respect to an *implicit*[2] state space describing future preferences. Kreps assumed that present choices determine future opportunity sets deterministically. Such an assumption is obviously very restrictive; for instance, if Flexia decides to wait, realistically she will need to reckon with the risk that seats may no longer be available later. The present paper characterizes EIU maximization in such more general situations in which the agent may be uncertain about the opportunity set she is going to face, and in which at least some of the uncertainty about future preferences is explicitly modeled.

Thus, at the heart of this paper's analysis is a distinction between explicit and implicit

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[1]assuming greater wealth to be preferred, of course.

[2]That is: part of the representation, not of the set-up.

state spaces. In various guises, this distinction is of significant conceptual interest. As shall be argued, it corresponds, roughly speaking, to distinctions between verifiable / non-verifiable and foreseen / anticipated unforeseen contingencies, between incentive-compatible and arbitrary Savage acts, and, finally, between preference for flexibility itself and intrinsic preference for freedom of choice. Before describing these alternative interpretations of the theory in greater detail in the second half of this introduction, we first sketch the main result of the paper.

In formal terms, we will study preferences over acts f ("*opportunity acts*") that map states $\theta \in \Theta$ to opportunity sets $A \in \mathcal{A} = 2^X \backslash \emptyset$ of alternatives $x \in X$. Both states as well as prizes (opportunity sets) are to be understood as described in terms of what is knowable ex-interim, rather than as complete descriptions of everything relevant, as "small worlds" rather than "grand worlds" in Savage's terminology (1972, pp.82). By consequence, belief cannot be fully disentangled from value. As with single-valued acts, this leads to state-dependent preferences. Thus, we take as point of departure for our characterization a state-dependent version of Savage's axioms (Karni-Schmeidler (1993)).

If preferences are defined over opportunity acts, incompleteness of description (specifically: the absence of future preferences from the definition of a state) manifests itself additionally in *conditional* preferences displaying a preference for flexibility. That is to say, it will typically not be the case, even for events E such that preferences are state-independent on E, that the following property holds.

Conditional IU-Property)

For all sets (constant acts) $A, B \in \mathcal{A}$, all acts $f \in \mathcal{F}$, and all events $E \subseteq \Theta$:

$$[A, E; f, E^c] \succeq [B, E; f, E^c] \text{ implies } [A, E; f, E^c] \succeq [A \cup B, E; f, E^c].^3$$

The key issue is to find axioms that yield an expected-utility representation with respect to the implicit uncertainty concerning future preferences. Partial solutions can be found in the work of Kreps (1979, 1992). Kreps (1979) characterizes the implications of EIU

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[3]$[A, E; f, E^c]$ denotes the act that coincides with the constant act A on E and with f outside E; cf. section 3.

maximization for preferences over constant acts, that is: opportunity *sets*. The obtained restrictions, described by the following condition, are weak.[4]

OSM)[5] $\forall A, B, C \in \mathcal{A} : A \succeq A \cup B$ implies $A \cup C \succeq A \cup B \cup C$.

Building on this result, Kreps (1992) introduces uncertainty explicitly into the model and shows that EIU maximization entails implications analogous to OSM for conditional preferences over opportunity *acts*. However, these implications are only ordinal in character and fail to characterize EIU maximization by far. Consequently, he obtains a representation in which the utility of an act is monotone but *not necessarily additive* in implicit-state utilities.

In an explicitly stochastic context, EIU maximization has additional and interesting *cardinal* implications. For example, it entails the following cardinal (conditional) version of OSM, which implies that opportunity subsets are of necessity substitutes for each other.

CSM)[6] For all sets, $A, B, C \in \mathcal{A}$, all acts $g, h \in \mathcal{F}$, and all events $E \subseteq \Theta$:
$$[A, E; g, E^c] \succeq [A \cup B, E; h, E^c] \text{ implies } [A \cup C, E; g, E^c] \succeq [A \cup B \cup C, E; h, E^c].$$

CSM asserts that the incremental value of a set B of additional alternatives never increases as further alternatives become available; note that a conditional version of OSM results by restricting CSM to g, h such that $g = h$. While coming much closer to characterizing EIU rationalizable preferences, CSM fails to capture all their cardinal implications.[7]

To characterize EIU-maximization, we introduce an axiom "Indirect Stochastic Dominance" (ISD*) that makes use of the "more-likely-relation" $\geq$ over events derived from

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[4]Conversely, an additive representation is of limited significance in this context, as Kreps (1979, p. 567) points out.

[5]OSM for "Ordinal Submodularity", cf. section 4.

[6]CSM stands for "Cardinal Submodularity".

[7]In addition, CSM does not seem to be fully satisfactory conceptually: that the incremental value of additional alternatives be non-increasing seems plausible enough, but how compelling can it be? Indeed, it can be shown that CSM may easily be violated if the decision maker is uncertainty-averse in the sense of Gilboa-Schmeidler (1989).

preferences over bets in the usual way. Roughly speaking, ISD* formalizes the notion that acts that "in expectation offer effectively more choice" are better. More specifically, ISD* requires that f be weakly preferred to g whenever, given *any* hypothetical (future) weak order R, the R-best available alternatives under f are $\geq$-more-likely to be R-better than under g ("first-order stochastically dominates in utility"; see sections two and three). For example, ISD* entails the following condition ISD$_2$ which evidently is closely related to CSM.

ISD$_2$) For any event E such that E is $\geq$-equally likely to its complement E^c and any $A, B, C \in \mathcal{A} : [A, E; A \cup B \cup C, E^c] \preceq [A \cup B, E; A \cup C, E^c]$.

Specialized to state-independent preferences, the main result (theorem 3) characterizes EIU-rationalizable preferences $\succeq$ as those satisfying the Savage axioms plus ISD*.

The notion of a "preference for flexibility" translates into decision theory, and thereby generalizes, one of the central ideas of financial economics, the notion that "options have value"; for example, the theory of option-pricing has profoundly affected the theory of investment in physical capital under the name of "real options" (see especially the recent monograph by Dasgupta/Pindyck (1994)). A more general decision-theoretic approach seems clearly desirable in contexts in which markets are thoroughly incomplete (as is the case for many investments in human capital, e.g. the comparatively "inflexible" decisions to obtain a Ph.D. training in economic theory rather than an M.B.A., for instance), or where markets are entirely absent; – in a somewhat playful and speculative vein, Dasgupta and Pindyck stress the irreversibility inherent in the decisions to marry and commit suicide.[8]

While the economic importance of preference for flexibility seems evident, an *axiomatic* approach to preference for flexibility is motivated by an interest in the distinction between explicit and implicit states fundamental to the opportunity act model. A polarity of this kind arises naturally from a variety of perspectives; while the second and third opposi-

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[8]See also Jones-Ostroy (1984) for a decision-theoretic model which relates the value of flexibility to the amount of information to be received.

tions described below are conceptually of a rather different nature, they turn out to be mathematically isomorphic in a precise way.

1. The distinction between explicit and implicit states captures at a primitive level important restrictions on the elicitation and even "construction" of an agent's preferences. In descriptive, especially in experimental applications, one may want to confine attention to acts defined in terms of *verifiable* or *contractible* contingences.[9] If one seriously wants to test experimentally whether and to what extent a subject's *behavior* conforms to SEU-maximization, one will elicit preferences over acts defined in terms of a finite set of contingencies that is coarse almost by definition. The phenomenon of preference for flexibility shows that implicit (not directly elicited) uncertainty matters in a sequential setting even to decisions between acts whose consequences are fully described in terms of the coarse explicit state-space.

From a first-person point of view, the explicit state-space may analogously be interpreted as the space of *foreseen* contingencies determining the class of thought-experiments relevant to the decision-maker's preference construction. Violation of the conditional IU property can then be viewed as reflecting *anticipated unforeseen* contingencies (Kreps 1992); for example, Flexia may explain her preference for flexibility by the expectation that "quite possibly *something* will interfere with my travel plans", without having a clear idea about specifically what is likely to interfere. For another example, we note that in discussions of the value of preserving biodiversity, the irreversibility of extinction occupies frequently a central place. Preserving species for another generation keeps the option of their continued existence open, an option which has significant value in view of our uncertainty about the preferences of future generations which presumably we cannot foresee in any detail.

A conceptualization of anticipated unforeseen contingencies in terms of a set of implicit decision-relevant states is interesting particularly in that it combines (anticipated) "unforeseenness" with notions of subjective probability and expected utility[10]; this contrasts

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[9]Note that many "Dutch book" argument for the sure-thing principle rely on the contractibility of contingencies.

[10]This is not to belittle the seriousness of the uniqueness issue in this context; see section 5 for further

with approaches in which "unforeseenness" is identified with ignorance of some kind, as in Ghirardato (1996) and Mukerjee (1995)[11]. [12]

2. It will be shown in section 7 that from a thorough-going revealed-preference perspective which identifies preferences with dispositions to choice-behavior, difficulties arise for a direct application of Savage's theorem in which future preferences are incorporated in the description of a state and in which the preference ordering is defined on the class of all Savage acts; in particular, it is not obvious which class of acceptable thought-experiments can support *arbitrary* acts (i.e. acts that depend on the decision-maker's future preferences in arbitrary ways). We will thus argue that a "revealed-preference" interpretation of future preferences implies *incentive-compatibility restrictions* on the domain of acts which in turn lead to models equivalent to the opportunity-act model studied in this paper.

3. Last but not least, if one assumes that all uncertainty is modelled in standard ways[13], and preference for flexibility is thus fully accounted for in terms of the explicit uncertainty, a failure of preferences to satisfy the conditional IU property can by definition no longer be attributed to uncertainty about future preferences; instead, it reveals an intrinsic "preference for freedom of choice." It is in fact this notion of freedom of choice which has been at

discussion.

[11]For recent epistemic work on the related notion of awareness, see Modica-Rustichini (1994) and Dekel-Lipman-Rustichini (1996).

[12]Finally, the issue of "coarse explicit state-spaces" is central to a related, but largely philosophical literature on Bayesian belief revision and "belief kinematics" (the locus classicus is Jeffrey (1965, ch. 11). Coarseness there corresponds to the notion that rational belief change cannot be fully accounted for by Bayesian updating on "explicitly given" evidence. The literature emphasizes the existence of evidence that may be "non-verifiable" (e.g. impressionistic judgements) and/or "unforeseen" (e.g. future insights) ; see also Binmore-Brandenburger (1990) who forcefully spell out the problematic nature of large-world assumptions. Reservations have been articulated towards inclusion of future beliefs (i.e. here: of future preferences over bets) in the definition of a state. If these reservations are fully taken to heart, a theory of the kind outlined in this paper is required to justify "as-if Bayesian updating" (which is what EIU maximization amounts to in this context) with respect to the implicit uncertainty; in such a theory (with appropriately enriched structure), beliefs about future beliefs are revealed by preferences over *sets of* future bets, i.e. by the "flexibility value of effective belief revision".

[13]Thus bracketing points 1 and 2.

the center of the recent wave of interest in the axiomatic study of ranking of opportunity sets[14]. The present paper is the first to simultaneously incorporate and distinguish within one model the two sources of preferences for opportunities.[15]

The key step in making the results of this paper equally applicable to a freedom-of-choice interpretation is a novel conceptualization of effective freedom of choice as a multi-attribute construct, with *component opportunities* (i.e. the opportunities to bring about particular consequences) defining the different attributes. The Indirect Stochastic Dominance axiom is reinterpreted accordingly as requiring that "in expectation more opportunity is better", and theorem 3 yields an additive multi-attribute representation with optimal uniqueness properties.

The remainder of the paper is organized as follows:

Section 2 considers a von Neumann-Morgenstern-type setting in which preferences are defined on the class of all (objective) probability distributions over opportunity sets; an objective version of the Indirect Stochastic Dominance axiom is introduced and used to characterize EIU rationalizable preferences (theorem 1). We present theorem 1 as a separate core result both to make it more accessible to the general reader unfamiliar with Savage's framework, and because from a mathematical point of view, the theorem is best understood as a result on mixture-spaces over opportunity sets.

In section 3, a subjective version of the Indirect Stochastic Dominance axiom is formulated; Karni-Schmeidler's (1993) generalization of Savage's theorem to state-dependent preferences is then combined with theorem 1 to obtain a characterization state-dependent EIU rationalizable preferences over opportunity acts.

Section 4 introduces the key technical tool of this paper, (dual) Möbius inversion which is taken from the literature on belief-functions (non-probabilistic representations of uncertainty). It is shown that EIU rationalizable preferences are characterized by a risk-aversion

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[14]It will become clear that our results are equally applicable to rankings of opportunity sets *purely* in terms of freedom of choice, without regard to the agent's indirect utility.

[15]The nature and legitimacy of the distinction is intensively debated: for example, while Sen (1988) affirms it emphatically, Arrow (1995) does not appear to see any meaning in it.

property with respect to the "size" of the opportunity set. Dual Möbius inversion is also shown to yield a direct and intuitive proof of Kreps's (1979) classic result.

The following section 5 describes the uniqueness properties of the representation. While these are significantly stronger than those obtainable in a standard setting without explicit uncertainty, they still fall short of what one might have hoped for. It becomes clear, however, what kind of structure needs to be added to obtain optimal results.

A reinterpretation of the results in terms of freedom of choice is given in section 6. Finally, section 7 discusses the difficulties of applying a direct Savage approach under a "revealed-preference" interpretation of future preferences. All proofs are collected in the appendix.

2. AN AXIOMATIZATION OF EXPECTED INDIRECT UTILITY

This section presents a characterization of Expected Indirect Utility maximization in a von Neuman-Morgenstern (vNM) context in which preferences are defined over "opportunity prospects" with numerically given probabilities and opportunity sets as prizes. It serves both as a simplified version as well as a key building block of the main result of the paper, theorem 3 of the following section.

Let X denote a finite non-empty set of *alternatives*, $\mathcal{A} = 2^X \setminus \emptyset$ the set of its non-empty subsets (*opportunity sets*), and $\Delta^{\mathcal{A}}$ denote the probability simplex in $\mathbf{R}^{\mathcal{A}}$ with typical element p. (Ex ante-) preferences are described by a relation $\succeq$ on the set of *opportunity prospects* $\Delta^{\mathcal{A}}$.

The chronology of decision-making and uncertainty-resolution is as follows: at date 1, an opportunity prospect p is chosen by the agent. Then, at some time between dates 1 and 2, say at date 1.5, the opportunity prospect is resolved, yielding with probability p_S the opportunity set S. Finally, at date 2, the agent selects one alternative among S. At date 1, the agent is uncertain of his preferences based on which date 2 choices are made; this uncertainty resolves before date 2.

The uncertainty concerning date-1.5 opportunity sets may arise "artificially" as result of

an agent's intentional randomization of set-choices, or of an experimenter's explicitly offering choices among "lotteries" with opportunity sets as prizes. Often, opportunity prospects also arise naturally, as in the following example modifying Kreps (1979).

Example 1 *At lunchtime, the agent has to make a reservation at a restaurant of her choice for dinner with a friend. She wants to choose the restaurant offering the best-tasting meal to her friend. Since she knows his tastes (at dinner) only incompletely, her choice among restaurants will exhibit a "preference for flexibility". Since she is also uncertain of the menu (set of meals) offered by each restaurant, a restaurant represents a (subjective) prospect over menus. To satisfy the domain assumption, one needs to ask the agent to imagine hypothetical "restaurants" corresponding to arbitrary subjective (but not yet decision-theoretically derived) probability distributions over menus.* □

To capture formally uncertainty about future tastes in the intended representation, let Ω denote a (finite) set of preference-determining contingencies ω with associated utility-function v_ω , and let $\lambda \in \Delta^\Omega$ denote a probability distribution over Ω. Note well that for opportunity prospects, i.e. (marginal) probability distributions over opportunity sets, to denote well-defined objects of preference, these distributions must be stochastically independent of the uncertainty governing future preferences. This assumption becomes explicit in a Savage setting (where it will be relaxed and further discussed); it is reflected here in the axiom ISD below, and motivates the following definition of the class of "Expected Indirect Utility" (EIU-) rationalizable preferences.

Definition 1 $\succeq$ *is* **EIU-rationalizable** *if there exists a finite set* Ω, $\lambda \in \Delta^\Omega$ *and utility-functions* $\{v_\omega\}_{\omega \in \Omega}$ *such that, for all* $p, q \in \Delta^{\mathcal{A}}$:

$$p \succeq q \quad \Leftrightarrow \quad \sum_{S \in \mathcal{A}} \sum_{\omega \in \Omega} p_S \lambda_\omega \max_{x \in S} v_\omega(x) \geq \sum_{S \in \mathcal{A}} \sum_{\omega \in \Omega} q_S \lambda_\omega \max_{x \in S} v_\omega(x).$$

Remark: In order to preserve generality, we have allowed in this definition the implicit state-space Ω to be arbitrary (finite), herein following Kreps (1979). It is debatable whether these are really meaningful; one may want to restrict attention to a canonical space of states

that is logically constructed from the data, i.e. ultimately from the universe of alternatives X. A natural candidate for such a canonical state-space is the set of all weak orders on X.[16]

Basic to the characterization of EIU-rationalizable preference relations are the von Neumann-Morgenstern axioms vNM .

Axiom 1 (vNM)

i) (**Completeness**) $p \succeq q$ *or* $p \preceq q$ *, for all* $p, q \in \Delta^{\mathcal{A}}$.

ii) (**Transitivity**) $p \succeq q$ *and* $q \succeq r$ *imply* $p \succeq r$ *, for all* $p, q, r \in \Delta^{\mathcal{A}}$.

iii) (**Independence**) $p \succeq q \Longleftrightarrow ap + (1-a)r \succeq aq + (1-a)r$ *, for all* $a : 0 < a < 1$ *and all* $p, q, r \in \Delta^{\mathcal{A}}$.

iv) (**Continuity**) $p \succeq q \succeq r \Longrightarrow \exists a : 0 \leq a \leq 1$ *such that* $ap + (1-a)r \sim q$ *, for all* $p, q, r \in \Delta^{\mathcal{A}}$.

The final axiom is based on an "Indirect Stochastic Dominance" relation defined as follows. For $\mathcal{S} \subseteq \mathcal{A}$, let $p(\mathcal{S}) = \sum_{T \in \mathcal{S}} p_T$ denote the probability of $\mathcal{S}$.

Definition 2 *The prospect* p indirectly stochastically dominates q *with respect to the weak order*[17] R *on* X *("$p \trianglerighteq_R q$") if and only if, for all* $y \in X$:

$$p(\{S \mid S \cap \{x \mid xRy\} \neq \emptyset\}) \geq q(\{S \mid S \cap \{x \mid xRy\}) \neq \emptyset\}.$$

p **indirectly stochastically dominates** q *("$p \trianglerighteq q$") if it indirectly stochastically dominates* q *with respect to every weak order* R *on* X.

In other words, p indirectly stochastically dominates q if, given any *hypothetical* weak preference ordering over alternatives R and any associated ordinal indirect utility-function u_R, the probability distribution of indirect utilities $p \circ u_R^{-1}$ induced from p first-order stochastically dominates (in the ordinary sense) the analogously defined probability distribution $q \circ u_R^{-1}$.

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[16]However, fixing Ω in this way is not enough to ensure essential uniqueness; see section 5 for further discussion.

[17]i.e.: complete and transitive relation.

Indirect Stochastic Dominance restricted to degenerate prospects that yield with probability one some opportunity set A (and written as $\mathbf{1}_A$) coincides with monotonicity with respect to set-inclusion; in a stochastic setting, it is however much richer in content.

Example 2 *Let* $X = \{x, y, z\}$, $p = \frac{1}{2}\mathbf{1}_{\{x,y\}} + \frac{1}{2}\mathbf{1}_{\{x,z\}}$ *, and* $q = \frac{1}{2}\mathbf{1}_{\{x\}} + \frac{1}{2}\mathbf{1}_{\{x,y,z\}}$ *. Then* $p \unrhd q$, *but not* $q \unrhd p$.

This is easily verified. If x is a best alternative with respect to R, it is available with probability one under p and q, and thus $p \unrhd_R q$ as well as $q \unrhd_R p$. If, on the other hand, x is not a best alternative with respect to R, the[18] *R-best alternative is available with probability one half under each. Under p, the at-least-second-best alternative is always available, and thus $p \unrhd_R q$ again. However, if x is worst with respect to R, with probability one half not even the second-best option is available under q , and thus not $q \unrhd_R p$ for such R. It follows that $p \unrhd q$, but not $q \unrhd p$.* $\square$

Axiom 2 (ISD) *$p \succeq q$ whenever p indirectly stochastically dominates q.*

Remark: Note that, for the use of the *un*conditional distributions over opportunity sets p and q to be legitimate in the definition of R-conditional dominance and of ISD, these have to coincide with the ω-conditional distributions; that is to say, the distributions of state-contingent preferences R_ω and opportunity sets must be subjectively independent.

The following characterization of Indirect Stochastic Dominance is a straightforward consequence of the adopted definitions.

Fact 1 *The following three statements are equivalent:*

i) $p \unrhd q$,

ii) $p(\{S \mid S \cap A \neq \emptyset\}) \geq q(\{S \mid S \cap A \neq \emptyset\})$ for all $A \in \mathcal{A}$,

iii) For all utility-functions v on X : $\sum\limits_{S \in \mathcal{A}} p_S \max_{x \in S} v(x) \geq \sum\limits_{S \in \mathcal{A}} q_S \max_{x \in S} v(x)$ *.*

Theorem 1 *$\succeq$ is EIU-rationalizable if and only if it satisfies vNM and Indirect Stochastic Dominance.*

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[18]breaking ties arbitrarily throughout.

Theorem 1 belongs to a family of decision-theoretic results that obtain an additively separable representation by appropriately augmenting the vNM axioms. These include in particular Harsanyi's (1955) Utilitarian representation theorem, as well as Anscombe-Aumann's (1963) characterization of SEU maximization. The role of ISD is played by a Pareto-condition in the former and by an (implicit, see Kreps (1988, p.107)) "only marginals matter" condition in the latter. The analogy to Harsanyi's theorem is particularly close, in that ISD functions as a monotonicity-condition analogous to the Pareto-condition there. Jaffray's (1989) mixture-space approach to belief-functions, by contrast, enhances the vNM axioms in a rather different direction.

3. PREFERENCE FOR FLEXIBILITY IN A SAVAGE FRAMEWORK

In this section, the characterization of EIU rationalizable preferences is extended to a fully subjective Savage-style formulation in which preferences are defined over acts that map states to opportunity sets. Theorem 1 can be translated to a Savage framework (with state-independent preferences) for the following two reasons: first, the ISD axiom uses probabilities only in ordinal, comparative way, and is thus straightforwardly put into subjective terms. Secondly, ISD thus translated retains its force due to the richness of Savage acts, specifically: to the fact that *any* subjective probability distribution over opportunity sets is generated by some opportunity act.[19] Besides providing an interpretation of theorem 1 in subjective terms, "going Savage" opens an important dimension of generality by explicitly raising the issue of state-independent preference. We will argue that state-independence is a rather restrictive assumption in an opportunity-act setting, and present an additive state-dependent generalization of Savage's theorem. We will then "subjectivize" ISD to obtain a subjective, state-dependent generalization of theorem 1.

Three basic types of explicit uncertainty can be distinguished in the present context: the agent may be uncertain as to which opportunity set results from a particular present

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[19]I.e., in the notation to follow, if μ denotes the agent's subjective probability measure on Ω, $\{\mu \circ f^{-1} \mid f \in \mathcal{F}\} = \Delta^{\mathcal{A}}$.

choice (e.g., in Flexia's case, the availability of a ticket if she does not buy one now), the agent may receive information about the comparative value of alternative final choices (e.g., if Flexia is worried about the health of her child, her final decision may depend on his body temperature), and thirdly the final choice itself may be one under uncertainty (e.g., at the time of her final decision, Flexia may still not know whether the child will fall seriously ill.). In this paper, we will deal with uncertainty that resolves before the final choice is made, i.e. with uncertainty of the first two kinds. Uncertainty not resolving before the final choice is not explicitly modeled; doing so promises to be a worthwhile (see the concluding remark of section 5) and non-trivial task for future research. Uncertainty of the first kind is associated with state-independent preferences, uncertainty of the second kind with state-dependent preferences. Thus, to assume global state-independence would be highly restrictive, as it effectively eliminates uncertainty of the second kind.

We first state an appropriate state-dependent generalization of Savage's theorem that comes tailor-made from the literature; this result is then combined with theorem 1 to yield the main result of the paper, a subjective state-dependent generalization of EIU rationalizable preferences over opportunity acts.

Some additional notation and definitions.

Θ : the space of explicit states θ.

$\mathcal{F}$: the class of opportunity acts $f : \Theta \to \mathcal{A}$.

$\mathcal{F}^{const}$: the subclass of constant acts, typically denoted by the constant prize.

$[f, E; g, E^c]$: the act h such that, for $\theta \in \Theta$,

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h_\theta = \begin{cases} f_\theta & \text{if} \quad \theta \in E \\ g_\theta & \text{if} \quad \theta \in E^c \end{cases} \quad (\text{``}f \text{ on } E \text{ and } g \text{ on } E^c\text{''}).
$$

$\succeq$: a preference relation on $\mathcal{F}$.

$f \succeq_E g$: whenever $[f, E; h, E^c] \succeq [g, E; h, E^c]$ for some $h \in \mathcal{F}$ ("f is weakly preferred to g *given* the event E").

E is *null* if $f \succeq_E g$ for all $f, g \in \mathcal{F}$.

The following three axioms are exactly Savage's P1, P2 (the "sure-thing principle" in standard, if not Savage's, terminology), and the richness and continuity condition P6.

Axiom 3 (P1) $\succeq$ *is transitive and complete, i.e. a weak order.*

Axiom 4 (P2) *For all* $f, g, h, h' \in \mathcal{F}$, $E \subseteq \Theta$: $[f, E; h, E^c] \succeq [g, E; h, E^c]$ *if and only if* $[f, E; h', E^c] \succeq [g, E; h', E^c]$.

Axiom 5 (P6) *For all* $f, g \in \mathcal{F}$ *such that* $f \succ g$ *and all* $h \in \mathcal{F}^{const}$, *there exists a finite partition* $\mathcal{H}$ *of* Θ *such that, for all* $H \in \mathcal{H}$:

i) $[h, H; f, H^c] \succ g$,

ii) $f \succ [h, H; g, H^c]$.

The generalization of Savage's theorem to be used assumes "finitary state-dependence".

Definition 3 *An event* G *is a state-independent preference (s.i.p.) event with respect to* $\succeq$ *if the following three conditions are satisfied:*

i) *For non-null* $E \subseteq G$, *and all* $f, g \in \mathcal{F}^{const}$: $[f, E; h, E^c] \succeq [g, E; h, E^c]$ *if and only if* $f \succeq_G g$.

ii) *For all* $E, F \subseteq G$ *and* $f, g, f', g' \in \mathcal{F}^{const}$ *such that* $f \succ_G g$ *and* $f' \succ_G g'$: $[f, E; g, E^c] \succeq [f, F; g, F^c]$ *if and only if* $[f', E; g', E^c] \succeq [f', F; g', F^c]$.

iii) *There exist* $f, g \in \mathcal{F}^{const}$: $f \succ_G g$.

Condition iii) requires G to be non-null, i) and ii) are Savage's state-independence axiom P3 and P4 restricted to G. The preference relation $\succeq$ is *finitely state-dependent* if there exists a finite partition[20] $\{\Theta_i\}_{i \in I}$ of Θ such that each Θ_i (for $i \in I$) is an s.i.p. event.

Axiom 6 (P345*) $\succeq$ *is finitely state-dependent.*

The assumption of finite state-dependence can be viewed as having two parts: conditional on each Θ_i, there is a rich, *non-atomic* set of contingencies within which preferences are

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[20] For transitive $\succeq$, it is easily verified that one might have equivalently replaced "partition" by "collection"; we choose the former for greater specificity.

state-independent; this follows from Θ_i being non-null and P6. Secondly, state-dependence can be described in terms of a *finite* partition. The second of these assumptions is made for technical convenience; the first, however, has substantive content, as it is indispensable for a characterization of subjective EIU-maximization based on an ISD type axiom. Note that state-independence of preferences conditional on $E \subseteq \Theta_i$ requires in effect that, conditional on Θ_i, any implicit uncertainty about future preferences is subjectively stochastically independent of the explicit uncertainty θ. For simplification of language, we take in the following the partition $\{\Theta_i\}_{i \in I}$ as given and will abbreviate $\succeq_{\Theta_i}$ to $\succeq_i$; theorems 2 and 3 are to be read accordingly.

For any finitely-ranged function $x : \Theta \to \mathbf{R}$, define
$$\int x(\theta)d\mu = \sum_{\xi \in x(\Theta)} \xi \mu(\{\theta \in \Theta | x(\theta) = \xi\}).$$

Theorem 2 (Karni-Schmeidler) $\succeq$ *on $\mathcal{F}$ satisfies P1, P2, P345* and P6 if and only if there exists a collection of finitely additive, convex-ranged[21] probability measures[22] $\{\mu_i : 2^\Theta \to \mathbf{R}\}_{i \in I}$ such that $\mu_i(\Theta_i) = 1$ and a collection of non-constant utility-functions $\{u_i\}_{i \in I}$ such that*
$$f \succeq g \text{ if and only if } \sum_{i \in I} \int u_i(f(\theta))d\mu_i \geq \sum_{i \in I} \int u_i(g(\theta))d\mu_i \text{ , for all } f, g \in \mathcal{F}.\text{[23]}$$

It remains to "subjectivize" ISD as ISD*. ISD* is naturally formulated here as an assumption on conditional preferences $\succeq_i$, since comparative probability relations can meaningfully be defined only conditional on s.i.p. events Θ_i. Thus, let $\geq_i$ be the conditional more-likely-than relation on 2^Θ defined by

$E \geq_i F$ if, for any constant acts f, g such that $f \succ_i g : [f, E; g, E^c] \succeq_i [f, F; g, F^c]$.

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[21] μ is said to be *convex-ranged* if, for all $E \subseteq \Theta$ and all $\rho : 0 \leq \rho \leq 1$, there exists $F \subseteq E$ such that $\mu(F) = \rho\mu(E)$.

[22] For notational convenience, the measures μ_i are defined on 2^Θ instead of on 2^{Θ_i}; in view of the fact that $\mu_i(\Theta_i) = 1$, they can nonetheless be interpreted as subjective conditional probability measures. Analogous remarks apply to the subsequently defined relations $\geq_i$.

[23] Karni-Schmeidler assume P3, but their proof is easily modified to a partition-relativized version of P3.

Note that by part ii) of the definition of an s.i.p. event, "any" can be replaced by "all" in the definition of $\geq_i$, and that $E \geq_i F$ if and only if $\mu_i(E) \geq \mu_i(F)$.

Moreover define

Definition 4 $f \trianglerighteq_i g$ ("f indirectly stochastically dominates g conditional on Θ_i"), iff, for all weak orders R on X and all $x \in X$:

$$\{\theta \in \Theta | f(\theta) \cap \{y \in X | yRx\} \neq \emptyset\} \geq_i \{\theta \in \Theta | g(\theta) \cap \{y \in X | yRx\} \neq \emptyset\}.$$

The following is a subjective, conditional version of ISD.

Axiom 7 (ISD*, Indirect Stochastic Dominance) *For all $f, g \in \mathcal{F}$ and all $i \in I$:* $f \succeq_i g$ *whenever* $f \trianglerighteq_i g$.

ISD* can be expressed purely in preference terms: if f and g coincide outside Θ_i, and if any bet on attaining under f any level set of any weak order conditional on Θ_i, i.e. the bet on the event $\{\theta \in \Theta | f(\theta) \cap \{y \in X | yRx\} \neq \emptyset\} \cap \Theta_i$, is preferred to the corresponding bet based on g, then f itself is weakly preferred to g.

The following result, the main theorem of the paper, is a straightforward consequence of theorems 1 and 2. Note that in the representation, the implicit probability distributions λ^i over future preferences are allowed to depend on Θ_i.

Theorem 3 *A preference relation over opportunity acts $\succeq$ satisfies P1, P2, P345*, P6 as well as ISD* if and only if there exist $\{\mu_i\}_{i \in I}$ and $\{u_i\}_{i \in I}$ as in theorem 2 and such that each u_i has the form $u_i(A) = \sum_{\omega_i \in \Omega_i} \lambda^i_{\omega_i} \max_{x \in A} v^i_{\omega_i}(x)$ (for appropriate Ω_i, $\lambda^i \in \Delta^{\Omega_i}$, and $\{v^i_{\omega_i}\}_{\omega_i \in \Omega_i}$).*

As remarked before, the richness of the state-space implied by P6 is critical to the validity of the result. The result would cease to hold with additively separable preferences and a finite state space as in Kreps (1992); it is easily verified, for instance, that the result is false

if Θ consists of only one state, since then ISD* coincides with monotonicity with respect to set-inclusion which is not enough according to theorem 5 below.

4. THE SIMPLE ALGEBRA OF EXPECTED INDIRECT UTILITY

Sections 2 and 3 have left two bits of unfinished business. The uniqueness properties of the representation have not been discussed. One would also like to know more explicitly the nature of the restrictions imposed by EIU-rationalizability on preferences over opportunity prospects, and especially the restrictions on the cardinal utility-functions u representing those preferences ("EIU functions"). Both of these issues will now be addressed based on a preceding exposition of the algebra of EIU functions u . The basic novel insight of this section is the observation (fact 2) that the structure of EIU functions is closely related to that of "plausibility-functions" (conjugate belief-functions) in the literature on non-probabilistic belief representations; as a result, the key technical tool of that literature, Möbius inversion (originally due to Rota 1964), becomes applicable and central here as well. It has in fact been used already in the proof of theorem 1; among other applications, Möbius inversion proves its mettle at the end of this section by yielding a particularly transparent proof of Kreps's (1979) main result.[24]

Let $\mathcal{A}^* = 2^X \backslash (\emptyset \cup \{X\})$. $\#S$ is the cardinality of the set S, with $\#X = n,$ and $\subset$ denotes the *strict* subset relation. $\mathbf{1}: \mathcal{A} \to R$ is the constant function equal to 1, $\mathbf{1}_{\mathcal{S}} : \mathcal{A} \to \mathbf{R}$ is the indicator-function of the set of sets $\mathcal{S}$. Functions from $\mathcal{A}$ to $\mathbf{R}$ will often be viewed as vectors in $\mathbf{R}^{\mathcal{A}}$.

A function $u : \mathcal{A} \to \mathbf{R}$ is an **indirect utility (IU) function** if it has the form $u(A) = \max_{x \in A} u(\{x\})$ for all $A \in \mathcal{A}$. An function $u : \mathcal{A} \to \mathbf{R}$ is an **expected indirect utility (EIU) function** if it is a convex combination of IU-functions: $u(A) = \sum_{\omega \in \Omega} \lambda_\omega v_\omega(A) = \sum_{\omega \in \Omega} \lambda_\omega \max_{x \in A} v_\omega(\{x\})$ for all $A \in \mathcal{A}$, for some finite collection of IU-functions $\{v_\omega\}_{\omega \in \Omega}$ and some set of coefficients $\{\lambda_\omega\}_{\omega \in \Omega}$ such that $\lambda_\omega \geq 0$ for all $\omega \in \Omega$ and $\sum_{\omega \in \Omega} \lambda_\omega = 1$. Thus,

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[24]The classical references on belief-functions are Dempster (1967) and Shafer (1976); for a recent thorough study of Möbius inversion, the key technical tool, see Chateauneuf-Jaffray (1989).

preferences over opportunity prospects / acts are EIU-rationalizable if and only if they have a vNM / Savage representation in terms of an EIU function u.

An IU function is **dichotomous** (and 0-1 normalized) (DIU) if it takes the values 0 and 1 only, i.e. if $u(\mathcal{A}) \subseteq \{0,1\}$. Finally, a function $u : \mathcal{A} \to \mathbf{R}$ is **simple** if $u = v_S$ for some $S \in 2^X$, with $v_S : \mathcal{A} \to \mathbf{R}$ defined by

$$v_S(A) = \begin{cases} 1 & \text{if } A \cap S \neq \emptyset, \\ 0 & \text{if } A \cap S = \emptyset, \end{cases} \quad \text{for } A \in \mathcal{A}.$$

The following observation characterizes EIU functions as equivalent to certain linear combinations of dichotomous IU (respectively simple) functions.

Fact 2 *i)* u *is a DIU-function if and only if* u *is simple.*

ii) u *is an IU-function if and only if*

$$u = \sum_{S \in \mathcal{A}} \lambda_S v_S \ , \ \text{for } \lambda \in \mathbf{R}^{\mathcal{A}} \text{ such that } \lambda_S \geq 0 \text{ for all } S \neq X, \text{ and such that } \lambda_S > 0 \text{ and }$$
$\lambda_T > 0$ *imply* $S \subseteq T$ *or* $S \supseteq T$.

iii) u *is an EIU-function if and only if*

$$u = \sum_{S \in \mathcal{A}} \lambda_S v_S, \text{ for } \lambda \in \mathbf{R}^{\mathcal{A}} \text{ such that } \lambda_S \geq 0 \text{ for all } S \neq X.$$

Example 3 *Let* $X = \{1, 2, 3\}$ *and* u *the IU-function defined by* $u(S) = \max_{x \in S} x^2$. *Then* $u = v_{\{1,2,3\}} + 3v_{\{2,3\}} + 5v_{\{3\}}$.

Mathematically, the key to the following analysis is the observation that the set of DIU functions is a linear basis of the space $\mathbf{R}^{\mathcal{A}}$. How DIU-functions combine (in particular to yield EIU functions) is described by the *"dual Möbius operator"*[25] $\Psi : \mathbf{R}^{\mathcal{A}} \to \mathbf{R}^{\mathcal{A}}$ defined by $\lambda \mapsto u = \sum_{S \in \mathcal{A}} \lambda_S v_S$, and thus $u(A) = \Psi(\lambda)(A) = \sum_{S \in \mathcal{A}: S \cap A \neq \emptyset} \lambda_S$, for $A \in \mathcal{A}$.

Basic is the following fact.

Fact 3 $\Psi : \mathbf{R}^{\mathcal{A}} \to \mathbf{R}^{\mathcal{A}}$ *is a bijective linear map. Its inverse* Ψ^{-1} *is given by*

$$\Psi^{-1}(u)(A) = \sum_{S \in 2^X : S \subseteq A} (-1)^{\#(A \setminus S) + 1} u(S^c) \text{ for } A \in \mathcal{A} \ , \text{ with } u(\emptyset) = 0 \text{ by convention.}$$

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[25]For the choice of terminology, consult the proof of fact 3.

The fact allows a straightforward characterization of EIU-functions in terms of $2^n - 2$ linear inequalities.

Corollary 1 *u is an EIU function if and only if $\Psi^{-1}(u)(A) \geq 0$ for all $A \in \mathcal{A}^*$.*

Drawing on the literature on belief-functions, the characterizing condition is made more intelligible by generalizing it to the following effectively equivalent pair of conditions.

Definition 5 *i)* $u : \mathcal{A} \to \mathbf{R}$ *is* **monotone** *if $A \subseteq B$ implies $u(A) \leq u(B)$ $\forall A, B \in \mathcal{A}$.*

ii) $u : \mathcal{A} \to \mathbf{R}$ *is* **uniformly submodular** *if, for any finite collection $\{A_k\}_{k \in K}$ in $\mathcal{A}$ such that $\bigcap\limits_{k \in K} A_k \neq \emptyset$,*

$$u\left(\bigcap_{k \in K} A_k\right) \leq \sum_{J : \emptyset \neq J \subseteq K} (-1)^{\#J+1} u\left(\bigcup_{k \in J} A_k\right).{}^{26}$$

Uniform Submodularity is easiest understood by considering the case of $\#K = 2$, where it specializes to the following standard "submodularity" condition:

$$u(A \cap B) + u(A \cup B) \leq u(A) + u(B) \quad \forall A, B \in \mathcal{A} \text{ such that } A \cap B \neq \emptyset , \qquad (1)$$

or equivalently:

$$u(A \cup B) - u(A) \geq u(A \cup B \cup C) - u(A \cup C) \quad \forall A, B, C \in \mathcal{A}.$$

In this version, submodularity says that the incremental value of adding some set a given set of alternatives (the set B to A) never increases as other alternatives (the set C) are added. Submodularity implies that opportunity subsets are substitutes in terms of flexibility value.

Theorem 4 *u is an EIU function if and only if it is monotone and uniformly submodular.*

Theorem 4 translates immediately into a characterization of the risk attitudes towards opportunity prospects implied by EIU maximization.

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[26]The conjunction of monotonicity and uniform submodularity differs from "infinite monotonicity" in the sense of Choquet (1953) in two ways: the latter condition would result if in the definition of uniform submodularity the inequality would be reversed and if the non-empty-intersection clause be dropped.

Definition 6 *i)* $\succeq$ *is* **monotone** *if* $\mathbf{1}_{\{A\}} \succeq \mathbf{1}_{\{B\}}$ *for all* A, B *in* $\mathcal{A}$ *such that* $A \supseteq B$.

ii) $\succeq$ *is* **opportunity risk-averse** *if, for any finite collection* $\{A_k\}_{k \in K}$ *in* $\mathcal{A}$ *such that* $\bigcap_{k \in K} A_k \neq \emptyset$, *and any* q, p *such that*

q is defined by $q_S = 2^{-n+1} \cdot \#\{J \leq K \mid \#J$ *is even and strictly positive and* $S = \bigcup_{k \in J} A_k$,
or $J = \emptyset$ *and* $S = \bigcap_{k \in K} A_k\}$, *and*

p is defined by $p_S = 2^{-n+1} \cdot \#\{J \leq K \mid \#J$ *is odd and* $S = \bigcup_{k \in J} A_k\}$,

then $p \succeq q$.

The connection of this definition with an intuitive notion of risk-aversion emerges from considering prospects of two opportunity sets. Opportunity risk-aversion then specializes[27] to the condition that, for all $A, B, C \in \mathcal{A}$ such that $A \supset B \cup C$ and $B \cap C = \emptyset$:

$$(\frac{1}{2}\mathbf{1}_{\{A \backslash B\}} + \frac{1}{2}\mathbf{1}_{\{A \backslash C\}}) \succeq (\frac{1}{2}\mathbf{1}_{\{A\}} + \frac{1}{2}\mathbf{1}_{\{A \backslash (B \cup C)\}}). \tag{2}$$

Thus, losing one of the opportunity subsets B or C for sure (each with equal odds) is weakly preferred to facing a fifty-present chance of losing both B and C. All instances of opportunity risk-aversion share the following two characteristics which together lend some minimal justification to viewing them as genuine instances of risk-aversion:

i) $p(\{S \mid S \ni x\}) = q(\{S \mid S \ni x\}) \; \forall x \in X$, and

ii) for some S with $q_S > 0 : p_T > 0 \Rightarrow T \supset S \; \forall T \in \mathcal{A}$.

Theorem 4 yields the following corollary:

Corollary 2 $\succeq$ *on* $\Delta^{\mathcal{A}}$ *is EIU-rationalizable if and only if it satisfies vNM and is monotone and uniformly risk-averse.*

Remark: While opportunity risk-aversion emerges as a natural characterizing property from a purely descriptive point of view, it is not very appealing decision-theoretically as a conceptually fundamental axiom. It is too complex to be particularly intuitive; more importantly, its link to an intuitive notion of flexibility / indirect utility stands in need of clarification; finally, in contrast to ISD, the role of stochastic independence remains hidden.

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[27]by considering collections of the form $\{A \backslash B, A \backslash C\}$, with $B \cap C = \emptyset$.

We conclude this section by providing a new and simplified proof of Kreps's (1979) classic result which characterizes EIU rationalizable preference orders defined on the class of opportunity *sets* $\mathcal{A}$. The new proof is based on dual Möbius inversion and given in the appendix; we hope that it significantly clarifies the logic of Kreps's result. For the remainder of this section only, assume $\succeq$ to be a weak order on $\mathcal{A}$.

Definition 7 *i)* $\succeq$ *is* **monotone** *if* $A \supseteq B$ *implies* $A \succeq B$ *, for all* $A, B \in \mathcal{A}$.

ii) $\succeq$ *is* **ordinally submodular** *if* $A \succeq A \cup B$ *implies* $A \cup C \succeq A \cup B \cup C$ *, for all* $A, B, C \in \mathcal{A}$.

iii) $\succeq$ *is* **ordinally EIU-rationalizable** *if there exists an EIU function* $u : \mathcal{A} \to \mathbf{R}$ *such that* $A \succeq B$ *if and only if* $u(A) \geq u(B)$ *for all* $A, B \in \mathcal{A}$.

Theorem 5 (Kreps) *A weak order* $\succeq$ *is ordinally EIU-rationalizable if and only if it is monotone as well as ordinally submodular.*

The sufficient conditions of the theorem seem surprisingly weak. In particular, Kreps' result implies that whenever a preference relation is "*strictly monotone*" (i.e. satisfies the condition "$A \supset B \Rightarrow A \succ B$ for all $A, B \in \mathcal{A}$"), it is ordinally EIU-rationalizable. To facilitate the discussion, we restate the result as one about ordinal utility–functions.

Condition 1 (OSM) $u(A) \geq u(A \cup B) \Rightarrow u(A \cup C) \geq u(A \cup B \cup C) \ \ \forall A, B, C \in \mathcal{A}$.

Theorem 6 (Kreps, restated) *For any function* $u : \mathcal{A} \to \mathbf{R}$*: there exists a strictly increasing transformation* $\tau : \mathbf{R} \to \mathbf{R}$ *such that* $\tau \circ u$ *is an EIU function if and only if* u *is monotone and satisfies OSM.*

Consider any utility function with the property $A \supset B \Rightarrow u(A) > u(B)$ for all $A, B \in \mathcal{A}$. According to theorem 6, for an appropriate τ, $\tau \circ u$ is uniformly submodular. The concave flavor of uniform submodularity suggests that to achieve this one needs to define transformations τ that concavify u "sufficiently strongly." The actual proof in the appendix follows this line of argument (lemma 5), and verifies that indifferences are adequately taken care of by condition OSM (lemma 4).

5. ON THE UNIQUENESS OF THE REPRESENTATION

So far, the uniqueness properties of the EIU representation in theorem 1 and 3 have not been discussed. This task will be addressed now, with dual Möbius inversion as the key tool. The story line goes as follows. There is an essentially unique representation in terms of *dichotomous* IU functions. Dichotomous IU representations can be reinterpreted as additive *multi-attribute* representations (eliminating the reference to an implicit state-space). This allows one to characterize the exact extent of the non-uniqueness problem; in particular, it becomes evident what kind of structure needs to be added to achieve uniqueness. For the sake of specificity, we will explicitly focus on the uniqueness properties of preferences over opportunity prospects in a vNM setting; the extension to preferences over opportunity acts is immediate.

From fact 2iii), EIU-rationalizability is equivalent to rationalizability by a set of *dichotomous* IU-functions; using dual Möbius inversion, it is easy to see that "dichotomous EIU-representations" enjoy optimal uniqueness properties. $\succeq$ is *nontrivial* if $X \succ \{x\}$ for some $x \in X$.

Proposition 1 *i)* $\succeq$ *is EIU-rationalizable if and only if there exists* $\lambda \in \mathbf{R}^{\mathcal{A}}$ *with* $\lambda_T \geq 0$ *for all* $T \neq X$ *such that:*

$$\forall p, q \in \Delta^{\mathcal{A}}: \quad p \succeq q \quad \Leftrightarrow \quad \sum_{S \in \mathcal{A}} \sum_{T \in \mathcal{A}} p_S \lambda_T v_T(S) \geq \sum_{S \in \mathcal{A}} \sum_{T \in \mathcal{A}} q_S \lambda_T v_T(S). \tag{3}$$

ii) If λ *satisfies condition (3), then* λ' *satisfies condition (3) as well if and only if, for some* $c > 0 : \lambda'_T = c\lambda_T$ *for all* $T \neq X$.

iii) If $\succeq$ *is nontrivial, there exists a unique* $\lambda \in \Delta^{\mathcal{A}^*}$ *satisfying (3)[28].*

In the remainder of this section, we will maintain the assumption that $\succeq$ is nontrivial and refer to $\lambda \in \Delta^{\mathcal{A}^*}$ satisfying (3) as the "weight vector" or "measure" representing $\succeq$.

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[28]Strictly speaking, $\lambda \in \Delta^{\mathcal{A}}$ such that $\lambda_X = 0$.

While uniqueness of *dichotomous* EIU representations in the present context may not seem to amount to that much, it is a significant improvement over what is achievable when preferences are defined over opportunity sets. This improvement is obviously due to the fact that the utility-functions representing preferences are unique up to positive affine rather than merely strictly increasing transformations. In the latter case, not even the support of λ is uniquely determined. Moreover, proposition 1 gives sufficient indication for what needs to be assumed of the class of possible future preferences in order to ensure optimal uniqueness properties.

Only in very rare situations, of course, will the decision maker in fact have dichotomous date-2 preferences, as in the following example in which λ may be interpreted as a subjective probability measure.

Example 4 *Flex needs to open a lock; he can choose among closed boxes with uncertain contents. Specifically, any box contains with probability p_S exactly the non-empty set S of keys $x \in X$; a box can thus be identified with a probability measure $p \in \Delta^{\mathcal{A}}$. Having chosen the box, Flex will attempt to open the lock, trying out all keys in the chosen box. He cares only about the chance of success in opening the lock, and does not know which keys if any will fit. In this case, the relevant state space is 2^X, with $T \in 2^X$ denoting the set of keys that in fact open the lock; in state T, Flex's preferences over sets of keys are given by the DIU-function v_T; in other words, Flex is successful ($v_T(S) = 1$) if the box S contains at least one key in T.*

Here, $\mathcal{A}^*$ denotes also the event that some keys fit but not all ($T \neq \emptyset, X$). By proposition 1, Flex's preference ordering $\succeq$ over hypothetical boxes $p \in \Delta^{\mathcal{A}}$ reveals unambiguously his subjective probability measure $\lambda \in \Delta^{\mathcal{A}^*}$ over the sets of keys that fit, conditional on some but not all keys in X fitting, that is: conditional on $\mathcal{A}^*$ (the conditional probability that exactly the keys in T fit is given by λ_T) . On the other hand, $\succeq$ contains no information about the subjective probability of the conditioning event $\mathcal{A}^*$ itself (beyond its being non-zero), since if either all keys work or none, Flex's choice of a box does not matter. □

In the general case, in which future preferences may be non-dichotomous, the coefficients

of a dichotomous EIU-representation yield only highly "compounded" information about the decision maker's beliefs about future preferences. The representation of proposition 1 then needs to be rewritten a bit to become meaningfully interpretable. The starting point is the observation that the interpretation of v_T as a utility-function is unnecessary and, in this case, unhelpful. Alternatively, v_T can be viewed as indicator-function of the class of sets that intersect with $T, v_T = \mathbf{1}_{\{S|S \cap T \neq \emptyset\}}$. Correspondingly, T can viewed as parametrizing not a state but an attribute the "*component opportunity*" T. S *realizes* the component opportunity T if and only if its intersects with T (or, equivalently, iff $v_T(S) = 1$), in other words: if S permits to realize *some* alternative in T.

Accordingly, the vNM utility of S can uniquely be written as the sum of the values λ_T of all component opportunities that it realizes: $u(S) = \sum\limits_{T:T \cap S \neq \emptyset} \lambda_S$, thus yielding an *additive multi-attribute representation* in terms of which essential uniqueness is always ensured.

$$\forall p, q \in \Delta^{\mathcal{A}}: \quad p \succeq q \quad \Leftrightarrow \quad \sum_{S \in \mathcal{A}} p_S \left(\sum_{T:T \cap S \neq \emptyset} \lambda_S \right) \geq \sum_{S \in \mathcal{A}} p_S \left(\sum_{T:T \cap S \neq \emptyset} \lambda_S \right). \quad (4)$$

If one is willing to postulate that the decision maker "in fact" maximizes expected indirect utility with given $\{v_\omega\}_{\omega \in \Omega}$ and subjective probabilities $\{\pi_\omega\}_{\omega \in \Omega}$, further explanation of the attribute weights λ_T can be given. In view of fact 2ii), it is easily verified that the (non-normalized) coefficients λ_T in (4) that correspond to the EIU-function $\sum\limits_{\omega \in \Omega} \pi_\omega v_\omega$ satisfy:

$$\lambda_X = \sum_{\omega \in \Omega} \pi_\omega \min_{x \in X} v_\omega(\{x\}) \quad,$$

and, for T different from X,

$$\lambda_T = \sum_{\omega \in \Omega} \pi_\omega [\min_{x \in T} v_\omega(\{x\}) - \max_{T' \supset T} \min_{x \in T'} v_\omega(\{x\})]. \quad (5)$$

Note that the expression "$\min_{x \in T} v_\omega(\{x\}) - \max_{T' \supset T} \min_{x \in T'} v_\omega(\{x\})$" differs from zero (being then in fact positive) if and only if T is a level set of v_w, i.e. if $T = \{x \in X \mid v_\omega(\{x\}) \geq v_\omega(T)\}$. Thus λ_T is the *expected incremental utility* from reaching the level set T rather than the next lower one.

Together with proposition 1, (5) precisely describes the extent of non-uniqueness of the EIU-representation. Preferences thus fail to reveal the agent's subjective probability distribution over IU-functions v_ω for two reasons. First, even if all IU-functions with positive probability are in fact dichotomous, their coefficients combine a subjective-probability and a utility-scale factor, as typical for state-contingent preferences. Secondly, the same EIU-function can typically be generated as convex combination of non-dichotomous IU-functions in many different ways.

However, (5) also suggests that the second source of non-uniqueness is not inescapable. In particular, uniqueness will obtain if either due to additional conditions on $\succeq$ or simply by an external "identifying" assumption, future preferences R_ω are known to belong to some class $\mathcal{R}$ with the property that any non-degenerate level set is associated with at most one preference ordering in that class, i.e. formally that, for any $x \in X$ and any R, $R' \in \mathcal{R}, \{y|yRx\} = \{y|yR'x\} \neq X$ implies $R = R'$. Such $\mathcal{R}$ will be referred to as *identified*.

If $\mathcal{R}$ is identified, it can be made the canonical state space; $\succeq$ has then a representation of the form

$$p \succeq q \Leftrightarrow \sum_S p_S \left(\sum_{R \in \mathcal{R}} \max_{x \in S} v_R(x) \right) \geq \sum_S q_S \left(\sum_{R \in \mathcal{R}} \max_{x \in S} v_R(x) \right) \ .$$

In view of (5), the v_R in this representation are essentially unique: specifically, if $\{v_R\}_{R \in \mathcal{R}}$ represents $\succeq$, then $\{v'_R\}_{R \in \mathcal{R}}$ represents $\succeq$ as well if and only if there exists $c > 0$ and $\{d_R\}_{R \in \mathcal{R}}$ such that $v'_R = cv_R + d_R$ for all $R \in \mathcal{R}$.

An obvious example of an identified class has already been introduced, that of weak orders R with only two level sets. More interestingly, identified classes arise quite naturally with infinite domains X; examples are the class of quasi-linear preferences on a domain X of the form $X = Y \times \mathbf{R}$, and the class of EU preferences on a lottery space X of the form $X = \Delta^Y$. Of course, the restrictions on preferences over opportunity sets implied by additional structure of this kind remain to be worked out. Note also that in a Savage framework, preferences of the latter class arise naturally from uncertainty that is not resolved at date 2.

6. FREEDOM OF CHOICE

We will now consider situations in which *all* relevant uncertainty is explicitly modelled in the manner of section 3, including uncertainty about future preferences. A failure of conditional preferences to satisfy the IU property can then by definition no longer be attributed to uncertainty about future preferences, but reveals an intrinsic "preference for freedom of choice." [29]

As a sound intuitive basis for imposing consistency conditions on preferences for opportunities, only the notion that "more opportunity is better" seems to remain.[30]We will argue in this section that, properly conceived, this notion is rich enough to provide the basis for an well-behaved theory of intrinsic preference for freedom of choice, and that in fact one merely needs to reinterpret the results above to obtain such a theory. By contrast, the bulk of the literature has relied on independence conditions to obtain additional structure; these, however, are very restrictive and preclude consideration of the diversity of alternatives in an opportunity set[31].

The key is an answer to the question: more precisely *of what* is better? To address it, we take as point of departure an interpretation of "freedom of choice" as the freedom to do this or that, to choose something particular, to bring about specific consequences such as living in a particular place, entering a particular profession, etc. . Thus, the freedom of choice offered by some opportunity set can be analyzed in terms of its *component opportunities* to bring about particular consequences, and *effective freedom of choice* is naturally viewed as multi-attribute construct, with the component opportunities as its relevant attributes. By "*effective* freedom of choice" we mean an agent's *inclusive* valuation of opportunity sets that combines indirect utility and freedom of choice considerations; the notion of "effective

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[29]We leave to philosophy the task of explicating this intuitive appealing concept in a rigorous manner; for a justification based on the notion that agents autonomously choose their own preferences, see Sugden (1996).

[30]Of course, this requires to keep abstracting from phenomena such as weakness of will, etc.

[31]See Pattanaik-Xu (1990), Puppe (1995), Nehring-Puppe (1996a) as well as Sugden (1996) for criticisms along this line.

freedom of choice" is thus understood to comprise as a special case the preferences of agents who do not intrinsically value freedom of choice, i.e. whose conditional preferences $\succeq_i$ satisfy the conditional IU-property; in this case, the valued component opportunities are those associated with the attainment of some level-set of the form $\{y|\{y\} \succeq_i \{x\}\}$. Up to the issue of extensionality raised below, the notion of component opportunity coincides with that of section 5 which had been introduced there for largely technical reasons.

A significant strand in the axiomatic literature on the ranking of opportunity sets is interested in "measuring" freedom of choice *exclusive* of considerations concerning an agent's welfare. The notion of a component opportunity and the following analysis based on it are equally applicable under such an exclusive freedom of choice interpretation, as illustrated by the following example. We do not pursue this interpretation further here, especially since its conceptual coherence stands in need of further clarification.[32]

Example 5 *Renate is a young East German woman currently living in the GDR*[33] *in the 1980s. An alternative consists of a place where she might live (East or West Berlin, E or W), and of a profession she might enter (becoming a medical doctor or a journalist, D or J). The relevant universe of alternatives is* $X = \{w, x, y, z\}$, *with* $w = (E, J)$, $x = (E, D)$, $y = (W, D)$, *and* $z = (W, J)$. *At the level of consequences, basic component opportunities are the opportunity to live in East Berlin, the opportunity to live in West-Berlin, that of becoming a doctor, and that of becoming a journalist. Typically also logically derived component opportunities are relevant, such as that of becoming a doctor while living in West-Berlin.*

To express component opportunities defined via consequences in terms of the primitives of the model (i.e. alternatives), they need to be translated into their *consequence extensions;* the extension $E \subseteq X$ of a consequence is the set of alternatives that bring about that consequence[34]. (Obviously, some information may be lost in translation, since different

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[32]Sugden (1996) for one is highly skeptical.

[33]German Democratic Republic, R.I.P..

[34]In logic, the extension of a predicate is defined as the set of objects that satisfy it. In Frege's famous

consequences may happen to have the same extensions). The following matrix associates component opportunities and their extensions in the example above.

Component Opportunity	E	W	D	J	W&J
Extension	$\{w, x\}$	$\{y, z\}$	$\{x, y\}$	$\{w, z\}$	$\{z\}$

Component opportunities will be described in the following extensionally, as the opportunity to bring about membership of the chosen alternative in E, to "realize E", and will be referred to by their extensions. As in section 5, an opportunity set A *realizes the component opportunity E* if and only if A contains one alternative realizing E, in other words, if and only if $A \cap E \neq \emptyset$.

In a stochastic setting, this suggests the following definition of "more opportunity in expectation".

Definition 8 *The opportunity act f offers* more opportunity in expectation *than the opportunity act g ($f \trianglerighteq g$) if and only if, for all component opportunities $E \in 2^X$ and all $i \in I$:* $\{\theta \in \Theta | f(\theta) \cap E \neq \emptyset\} \geq_i \{\theta \in \Theta | g(\theta) \cap E \neq \emptyset\}$.

Thus, f offers *more opportunity in expectation* than g if, for any component opportunity $E \in 2^X$ and conditional on any Θ_i, it is at least as likely for E to be realized under f as it is under g.

It is easily verified that the "offers more opportunity than" relation coincides with the Indirect Stochastic Dominance; it is therefore denoted by the same symbol $\trianglerighteq$. It follows from the above that the indirect stochastic dominance axioms ISD* and ISD capture the notion that more opportunity is better. State-dependence of preference is highly plausible, again. For example, if $\Theta = \{\Theta_1, \Theta_2\}$, with Θ_1 denoting the event "Renate has married someone unwilling to leave East-Berlin", Renate's valuation of the component opportunity W of living in West-Berlin will most probably depend on whether Θ_1 is realized or its complement; correspondingly, her preferences over opportunity sets conditional on Θ_1 and Θ_2

example, the predicates "is the morning star" and "is the evening star" have the same extension, the planet Venus.

will differ. Moreover, to account for a co-existing preference for flexibility in terms of explicit uncertainty, preferences need to be state-dependent. To establish the relevance of theorems 3 and 1 in a freedom-of-choice context, it remains to reinterpret the representations.

Theorem 3 yields a state-dependent additive *multi-attribute representation*, in which the utility-functions $u_i : \mathcal{A} \to \mathbf{R}$ of theorem 2 have the form $u_i(A) = \sum\limits_{E \in \mathcal{A}^*: E \cap A \neq \emptyset} \lambda_E^i$ for appropriate state-dependent attribute weights λ_E^i .[35] The collection of weights $\{\lambda_E^i\}_{E \in \mathcal{A}^*}$ defines an additive measure λ^i on $\mathcal{A}^*$, and the utility-representation can be rewritten as

$$u_i(A) = \lambda^i(\{E | E \cap A \neq \emptyset\}).$$

In view of the great popularity of proposals to measure pure (IU-exclusive) freedom of choice by counting alternatives, a measure representation is of some interest. It shows that the notion of counting makes sense after all, provided it is applied to the right type of objects, component opportunities rather than alternatives.

Remark 1: The counting of *alternatives* (with possibly asymmetric weights) emerges as a special case in which λ^i is concentrated on singletons, since then $u_i(A) = \lambda^i(\{\{x\} | x \in A\})$. However, concentration of λ^i on singletons means that the only valued consequences are those that can exclusively be realized by a single alternative. This seems to be a remarkably implausible implication even on a pure freedom-of-choice interpretation; for instance, in example 5, it means that realization of no basic component opportunity has value by itself. In fact, it is recognized by its apparent proponents that the counting of alternatives is not entirely satisfactory, and that, in particular, it fails to take properly into account the diversity of an opportunity set (see Pattanaik-Xu (1990) and Gravel-Laslier-Trannoy (1996)).

Remark 2: It is worth noting that the manner of counting has been motivated decision-theoretically rather than mathematically. On purely mathematical grounds, one might

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[35]Note that the sum is taken over $T \in \mathcal{A}^*$ rather than $T \in 2^X$. While conceptually perfectly sensible, the consequence extensions $\emptyset$ and X have been "normalized out" in the representation due to their irrelevance to preferences.

consider a dual measure based on all sets a given set contains, leading to uniformly *super*modular (rather than *sub*modular) vNM utility-functions of the form $u_i(A) = \lambda^i(\{E | E \subseteq A\})$. Such a measure is evidently devoid of decision-theoretic content.

From an inclusive valuation perspective, the most promising strand in the literature is the emerging multi-preference approach in which opportunity sets are compared in terms of a range of "relevant" (or "reasonable") preferences; see in particular Jones-Sugden (1982), Pattanaik-Xu (1995), and Sugden (1996). The results of this paper fit naturally into this line of research; one simply needs to reinterpret an EIU rationalization as follows: In the representing expression $\sum_{\omega_i \in \Omega_i} \lambda^i_{\omega_i} \max v^i_{\omega_i}(.)$ of theorem 3, Ω_i indexes the set of "reasonable" utility-functions (with $\lambda^i_{\omega_i} > 0$), conditional on Θ_i, and $\lambda^i_{\omega_i}$ is naturally interpreted as the *relevance-weight* of $v^i_{\omega_i}$; preferences satisfy the conditional IU property whenever all weight is concentrated on just one ordering. Of course, just as under the flexibility interpretation, there is the problem of non-uniqueness of the representation, and in particular that of disentangling relevance-weights from utility-scales. In this context, the present paper contributes the first *cardinal* representation and, more specifically, an additive aggregation rule. By comparison, the aggregation rules proposed in the literature are ordinal and entirely different in character (see Pattanaik-Xu (1995), Puppe-Xu (1995)). Moreover, with the exception of Nehring-Puppe (1996b), the set of relevant preferences is taken as given rather than derived from a representation theorem.

It is clear from the discussion of section 5 that within the framework studied here, the multi-attribute and the multi-preference interpretations are "observationally equivalent". It remains to be seen whether the two can be distinguished in interesting ways if more structure is assumed.

7. INCENTIVE-COMPATIBILITY

In this section, we discuss the viability of a *direct Savage-style approach* in which future preferences enter the description of a state. We will argue that whether or not any mileage is gained by such a move depends critically on one's willingness to accept certain types of

counterfactuals.

For expositional simplicity only, we will illustrate the problem by means of a simple example with two alternatives ($X = \{x, y\}$) and with only preference-uncertainty. In a direct Savage-style approach, the state-space is then given by the set of conceivable future preference-orderings, i.e.[36] by the two linear orders P_1, P_2 with xP_1y and yP_2x, as illustrated in the following table.

A natural subclass of Savage acts are those induced by the agent's future choice from some opportunity sets A. Such acts have the form $f_A : P_i \longmapsto \arg\max_{P_i} A$, assigning to each state as "prize" the finally chosen alternative; they will be referred to as "generated by the opportunity set A", and their class denoted by $\mathcal{F}^{opp}$. Note that on $\mathcal{F}^{opp}$ the sure-thing principle is satisfied vacuously (this holds true in general, irrespective of the cardinality of X). Note also that due to the inherent state-dependence of conditional preferences, Savage's other key axioms P_3 and P_4 do not apply here in any case.

act	P_1	P_2	generating set
(x,x)	x	x	$\{x\}$
(y,y)	y	y	$\{y\}$
(x,y)	x	y	$\{x,y\}$
(y,x)	y	x	–

Thus, if one considers preferences over $\mathcal{F}^{opp}$, the direct Savage approach entails no additional restrictions. It follows that in order to give the sure-thing principle, and thus the direct Savage approach, any bite, *arbitrary* acts (like (y, x)) need to be supported by appropriate (counterfactual) gedanken-experiments; the question is whether acceptable ones exist.

The most straightforward justification for the admissibility of arbitrary acts derives from postulating a *perfectly mind-reading referee* who awards the prize based on the agent's future preference ordering. The mind-reading might be that of an empathetic but potentially spiteful wife, or that of a brain-scientist in possession of a perfect "preference detector".

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[36] disregarding the possibility of future indifference for simplicity.

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Gedanken-experiments of this kind seem not only rather contrived, but also very much to go against the grain of the "revealed-preference" approach central to the decision-theoretic tradition, within which preferences are identified with dispositions to choice-behavior. Thus perhaps somewhat more plausible is a story in which the referee obtains knowledge of the agent's preference through the agent's own truthful revelation. This, however, leads to a severe incentive-compatibility problem, since honest reporting will often be contrary to the agent's current interest, i.e. to acting in accordance with the choice-function *defining* the state. In the above example, for instance, an agent faced with the "prima-facie act" (y, x) in terms of reported preferences will report P_2 if his true preferences are P_1, and vice versa, thus inducing the act (x, y) in terms of his true preferences. Note that at no point in the argument have we denied that the agent himself has introspective access to his own preferences; that simply is not enough to support arbitrary acts. The issue is rather whether it may be feasible for the agent to commit himself at present to make choices in the future contrary to his preferences at that time.

If this is doubted, the discussion suggests that only *incentive-compatible* acts correspond to *plausible* thought-experiments; an act is *incentive-compatible* if, for any pair of weak orders R and R' on X, $f(R) \ R \ f(R')$[37]; in other words, if, for all weak orders R,

$$f(R) \in \arg\max_R \{f(R')|R' \text{ is weak order on } X\}.$$

Thus, incentive-compatible acts are precisely those induced by some opportunity set; as a result the sure-thing principle is vacuously satisfied on the class of incentive-compatible acts, and the state-structure turns out to be redundant. In other words, a preference relation over opportunity acts[38] is effectively as primitive as can be.

It should be noted that analogous revealed-preference / incentive-compatibility considerations do *not* undermine the decision-theoretic approach to game-theory in which players are assumed to have beliefs about others' beliefs (respectively preferences on a state-space that

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[37]In a general deterministic model, an act f maps weak orders to alternatives.

[38]that is, in the absence of non-preference uncertainty assumed here, a preference relation over opportunity *sets.*

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includes others' preferences). In a nutshell, the difference to the intrapersonal intertemporal case is that in a game with *self-interested* players, player i's belief that another player j has placed some bet on i's own betting behavior does not interfere with i's actual betting behavior since, by self-interestedness, i does not care whether j wins or loses his bet. By contrast, an agent's "future self" i will typically care about whether the "initial self" j wins his bet or not: not only do the interests of the initial and future selves typically coincide, but also the initial self's bets can only be physically paid out to the future self!

We have argued that incentive-compatibility constraints may be of concern even on a normative interpretation on which acts correspond to a decision-maker's *thought*-experiments. On the other hand, on a behavioral interpretation concerned with *real* experiments, these constraints seem to be binding *in principle*. Fortunately, theorem 3 has shown that the hypothesis of EIU maximization with respect to all uncertainty remains testable in principle nonetheless.

APPENDIX: PROOFS

Proof of Fact 1.

i) $\Rightarrow$ iii). True since $p \unrhd_{R_u} q$ implies $\sum_{S \in \mathcal{A}} p_S u(S) \geq \sum_{S \in \mathcal{A}} q_S u(S)$, with R_u defined by $x R_u y \Leftrightarrow u(x) \geq u(y)$.

iii) $\Rightarrow$ ii). True since $p(\{S \mid S \cap A \neq \emptyset\}$ coincides with expected utility from p under the indirect-utility function v_A given by $\sum_{S \in \mathcal{A}} p_S v_A(S)$.

ii) $\Rightarrow$ i). True since $p \unrhd_R q$ is equivalent by definition to $p(\{S \mid S \cap A \neq \emptyset\} \geq q(\{S \mid S \cap A \neq \emptyset\}$ for all A of the form $\{x \mid x R y\}$, for some $y \in X$. ∎

Proof of Theorem 1:

It is well-known that vNM implies the existence of a vNM utility-function $u : \mathcal{A} \rightarrow \mathbf{R}$ such that

$$p \succeq q \Longleftrightarrow \sum_{S \in \mathcal{A}} p_S u(S) \geq \sum_{S \in \mathcal{A}} q_S u(S) \text{ , for all } p, q \in \Delta^{\mathcal{A}}. \tag{6}$$

In view of fact 3 below, we need to show that $\succeq$ satisfies ISD if and only if $\lambda_A = \Psi^{-1}(u)(A) \geq 0$ for all $A \in \mathcal{A}^*$.

Note first that $p(\{S \mid S \cap A \neq \emptyset\}) = \Psi(p)(A)$. The dual Möbius operator Ψ thus maps opportunity prospects p to their *characteristic profiles* $\Psi(p)$, establishing a linear isomorphism between $\Delta^{\mathcal{A}}$ and the space of characteristic profiles $\Gamma^{\mathcal{A}} := \Psi(\Delta^{\mathcal{A}}) = \{\mu \in \mathbf{R}^{\mathcal{A}} \mid \mu$ is monotone, uniformly submodular and $\mu(X) = 1\}$. The desired result is obtained by studying the induced preferences over characteristic profiles.

$\succeq$ defined on $\Delta^{\mathcal{A}}$ induces $\widehat{\succeq}$ on $\Gamma^{\mathcal{A}}$ according to

$$\mu \widehat{\succeq} \mu' \Longleftrightarrow \Psi^{-1}(\mu) \succeq \Psi^{-1}(\mu').$$

$\widehat{\succeq}$ is said to be *monotone* if $\mu \geq \mu' \Rightarrow \mu \widehat{\succeq} \mu'$. Fact 1 implies $p \rhd q \Longleftrightarrow \Psi(p) \geq \Psi(q)$. This yields part i) of the following fact. In view of facts 2, iii) and equation (6), one also easily verifies its second part.

Fact 4 *i)* $\widehat{\succeq}$ *is monotone if and only if* $\succeq$ *satisfies ISD.*

ii) $\mu \widehat{\succeq} \mu'$ *if and only if* $\sum_{A \in \mathcal{A}^*} \lambda_A \mu(A) \geq \sum_{A \in \mathcal{A}^*} \lambda_A \mu'(A)$ *, for all* $\mu, \mu' \in \Gamma^{\mathcal{A}}$.

In view of fact 4, i) , the theorem follows from the following lemma.

Lemma 1 $\widehat{\succeq}$ *is monotone if and only if* $\lambda_A \geq 0$ *for all* $A \in \mathcal{A}^*$.

Proof of lemma.

Only if: define $\overline{\mu}$ by $\overline{\mu}(A) = \frac{\#\{S \in \mathcal{A} \mid S \cap A \neq \emptyset\}}{\#\{S \in \mathcal{A}\}} = 1 - \frac{2^{n-\#A}-1}{2^n - 1}$, for all $A \in \mathcal{A}$.

$\overline{\mu}$ is in the interior of $\Gamma^{\mathcal{A}}$, since Ψ is a homeomorphism and $\Psi^{-1}(\overline{\mu}) = \frac{1}{2^n - 1} \cdot \mathbf{1}$ is in the interior of $\Delta^{\mathcal{A}}$. Thus, for any $A \in \mathcal{A}^*$ and small enough ε, $\overline{\mu} + \varepsilon \mathbf{1}_{\{A\}} \in \Gamma^{\mathcal{A}}$. By the monotonicity of $\widehat{\succeq}$ and fact 4, ii) , $\lambda_A \geq 0$.

The converse is immediate, noting that $\Psi(p)(X) = 1$ for all $p \in \Delta^{\mathcal{A}}$. $\square$ ∎

Remark: Characteristic profiles are, from the mathematical point of view, "plausibility functions" in the sense of the theory of belief-functions (Shafer (1976)); however, in contrast to the intended interpretation of that theory, a characteristic profile does not express a non-additive belief about the state-space $\mathcal{A}^*$, but rather probabilistic beliefs about events in $\mathcal{A}^*$ of the form $\{S \mid S \cap T \neq \emptyset\}$.

Proof of Theorem 3:

Let $\{\mu_i\}_{i \in I}$ and $\{u_i\}_{i \in I}$ as in theorem 2, which implies in particular

$$f \succeq_i g \text{ if and only if } \int u_i(f(\theta)) d\mu_i \geq \int u_i(g(\theta)) d\mu_i, \text{ for all } f, g \in \mathcal{F}. \qquad (7)$$

For given $i \in I$, define $\succeq_i^*$ on $\Delta^{\mathcal{A}}$ according to $\mu_i \circ f^{-1} \succeq_i^* \mu_i \circ g^{-1}$ if and only if $f \succeq_i g$, for all $f, g \in \mathcal{F}$. By the representation (7), $\succeq_i^*$ is well-defined. From the convex-rangedness of μ_i and the definition of $\mathcal{F}$, $\{\mu_i \circ f^{-1} \mid f \in \mathcal{F}\} = \Delta^{\mathcal{A}}$; $\succeq_i^*$ is therefore complete. From (7), it follows that $\succeq_i^*$ satisfies all of the vNM axioms and is represented by the vNM utility-function u_i. [These facts have been in fact derived by Savage as a key step in obtaining his representation theorem in the first place].

ISD* of $\succeq$ is clearly equivalent to ISD of $\succeq_i^*$. By theorem 1, this in turn is equivalent to a representation of u_i according to $u_i(A) = \sum\limits_{\omega_i \in \Omega_i} \lambda_{\omega_i}^i \max_{x \in A} v_{\omega_i}^i(x)$, for appropriate Ω_i, $\lambda^i \in \Delta^{\Omega_i}$, and $\{v_{\omega_i}^i\}_{\omega_i \in \Omega_i}$. $\blacksquare$

Proof of Fact 2.

1. i) $\Longrightarrow$. If u is a DIU-function, then $u = v_{\{x \in X \mid u(x) = 1\}}$.

2. i) $\Longleftarrow$. By definition of a simple function, $v_S(A) = 1$ if and only if $\exists x \in X : x \in A \cap S$, which in turn holds if and only if $\exists x \in A : v_S(\{x\}) = 1$.

3. ii) $\Longleftarrow$. Consider $u = \sum\limits_{S \in \mathcal{A}} \lambda_S v_S$, for $\lambda \in \mathbf{R}^{\mathcal{A}}$ such that $\lambda_S \geq 0$ for $S \neq X$, and such that $\lambda_S > 0$ and $\lambda_T > 0$ imply $S \subseteq T$ or $S \supseteq T$. Define $\Lambda = \{S \in \mathcal{A} \mid \lambda_S > 0 \text{ or } S = X\}$. Then $u(\{x\}) = \sum\limits_{S \in \Lambda : S \ni x} \lambda_S$, for all $x \in X$, and $u(A) = \sum\limits_{S \in \Lambda : S \cap A \neq \emptyset} \lambda_S = u(\{y\})$ for

any $y \in \cap\{S \in \Lambda \mid S \cap A \neq \emptyset\}$; such y exist by the assumed ordering property of Λ. Since clearly $u(A) \geq u(\{z\})$ for all $z \in A$, $u(A) = \max_{x \in A} u(\{x\})$; u is thus an IU-function.

4. ii) $\Longrightarrow$. If u is an IU-function, let $\{x_k\}_{k=1,..,n}$ be an enumeration of X such that $u(\{x_k\}) \geq u(\{x_{k+1}\})$ for $k = 1, .., n$. Then

$$w = \sum_{k=1}^{n-1} (u(\{x_k\}) - u(\{x_{k+1}\}))v_{\{x_j \mid j \leq k\}} + u(\{x_n\})v_X$$

denotes a function of the desired form. By part 3., w is an IU-function. To show its equality to u, it thus suffices to show equality for singleton-sets, as follows: $w(\{x_l\}) = \sum_{k=l}^{n-1} (u(\{x_k\}) - u(\{x_{k+1}\})) + u(\{x_n\}) = u(\{x_l\})$.

5. iii) $\Longleftrightarrow$. Immediate from 3. and 4. . ■

Proof of Fact 3.

Extend u to $\mathbf{R}^{2^X}$ by setting $u(\emptyset) = 0$, and set $\lambda_\emptyset = 0$ as well. Define $\Theta : \mathbf{R}^\mathcal{A} \times \{0\} \rightarrow \mathbf{R}^\mathcal{A} \times \{0\}$, $u \longmapsto \Theta(u) = l$ by $l(A) = u(X) - u(A^c)$.

By construction, $l(A) = \sum_{S \in 2^X : S \cap X \neq \emptyset} \lambda_S - \sum_{S \in 2^X : S \cap A^c \neq \emptyset} \lambda_S = \sum_{S \in 2^X : S \subseteq A} \lambda_S$.

Let $\Phi : \mathbf{R}^\mathcal{A} \times \{0\} \rightarrow \mathbf{R}^\mathcal{A} \times \{0\}$ denote the linear ("Möbius") operator that maps λ to l as just described. Shafer (1976) has shown the following.

Proposition 2 (Shafer) $\Phi : \mathbf{R}^\mathcal{A} \times \{0\} \rightarrow \mathbf{R}^\mathcal{A} \times \{0\}$ *is a bijective linear map. Its inverse* Φ^{-1} *is given by*

$$\Phi^{-1}(l)(A) = \sum_{S \in 2^X : S \subseteq A} (-1)^{\#(A \setminus S)} l(S) \text{ for } A \in 2^X.$$

Since Θ is invertible (with inverse $\Theta^{-1} = \Theta$; this follows from noting that $l(l(A)) = u(A)$), one can write $\Psi = \Theta^{-1} \circ \Phi$, and thus also $\Psi^{-1} = \Phi^{-1} \circ \Theta$. Specifically, in view of proposition 2, one obtains $\Psi^{-1}(u)(A) = \sum_{S \in 2^X : S \subseteq A} (-1)^{\#(A \setminus S)}(u(X) - u(S^c))$, for $A \in \mathcal{A}$. Since $\sum_{S \in 2^X : S \subseteq A} (-1)^{\#(A \setminus S)} = 0$ (cf. Shafer (1976, p.47)) , one can simplify to $\Psi^{-1}(u)(A) = \sum_{S \in 2^X : S \subseteq A} (-1)^{\#(A \setminus S)+1} u(S^c).$ ■

Proof of Theorem 4.

In view of corollary 1 and the decomposition $\Psi^{-1} = \Phi^{-1} \circ \Theta$ as in the proof of fact 3, the theorem is an immediate consequence of the following two lemmas. Let $\mathcal{A}^{**} = 2^X \backslash \{X\}$.

Lemma 2 *i) u is monotone (on $\mathcal{A}$) if and only if the associated loss-function $l = \Theta(u)$ is monotone on $\mathcal{A}^{**}$.*

ii) u is uniformly submodular (on $\mathcal{A}$) if and only if the associated loss-function $l = \Theta(u)$ is uniformly supermodular on $\mathcal{A}^{**}$, i.e. if, for any finite collection $\{A_k\}_{k \in K}$ in $\mathcal{A}^{**}$ such that $\bigcup_{k \in K} A_k \subset X$, $l\left(\bigcup_{k \in K} A_k\right) \geq \sum_{J: \emptyset \neq J \subseteq K} (-1)^{\#J+1} l\left(\bigcap_{k \in J} A_k\right)$.

Proof of lemma. Part i) is obvious from the definition of l.

For part ii), we shall prove the "only-if" part; the "if" part follows from reading the proof given backwards. Thus, consider a finite collection $\{A_k\}_{k \in K}$ in $\mathcal{A}^{**}$ such that $\bigcup_{k \in K} A_k \subset X$; it needs to be shown that $l\left(\bigcup_{k \in K} A_k\right) \geq \sum_{J: \emptyset \neq J \subseteq K} (-1)^{\#J+1} l\left(\bigcap_{k \in J} A_k\right)$. This follows from the equivalence of the three inequalities just below, as well as the equivalence of " $\bigcup_{k \in K} A_k \subset X$ "and " $\bigcap_{k \in K} A_k^c \neq \emptyset$".

$$l\left(\bigcup_{k \in K} A_k\right) \geq \sum_{J: \emptyset \neq J \subseteq K} (-1)^{\#J+1} l\left(\bigcap_{k \in J} A_k\right)$$

is by the definition of l and computation of complements equivalent to

$$u(X) - u\left(\bigcap_{k \in K} A_k^c\right) \geq \sum_{J: \emptyset \neq J \subseteq K} (-1)^{\#J+1}\left(u(X) - u\left(\bigcup_{k \in J} A_k^c\right)\right),$$

which, due to the equality $\sum_{J: \emptyset \neq J \subseteq K} (-1)^{\#J+1} = 1$, is equivalent to

$$u\left(\bigcap_{k \in K} A_k^c\right) \leq \sum_{J: \emptyset \neq J \subseteq K} (-1)^{\#J+1} u\left(\bigcup_{k \in J} A_k^c\right). \qquad \square$$

Lemma 3 *l is monotone and uniformly supermodular on $\mathcal{A}^{**}$ if and only if $\Phi^{-1}(l)(A) \geq 0$ for all $A \in \mathcal{A}^{**}$.*

Proof of lemma. For $x \in X$, let $\mathcal{A}_x = 2^{X \setminus \{x\}}$. It follows from Shafer's (1976) theorem 2.1 (see also Chateauneuf-Jaffray (1989), Corollary 1) for any $x \in X$ that l is uniformly supermodular on $\mathcal{A}_x$ if and only if $\Phi^{-1}(l)(A) \geq 0$ for all $A \in \mathcal{A}_x$. The claim follows from noting that $\bigcup_{x \in X} \mathcal{A}_x = \mathcal{A}^{**}$. $\square$ $\blacksquare$

Proof of Corollary 2.

By standard arguments , the vNM axioms ensure the existence of a vNM representation $p \succeq q \Leftrightarrow \sum_{S \in \mathcal{A}} p_S u(S) \geq \sum_{S \in \mathcal{A}} q_S u(S)$. Monotonicity and opportunity risk–aversion of $\succeq$ are then easily verified to be equivalent to monotonicity and uniform submodularity of the utility–function u. $\blacksquare$

Proof of Theorem 6.

Necessity is straightforward.

For sufficiency, assume w.l.o.g. that $u(X) = 0$, and hence that $u(S) \leq 0$ for all $S \in \mathcal{A}$. Let $u_m : \mathcal{A} \to \mathbf{R}$ defined by $u_m(S) = -(-u(S))^m$. Let λ^m denote the associated coefficient vector $\lambda^m = \Psi^{-1}(u_m)$; note that $-u_m(A) = \sum_{S \subseteq A^c} \lambda_S^m$.

We want to show that, for some sufficiently large m, u_m is an EIU-function. By fact 2,iii) , it thus needs to be shown that for some sufficiently large $m : \lambda_S^m \geq 0$ for all $S \neq X$. Since X is finite, it suffices to show that for all $S \in \mathcal{A}^*$, $\lambda_S^m \geq 0$ for all sufficiently large m. Take $S \in \mathcal{A}^*$.

Case 1: For some $x \in S : u(S^c \cup \{x\}) = u(S^c)$. Then $u^m(S^c \cup \{x\}) = u^m(S^c)$; since, moreover, u^m satisfies OSM because u does, it follows that $\lambda_S^m = 0$ by lemma 4 below.

Case 2: For all $x \in S : u(S^c \cup \{x\}) > u(S^c)$. Then $\lambda_S^m > 0$ for sufficiently large m by lemma 5 below. $\square$

Lemma 4 *For any w satisfying OSM, $A \in \mathcal{A}$, and $x \in X$:*

$w(A \cup \{x\}) = w(A)$ implies $\lambda_T = 0$ for all T such that $x \in T \subseteq A^c$.

Proof of lemma.   Fix $x \in X$. The claim is shown by downward induction on the size of A. It holds vacuously for $A = X$.

Thus, assume the claim to be true for all B such that $B \supset A$, and assume also

$$w(A \cup \{x\}) = w(A). \tag{8}$$

By OSM, $w(B \cup \{x\}) = w(B)$ for all $B \supset A$;

hence by induction assumption, $\lambda_T^m = 0$ for all T such that $x \in T \subset A^c : \lambda_T = 0$.

Since equation 8 implies by the definition of λ: $\sum\limits_{T:x \in T \subseteq A^c} \lambda_T = w(A \cup \{x\}) - w(A) = 0$,

it follows that $\lambda_{A^c} = 0$. $\square$

Lemma 5 *i)* $m \to \infty \lim\sup \frac{|\lambda_S^m|}{-u_m(S^c)} < \infty$.

ii) In case 2: $m \to \infty \lim \frac{\lambda_S^m}{-u_m(S^c)} = 1$.

Proof of lemma.   From $\lambda_S^m = -u_m(S^c) - \sum\limits_{T \subset S} \lambda_T^m$, one obtains

$$\frac{\lambda_S^m}{-u_m(S^c)} = 1 - \sum_{T \subset S} \frac{\lambda_T^m}{-u_m(T^c)} \frac{-u_m(T^c)}{-u_m(S^c)}. \tag{9}$$

Let $\eta_S = m \to \infty \lim\sup \frac{|\lambda_S^m|}{-u_m(S^c)}$. Due to the monotonicity of u, $\mid \frac{-u_m(T^c)}{-u_m(S^c)} \mid \leq 1$ in equation (9). One thus obtains from equation (9), $\eta_S \leq 1 + \sum\limits_{T \subset S} \eta_T$ for $S \in \mathcal{A}$.

Part i) follows from this by induction on the size of S.

Part ii) follows from the validity of i) for $T \subset S$, and the fact that satisfaction of the condition "for all $x \in S : u(S^c \cup \{x\}) > u(S^c)$" implies $\frac{u_m(T^c)}{u_m(S^c)} \to 0$ as $m \to \infty$, for all T such that $T \subset S$. $\square$ ∎

<u>Proof of Proposition 1.</u>

i) follows from theorem 1 and fact 2,iii) .

ii) By the unique determinacy of vNM-utility functions up to positive affine transformations, $u = \sum\limits_{T \in \mathcal{A}} \lambda_T v_T$ and $u' = \sum\limits_{T \in \mathcal{A}} \lambda_T' v_T$ must be positive affine transformations of each

other. The claim is thus a straightforward implication of the behavior of Ψ^{-1} under affine transformations described by the following fact which itself follows at once from the linearity Ψ^{-1} and the definition of Ψ; note that changes in the "level" of u affect only the coefficient on v_X representing global indifference.

Fact 5 *For any $u \in \mathbf{R}^{\mathcal{A}}$, $c > 0$ and $d \in \mathbf{R}$:*

$$\Psi^{-1}\left(cu + d\mathbf{1}\right)(T) = \begin{cases} c\Psi^{-1}(u)(T) & \text{if } T \neq X \\ c\Psi^{-1}(u)(T) + d & \text{if } T = X. \end{cases}$$

iii) is straightforward from ii). ∎

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