# Bernoulli Without Bayes:

# A Theory of Utility-Sophisticated Preferences under Ambiguity

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**Abstract**

A decision-maker is utility-sophisticated if he ranks acts according to their expected utility whenever such comparisons are meaningful. Assuming that probabilistic beliefs are minimally precise, we characterize utility sophisticated preferences and show that preferences over general multi-valued acts are determined by consequence utilities and betting preferences via a non-linear expectation operator, the "Bernoulli integral". We provide a fully behavioral criterion for "revealed utility-sophistication" and, for preferences satisfying this criterion, propose a definition of revealed probabilistic beliefs that overcomes the limitations of existing definitions.

**Keywords:** expected utility, ambiguity, probabilistic sophistication, revealed probabilistic beliefs, Bernoulli integral, deliberative sure-thing principle.

**JEL Classification:** D81

## 1. INTRODUCTION

Expected utility theory rests on two pillars of consequentialist rationality: the existence of a unique subjective probability measure underlying all decisions (the "Bayes principle"), and the consistent use of cardinal utilities in the valuation of acts (the "Bernoulli principle"). Both of these assumptions have been challenged. On the one hand, as illustrated by the Ellsberg paradox, it is frequently not possible to represent a decision-maker's betting preferences in terms of a well-defined subjective probability measure; in such cases, decision-makers are said to view certain events as "ambiguous". On the other hand, faced with given probabilities, utilities and probabilities may not combine linearly, as in the Allais paradox and related phenomena; such decision-makers are sometimes referred to as exhibiting "probabilistic risk-attitudes".

While a descriptively fully adequate model of decision-making will need to incorporate both phenomena, for modelling purposes it is often desirable to zoom in on one of these two departures from the expected utility paradigm. To this purpose, Machina-Schmeidler (1992) have introduced the notion of probabilistic sophistication which precludes all phenomena of ambiguity but does not constrain the nature of probabilistic risk-attitudes. In the present paper, we introduce a complementary notion of utility sophistication which precludes all phenomena deriving from probabilistic risk-attitudes but does not constrain the decision-maker's attitudes towards ambiguity.

Besides this analytical motivation, the notion of utility sophistication has also an important normative purpose. Since the underlying Bernoulli principle is conceptually clearly distinct from the Bayes principle, one can formulate a normative position on which departures from the Bayes principle are rationally justifiable while departures from the Bernoulli principle are not. Such a position seems in fact quite attractive. On the one hand, it can be doubted that the precision of beliefs required by the Bayes principle is normatively mandated; indeed, it can even be argued that in situations of partial or complete ignorance rational decision making *cannot* rationally be based well-defined subjective probabilities (see the classical literature on complete ignorance surveyed in Luce-Raiffa (1957) as well the subsequent contributions of Jaffray (1989) and Nehring (1991,1992,2000)). On the other hand, while it is frequently argued that departures from the Bernoulli principle are rationally permissible, we are not aware of an argument that would rationally *mandate* departures from the Bernoulli principle, i.e. in particular, departures from expected utility in the presence of probabilities. Moreover, the typical examples of departures from the Bernoulli principle such as the Allais paradox can be interpreted as "real but not rational", by attributing them to cognitive distortions in the processing of probabilities as in Kahneman-Tversky's (1979) prospect theory, or as "rational but merely apparent", by appealing to the existence of implicit psychological payoffs (cf. for example Broome (1991) and Caplin-Leahy (2001)). The present paper articulates this normative "Bernoulli without Bayes" position axiomatically but will not defend its premises further.

Broadly speaking, an agent is "utility-sophisticated" if he compares acts in terms of their expected utility *whenever possible*. Since the existence of such comparisons depends on the agents' beliefs, utility sophistication must in the first instance be defined *relative* to a specified set of probabilistic beliefs. We shall thus model probabilistic beliefs as a distinct entity, specifically as partial orderings over events (likelihood relations) represented by a set of admissible probability measures $\Pi$. The specified likelihood relation can be viewed as describing all or merely a fragment of the decision maker's probabilistic beliefs; for example, the likelihood relation may reflect the existence of a continuous randomization device as implicit in the Anscombe-Aumann (1963) approach to decision making under uncertainty. This example exhibits the key richness property of "minimal precision" that requires that every event contains subevents with arbitrary precise conditional probability, and that is essential to much of our analysis.

Given a cardinal utility function $u$ (obtained from risk preferences), an agent is *utility-sophisticated* with respect to the set of admissible priors $\Pi$ if the agent prefers any act $f$ over another act $g$ whenever the expected utility of $f$ weakly exceeds that of $g$ for all admissible priors. The first and foremost task of the paper is to provide axiomatic foundations. The crucial axiom of "Trade-off Consistency"[1] captures the insight that expected-utility evaluations are possible not only for risky acts (acts with a precise induced distribution over outcomes), but also for the conditional risks embedded in a potentially ambiguous act.[2]

Trade-off Consistency (and utility sophistication more broadly) becomes especially powerful when there is an adequate supply of conditionally risky events, i.e. when the likelihood is minimally precise. The main result of the paper, Theorem 1, derives utility sophistication from Trade-off Con-

[1]Its basic idea can be described as follows. Consider two acts $f$ and $g$ whose outcomes differ on only two *equally likely* events $A$ and $B$ such that $f$ yields a better outcome in event $A$ and $g$ yields a better outcome in event $B$. Suppose also that we already have obtained a ranking of utility differences from the decision-maker's preferences over unambiguous acts. Tradeoff Consistency requires that if the utility gain from the outcome of $f$ over that of $g$ in the event $A$ exceeds the utility gain from $g$ over $f$ in the event $B$, then $f$ is preferred to $g$. More precisely, Tradeoff Consistency requires that preferences over acts can be rationalized consistently in this manner by an appropriate ranking of utility differences .

[2]In section 3, we justify the latter by reference to a "deliberative" version of Savage's Sure-Thing Principle.

sistency in the presence of weak regularity assumptions on preferences, assuming both a rich set of consequences and a minimally precise likelihood relation. Theorem 1 also shows that under these assumptions, utility-sophisticated preferences over general multi-valued acts are uniquely determined by event attitudes captured preferences over bets and cardinal utilities of outcomes via a non-linear expectation operator called the "Bernoulli integral". This powerful reduction property parallels that of the Choquet Expected Utility model. Since these models are based on fundamentally different conceptual starting points – Bernoullian rationality on the one hand and the rank-dependence heuristic on the other– it is not surprising that the two models generally lead to different rankings of multi-valued acts as we show in Proposition 6 of Section 5.

The reduction property serves a function somewhat analogous to that of probabilistically sophisticated preferences. Probabilistically sophisticated preferences are uniquely determined by preferences over lotteries and the decision-maker's subjective probability measure. As a result, any specific model of probabilistically sophisticated preferences can be characterized as a model of preferences over lotteries. Analogously, here the reduction property derived from the Bernoulli integral makes it possible characterize any specific model of utility sophisticated preferences in terms of betting preferences, a much more focused task. This is illustrated in the context of adaptations of the Minimum Expected Utility model due to Gilboa-Schmeidler (1989) and the variational preference model of Maccheroni et al. (2006).

In the first instance, utility sophistication is defined relative to a given likelihood relation. Is it possible to eliminate reference to beliefs as an independent, non-behavioral construct, and define utility sophistication in purely behavioral terms? A priori, in view of the belief relativity of the direct definition, this may seem difficult, if not impossible. However, assuming minimal precision, the belief relativity loses much of its sting once the relevant likelihood relation is at least minimally precise: the utility sophistication of a preference ordering is then largely independent of which particular minimally precise likelihood relation is employed to ascertain it (Proposition 2). As argued in more detail in section 6, this enables us to define a decision maker as "revealed utility sophisticated" if he is utility sophisticated relative to *some* minimally precise likelihood relation; an operational characterization of such decision makers is provided by Proposition 8.

As an important collateral benefit, refining and modifying earlier and related work (Ghirardato et al. (2004), Nehring (1996, 2001)), this criterion to suggests a natural behavioral definition of "revealed probabilistic beliefs" for revealed utility-sophisticated preferences. We argue that this renders the proposed definition immune to the interpretative ambiguities that have characterized these earlier contributions.

**Comparison to the existing literature.**—

While the existing literature has not yet attempted to define a distinct notion of utility sophistication, as we show in an **accompanying note (Nehring 2007b),** many models of decision making under ambiguity in the Anscombe-Aumann framework give rise to revealed utility-sophisticated preferences, starting from the seminal contributions of Schmeidler (1989) and Gilboa-Schmeidler (1989).

Other contributions, especially Ghirardato-Marinacci (2002), Ghirardato et al. (2003), Ghirardato et al. (2004), and Siniscalchi (2006) adopt an utility-sophisticated viewpoint by assuming in the *interpretation* of their definitions and axioms that all departures from expected utility can be attributed to ambiguity. However, as argued by Epstein-Zhang (2001) and discussed further in sections 6 and 7, such an interpretational assumption may be arbitrary or inappropriate.

**Organization of the paper.**—

The remainder of the paper is organized as follows. In section 2, we introduce likelihood relations and their multi-prior representation, as well as basic assumptions on preferences maintained throughout. We then define the notion of utility sophistication and characterize it axiomatically (section 3).

By not assuming Savage's axiom P4, our main representation theorem allows for betting preferences over events to depend on the "stakes" of the bets involved. This generality is important since in the presence of ambiguity, P4 cannot be taken to be a requirement of rationality; indeed, there is a live interest in stake-dependent preference models (see e.g. Epstein-Le Breton (1993), Klibanoff et al. (2005), Maccheroni et al. (2006)). Nonetheless, the Stake Invariance axiom P4 is a common and natural behavioral assumption. In section 4, we characterize the restrictions that stake-invariant betting preferences must satisfy to be consistent with utility sophistication and show that P4 is necessary and sufficient to achieve a separation of consequence and event attitudes as determinants of overall preferences.

In section 5, we study utility sophistication in various preference models in the literature, specifically the MEU, variational preference, alpha-MEU and CEU models. Section 6 quantifies out the likelihood relation to arrive at a definition of "revealed utility-sophisticated preferences" and provides an operational characterization. The definition naturally suggests an accompanying definition of "revealed probabilistic beliefs" as discussed in section 7.

The appendix contains a short statement of the multi-prior representation of minimally precise likelihood orderings obtained in Nehring (2007) and collects all proofs.

## 2. BACKGROUND

### 2.1. Coherent and Minimally Precise Likelihood Relations

Since utility sophistication is to be defined relative to a specified set of probabilistic beliefs, we shall model a decision maker in terms of two entities, a preference relation $\succsim$ over Savage acts and a comparative likelihood relation $\trianglerighteq$ describing some or all of his probabilistic beliefs. Formally, a likelihood relation is a partial ordering $\trianglerighteq$ on an algebra of events $\Sigma$ in a state space $\Omega$, with the instance $A \trianglerighteq B$ to be read as "$A$ is at least as likely as $B$" for the DM. We shall denote the symmetric component of $\trianglerighteq$ ("is as likely as") by $\equiv$. For now, we shall treat the likelihood relation as an independent primitive.

The likelihood relation $\trianglerighteq$ may be interpreted in different ways. First, $\trianglerighteq$ may stand for the sum total $\trianglerighteq^{jud}$ of all likelihood judgments the DM is prepared to make, where these judgments are understood as elicited separately from preferences. Second, $\trianglerighteq$ may stand for the sum total of the DM's probabilistic beliefs as revealed by (inferred from) his preferences $\trianglerighteq^{rev}$, where this revelation might be construed in various ways. Finally, $\trianglerighteq$ may represent pieces of probabilistic information in the possession of the agent $\trianglerighteq^{\inf}$. Such information is naturally viewed as a partial, "non-exhaustive" description (subrelation) of the his entire beliefs, whether these are construed as $\trianglerighteq^{jud}$ or $\trianglerighteq^{rev}$. While each of which has their advantages and limitations, for the purposes of this paper it turns out these interpretations can be used interchangeably as long as a minimum richness requirement of "minimal precision" is met that is central to the entire theory in any case. This is shown and further explained in section 3.2 right after the main result of the paper.[3]

At this point, we recommend the reader to adopt the interpretation (s)he feels most comfortable

[3] An incomplete rather than exhaustive interpretation of the likelihood relation is viable if the the likelihood relation $\trianglerighteq$ as rationally *constraining* preferences, but not as *determining* them. By contrast, imprecise probabilistic beliefs (modeled in different ways) (co-)determine preferences in contributions such as Jaffray (1989), Nehring (1992,2000), Ghibault et al. (2004,2006), Olszweski (2007), and Ahn (2005), and thus rely on an exhaustive interpretation of the imprecision. For more on the important distinction between between non-exhaustive and exhaustive interpretations of incompleteness/imprecision, see Walley (1991, sections 2.10.3 and 9.7.4).

with. For many, this could be the interpretation that fits best within the existing literature on decision-making under uncertainty, namely the interpretation of $\trianglerighteq$ as probabilistic information in the form of a continuous random device as embedded in the standard Anscombe-Aumann framework and described formally in Example 1 below.[4]

A *prior* $\pi$ is a finitely additive, non-negative set-function on $\Sigma$ such that $\pi(\Omega) = 1$. Given a likelihood relation $\trianglerighteq$, let $\Pi$ denote its set of *admissible priors* defined by

$$\pi \in \Pi \text{ if and only if, for all } A, B \in \Sigma,\ A \trianglerighteq B \text{ implies } \pi(A) \geq \pi(B).$$

For any $\trianglerighteq$, $\Pi$ is a closed convex set in the product (or weak$^*$) topology.

We will say that the likelihood relation $\trianglerighteq$ is *coherent* if, conversely, unanimity among admissibility priors implies comparative likelihood, that is, if

$$A \trianglerighteq B \text{ if and only if } \pi(A) \geq \pi(B) \text{ for all } \pi \in \Pi.$$

An axiomatization of coherent likelihood relations is given in Nehring (2007), which justifies their labeling as "coherent", that is: as closed under inferences from the logic of probability (plus some technical continuity requirements). In the following, a coherent likelihood relation will be referred to as a *likelihood ordering.* The axiomatization is briefly summarized in Appendix A.1.

This axiomatization relies on the following richness condition called "equidivisibility" or "minimal precision" that it is also central to the results of the present paper. The likelihood relation $\trianglerighteq$ is *equidivisible* if, all events $A \in \Sigma$, there exists an event $B \in \Sigma$ such that $B \subseteq A$ and $B \equiv A \backslash B$. On $\sigma-$algebras, equidivisibility of the likelihood relation is equivalent to convex-rangedness of the multi-prior representation in the following sense. The set of priors $\Pi$ is *convex-ranged* if, for any event $A \in \Sigma$ and any $\alpha \in (0, 1)$, there exists an event $B \in \Sigma$, $B \subseteq A$ such that $\pi(B) = \alpha\pi(A)$ for all $\pi \in \Pi$. If $\Sigma$ is merely an algebra, equidivisibility may not quite ensure convex-rangedness; in the following, we will refer to likelihood relations having a convex-ranged multi-prior representation as *minimally precise*[5].

Minimally precise likelihood relations are characterized by a rich set of risky and conditionally risky events. Say that $B \in \Sigma$ is *risky given* $A$ if, for some $\alpha \in [0, 1]$, $\pi(B) = \alpha\pi(A)$ for all $\pi \in \Pi$. Let $\Lambda_A$ denote the family of events that are risky given $A$; clearly, $\Lambda_A$ is closed under finite disjoint

[4]For further discussion of the general approach, see the companion paper Nehring (2007) where the framework of "decision-making in the context of probabilistic beliefs" has been introduced.

[5]Convex-rangedness on algebras arises naturally in the Anscombe-Aumann context $\trianglerighteq_{AA}$ defined in Example 1 below.

union and complementation, but not necessarily under intersection. An event $A$ is *null* if $A \equiv \varnothing$, or, equivalently, if $\pi(A) = 0$ for all $\pi \in \Pi$. For any non-null $A$ and any $\pi \in \Pi$, let $\overline{\pi}(./A)$ denote the restriction of $\pi(./A)$ to $\Lambda_A$, with $\overline{\pi}(B/A)$ denoting the precise conditional probability of $B$ given $A$. We will say that $B$ is *risky* if it is "risky given $\Omega$", and write $\Lambda$ for $\Lambda_\Omega$, as well as $\overline{\pi}$ for $\overline{\pi}(./\Omega)$.

In the following, when it is necessary to consider asymmetric likelihood comparisons, rather than using simply the asymmetric component $\rhd$ of $\unrhd$, it is often more appropriate to use the "uniformly more likely" relation $\rhd\rhd$, where $A \rhd\rhd B$ if $\min_{\pi\in\Pi}(\pi(A) - \pi(B)) > 0$. In general, $\rhd\rhd$ is a proper subrelation of $\rhd$. For further discussion and a characterization of $\rhd\rhd$ in terms of $\unrhd$ for minimally precise contexts, see Nehring (2007).

As a matter of further notation, let $\pi^-(A) = \min_{\pi\in\Pi}\pi(A)$ and $\pi^+(A) = \max_{\pi\in\Pi}\pi(A)$ denote the lower and upper probabilities of event $A$, respectively. Also, the indicator function associated with event $A$ will be denoted by $1_A$. Finally, the summation signs $+$ and $\sum$ will denote the disjoint union of sets.

**Example 1 (Continuous Randomization Device).** The following restates the widely used Anscombe-Aumann (1963) framework in terms of a likelihood relation. Consider a product space $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1$ is a space of "generic states" , and $\Omega_2$ a space of "random states" with associated algebras $\Sigma_1$ and $\Sigma_2$, respectively. Let $\eta$ denote a convex-ranged[6], finitely additive prior over random events $\Sigma_2$ . The "continuity" and stochastic independence of the random device are captured by the following coherent likelihood relation $\unrhd^{\text{rand}}$ defined on the product algebra $\Sigma = \Sigma_1 \times \Sigma_2$; note that any $A \in \Sigma_1 \times \Sigma_2$ can be written as $A = \sum_i S_i \times T_i$, where the $\{S_i\}$ form a finite partition of $\Omega_1$ :[7]

$$\sum_i S_i \times T_i \unrhd^{\text{rand}} \sum_i S_i \times T_i' \text{ if and only if } \eta(T_i) \geq \eta(T_i') \text{ for all } i.$$

Clearly, there exists a unique set of priors $\Pi^{\text{rand}}$ representing $\unrhd^{\text{rand}}$; indeed, $\Pi^{\text{rand}}$ is simply the set of all product-measures $\pi_1 \times \eta$ where $\pi_1$ ranges over all finitely additive measures on $\Sigma_1$. Note that the convex-rangedness of $\Pi^{\text{rand}}$ is a straightforward consequence of the convex-rangedness of $\eta$.

In general, a decision-maker will have additional probabilistic beliefs captured by a likelihood relation $\unrhd$ that strictly contains the likelihood ordering $\unrhd^{\text{rand}}$, reflecting for instance information about the composition of an Ellsberg urn; the relation $\unrhd$ evidently inherits the equidivisibility of

[6]That is, $\{\eta\}$ is convex-ranged in $\Sigma_2$.

[7]The relation $\unrhd^{\text{rand}}$ is easily characterized axiomatically; see Nehring (2007).

$\trianglerighteq^{\text{rand}}$. It is also clear that this setting allows to capture arbitrary probabilistic beliefs about the generic state-space $\Omega_1$. This observation illustrates the more general point that while Equidivisibility imposes substantial restrictions on the probabilistic beliefs when imposed on a given state space, it is entirely unrestrictive when imposed on a suitably enlarged state space.

**Example 2 (Limited Imprecision).** A particular way to formalize the intuitive notion of a limited extent of overall ambiguity is to assume that $\Pi$ is the convex hull of a finite set $\Pi'$ of non-atomic, countably additive priors on a $\sigma$-algebra $\Sigma$. Due to Lyapunov's (1940) celebrated convexity theorem, $\Pi$ is convex-ranged. The priors $\pi \in \Pi'$ can be interpreted as a finite set of hypotheses a decision-maker deems reasonable without being willing to assign precise probabilities to them. Finitely generated sets of priors occur naturally, for example, when an individual bases his beliefs on the views of a finite set of experts who have precise probabilistic beliefs $\trianglerighteq_i$ but disagree with each other. The decision maker may naturally want to respect all instances of expert agreement; these are represented by the unanimity relation $\trianglerighteq_I = \cap_{i \in I} \trianglerighteq_i$ which is evidently finitely generated.

## 2.2 Maintained Assumptions on Preferences

Consider now a DM described by a preference ordering over acts $\succsim$ and a coherent likelihood relation $\trianglerighteq$. Let $X$ be a set of *consequences*. An *act* is a finite-valued mapping from states to consequences, $f : \Omega \to X$, that is measurable with respect to the algebra of events $\Sigma$; the set of all acts is denoted by $\mathcal{F}$. A *preference ordering* $\succsim$ is a weak order (complete and transitive relation) on $\mathcal{F}$. An act is *risky* if it is measurable with respect to the system of risky events $\Lambda$; the set of all risky acts is denoted by $\mathcal{F}^{risk}$. The restriction of $\succsim$ to $\mathcal{F}^{risk}$ represents the decision maker's risk preferences.

We shall write $[x_1 \text{ on } A_1; x_2 \text{ on } A_2; ...]$ for the act with consequence $x_i$ in event $A_i$; constant acts $[x \text{ on } \Omega]$ are typically referred to by their constant consequence $x$. To prepare the ground for the subsequent analysis, we now introduce the basic substantive and regularity assumptions that will be maintained throughout.

The likelihood ordering constrains most directly preferences over bets. A *bet* is a pair of acts with the same two outcomes, i.e. a pair of the form $([x \text{ on } A; y \text{ on } A^c], [x \text{ on } B; y \text{ on } B^c])$. Fundamental is the following rationality requirement on the relation between betting preferences and probabilistic beliefs.

**Axiom 1 (Compatibility)** *For all $A, B \in \Sigma$ and $x, y \in X$ :*

$$i)\ [x \text{ on } A; y \text{ on } A^c] \ \succsim \ [x \text{ on } B; y \text{ on } B^c] \text{ if } A \trianglerighteq B \text{ and } x \succsim y, \text{ and}$$
$$ii)\ [x \text{ on } A; y \text{ on } A^c] \ \succ \ [x \text{ on } B; y \text{ on } B^c] \text{ if } A \rhd\!\rhd B \text{ and } x \succ y.$$

Throughout, preferences will be assumed to be eventwise monotone in the following weak version of Savage's axiom P3.

**Axiom 2 (Eventwise Monotonicity)** *For all acts $f \in \mathcal{F}$, consequences $x, y \in X$ and events $A \in \Sigma$ : $[x \text{ on } A; f(\omega) \text{ elsewhere}] \succsim [y \text{ on } A; f(\omega) \text{ elsewhere}]$ whenever $x \succsim y$.*

The following condition ensures that the set of consequences is sufficiently rich.

**Axiom 3 (Solvability)** *For any $x, y \in X$ and $T \in \Lambda$, there exists $z \in X$ such that $z \sim [x, T; y, T^c]$.*

For expositional simplicity, especially in the stake-dependent case, we shall assume throughout that consequences are bounded in utility.

**Axiom 4 (Boundedness)** *There exist $x^-, x^+ \in X$ such that, for all $x \in X$, $x^- \precsim x \precsim x^+$.*

To obtain a real-valued representation, some Archimedean property is usually assumed. The following is sufficiently strong to help deliver the main result, Theorem 1, below. Note that it is defined relative to the likelihood ordering and presumes its equidivisibility. Substantively, as confirmed by the upcoming representation result, Proposition 1, it asserts that if acts are changed on events of sufficiently small upper probability, strict preference does not change.

**Axiom 5 (Archimedean)** *For any $x, y \in X$ such that $x \succsim y$ and any acts $f = [x \text{ on } A,\ y \text{ on } B;$ $f$ otherwise$]$ and $g$ such that $f \succ g$ (resp. $f \prec g$) and such that $A$ is risky given $A + B$, there exists an event $C$ that is risky given $A + B$ such that $C \lhd\!\lhd A$ and $f' = [x \text{ on } C,\ y \text{ on } (A + B) \backslash C;\ f$ otherwise$] \succ g$ (resp. $f' \prec g$).*

Since axioms 3 through 5 will usually show up together in the following results, it is convenient to refer to a preference ordering satisfying these three axioms as **regular**. Let $\mathcal{Z}$ denote the set of finite-valued, $\Sigma$-measurable functions $Z : \Omega \to [0, 1]$. Using the above axioms, we will now establish a basic representation theorem that ensures the existence of a utility function $u$ mapping $X$ onto the unit interval together with an evaluation functional $I$: $\mathcal{Z} \to [0, 1]$ such that $f \succsim g$ if and only if $I(u \circ f) \geq I(u \circ g)$, for all $f, g \in \mathcal{F}$.

$I$ is *normalized* if $I(c1_\Omega) = c$ for all $c \in [0,1]$ and $I(1_T) = \overline{\pi}(T)$ for all $T \in \Lambda$. Note that for normalized $I$, $u$ is calibrated in terms of probabilities, i.e. satisfies $u(z) = \overline{\pi}(T)$ whenever $z \sim [x^+, T; x^-, T^c]$.[8] $I$ is *monotone* if $I(Y) \geq I(Z)$ whenever $Y \geq Z$ (pointwise); $I$ is *compatible with* $\trianglerighteq$ if $I(1_A) \geq I(1_B)$ whenever $A \trianglerighteq B$ and $I(1_A) > I(1_B)$ whenever $A \rhd\rhd B$; $I$ is *event-continuous* if, for any $x, y \in X$ , $Z \in \mathcal{Z}$, $E \in \Sigma$, $A \in \Lambda_E$ with $A \subseteq E$ and any increasing sequence $\{A_n\} \subseteq \Lambda_E$ of events contained in $A$ such that $\overline{\pi}\left(A_n/E\right)$ converges to $\overline{\pi}\left(A/E\right)$, $I(x1_{A_n} + y1_{E \setminus A_n} + Z1_{E^c})$ converges to $I(x1_A + y1_{E \setminus A} + Z1_{E^c})$.

**Proposition 1** *Let $\trianglerighteq$ be a minimally precise likelihood ordering. The following two statements are equivalent:*

*i) the preference ordering $\succsim$ is compatible with $\trianglerighteq$, eventwise monotone and regular (Archimedean, solvable, and bounded).*

*ii) there exist an onto utility function $u : X \to [0,1]$ and a functional $I : \mathcal{Z} \to [0,1]$ that is monotone, event-continuous and compatible with $\trianglerighteq$ such that*

$$f \succsim g \text{ if and only if } I(u \circ f) \geq I(u \circ g), \text{ for all } f, g \in \mathcal{F}.$$

*There is a unique pair $(u, I)$ satisfying ii) such that $I$ is normalized.*

In the sequel, preferences over bets will play a special role. We shall frequently but not always assume that preferences over bets depend only on the events involved, not on the stakes. This is captured by Savage's axiom P4.

**Axiom 6 (Stake Independence, P4)** *For all $x, y, x', y' \in X$ such that $x \succ y$ and $x' \succ y'$ and all $A, B \in \Sigma$ :*

$$[x \text{ on } A; y \text{ on } A^c] \succsim [x \text{ on } B; y \text{ on } B^c] \text{ iff } [x' \text{ on } A; y' \text{ on } A^c] \succsim [x' \text{ on } B; y' \text{ on } B^c].$$

We will frequently use the notation $A \succsim_{bet} B$ for the preference $[x^+ \text{ on } A; x^- \text{ on } A^c] \succsim [x^+ \text{ on } B; x^- \text{ on } B^c]$. This notation is primarily motivated by the stake-invariant case in which the relation $\succsim_{bet}$ completely summarizes the DM's beliefs and ambiguity attitudes. If preferences are utility-sophisticated, this turns out to be the case even when betting preferences are stake-dependent.

Compatibility of betting preferences with a given likelihood ordering ensures a ranking of bets on risky events $T$ according to their precise probability $\overline{\pi}(T)$. Under the assumptions of Proposition 1, there exists a unique set-function $\rho : \Sigma \to [0,1]$ representing $\succsim_{bet}$ that is additive on risky events

[8] To see this, $z \sim [x^+, T; x^-, T^c]$ implies $I(u(z)1_\Omega) = I(1_T)$. Thus by the two normalization conditions $u(z) = I(u(z)1_\Omega) = I(1_T) = \overline{\pi}(T)$.

and has $\rho(\Omega) = 1$; $\rho$ assigns to each event the probability $\overline{\pi}(T)$ of any risky event to which it is indifferent. If $I$ is normalized, clearly $\rho(A) = I(1_A)$. The properties on $I$ introduced above translate naturally into properties of $\rho$. In particular, $\rho$ is *compatible with* $\trianglerighteq$ if $\rho(A) \geq \rho(B)$ whenever $A \trianglerighteq B$ and $\rho(A) > \rho(B)$ whenever $A \rhd\!\rhd B$ ; finally, $\rho$ is *event-continuous* if, for any disjoint $B, E \in \Sigma$, any $A \in \Lambda_E$ with $A \subseteq E$ and any increasing (respectively decreasing) sequence $\{A_n\}$ of events contained in (resp. containing) $A$ such that $\overline{\pi}(A_n/E)$ converges to $\overline{\pi}(A/E)$, $\rho(A_n + B)$ converges to $\rho(A + B)$.

## 3. UTILITY SOPHISTICATED PREFERENCES

The fundamental goal of this paper is to provide axiomatic foundations for the intuitive notion of a decision maker who departs from expected-utility principles *only* for reasons of ambiguity. This idea can be formulated transparently with reference to exogenously specified likelihood ordering $\trianglerighteq$ in terms of the following property of utility sophistication.

**Definition 1 (Utility Sophistication)** *The preference relation $\succsim$ is utility-sophisticated with respect to the likelihood ordering $\trianglerighteq$ with multi-prior representation $\Pi$ if there exists $u : X \to \mathbf{R}$ such that $f \succsim g$ (resp. $f \succ g$) whenever $E_\pi u \circ f \geq E_\pi u \circ g$ (resp. $E_\pi u \circ f > E_\pi u \circ g$) for all $\pi \in \Pi$.*

To motivate the key axiom underlying utility sophistication, consider first the ranking of risky acts for which utility sophistication entails EU maximization with respect to the probability measure $\overline{\pi}$. Specifically, consider choices among risky acts $f$ and $g$ with two outcomes, each with subjective probability one half, and assume that $f = [x \text{ on } A; y \text{ on } A^c]$ and $g = [x' \text{ on } A; y' \text{ on } A^c]$ with $x \succ x'$ , $y' \succ y$ and $A \equiv A^c$. According to a classical ("Bernoullian") interpretation of expected utility theory, a DM should choose $f$ over $g$ exactly if he assesses the utility gain from $x$ over $x'$ to exceed the utility loss of obtaining $y$ rather than $y'$. Therefore, the preference of $f$ over $g$ by a DM committed to this principle reveals a greater utility gain from $x$ over $x'$ than from $y'$ over $y$. Thus, if the DM chooses $f = [x \text{ on } A; y \text{ on } A^c]$ over $g = [x' \text{ on } A; y' \text{ on } A^c]$, consistency requires that he also choose the act $[x \text{ on } E; y \text{ on } E^c]$ over $[x' \text{ on } E; y' \text{ on } E^c]$, where $E$ is any other event that is equally likely to its complement, $E \equiv E^c$.[9]

[9] This consistency requirement is in fact axiom 2 of Ramsey's (1931) seminal contribution. Conditions requiring consistency of trade-offs across choices have been used elsewhere in the axiomatizations of SEU and CEU theory; see in particular Wakker (1989).

The following "Trade-off Consistency" axiom generalizes this consistency requirement to choices of the form $f = [x \text{ on } A; y \text{ on } B; f(\omega) \text{ elsewhere}]$ versus $g = [x' \text{ on } A; y' \text{ on } B; f(\omega) \text{ elsewhere}]$ whenever the events $A$ and $B$ are judged equally likely ($A \equiv B$), whether or not $A$ and $B$ are risky themselves. Since the relative probabilities of the events $A$ and $B$ are judged to be equal, the comparison between the acts $f$ and $g$ boils down to a comparison of the respective utility gains. *On the Bernoulli principle, this comparison is decisive.* For the comparison of the acts $f$ and $g$, the payoffs in states outside $A+B$ and his (possibly imprecise) assessment of the likelihood of the union $A+B$ are simply irrelevant.

The Bernoulli principle motivates the following rationality axiom according to which the DM's preferences must be consistently rationalizable in terms of utility differences in the manner just described.

**Axiom 7 (Trade-off Consistency)** *For all $x, y, x', y' \in X$ such that $x \succsim x'$, acts $f, g \in \mathcal{F}$ and events $A$ disjoint from $B$ and $A'$ disjoint from $B'$ such that $A \equiv B \rhd\rhd \emptyset$ and $A' \equiv B'$ :*

*if $[x \text{ on } A; y \text{ on } B; f(\omega) \text{ elsewhere}] \succsim [x' \text{ on } A; y' \text{ on } B; f(\omega) \text{ elsewhere}]$,*

*then $[x \text{ on } A'; y \text{ on } B'; g(\omega) \text{ elsewhere}] \succsim [x' \text{ on } A'; y' \text{ on } B'; g(\omega) \text{ elsewhere}]$.*[10]

For Trade-off Consistency to allow for ambiguity, the restriction to equally likely rather than merely indifferent events $A$ and $B$ respectively $A'$ and $B'$ is crucial. Indeed, if one replaced this clause by a weaker one requiring these events to be indifferent as bets ($A \sim_{bet} B$ and $A' \sim_{bet} B'$), the resulting stronger axiom would force betting preferences to satisfy the additivity condition

$$A \sim_{bet} B \text{ if and only if } A + C \sim_{bet} B + C, \text{ for any } A, B, \text{ and } C, \tag{1}$$

and thereby impose SEU. By contrast, Trade-off Consistency implies (1) only if $A$ and $B$ are equally likely. But since then $A+C$ and $B+C$ are equally likely as well by coherence, their indifference as bets follows from Compatibility; in particular, even though both $A+C$ and $B+C$ may be ambiguous (non-risky), there is no room for (rationally justifiable) Ellsberg-style complementary effects.

[10] Note the restriction to events $A$ and $B$ of strictly positive lower probability; it ensures that the premise "$[x \text{ on } A; y \text{ on } B; f(\omega) \text{ elsewhere}] \succsim [x' \text{ on } A; y' \text{ on } B; f(\omega) \text{ elsewhere}]$" implies that the utility advantage of $x$ over $x'$ is not smaller than the advantage of $y'$ over $y$ if the latter is positive.

Note also that, for equidivisible contexts $\trianglerighteq$, Trade-off Consistency entails Eventwise Monotonicity. Indeed, for equivisible contexts, Eventwise Monotonicity is simply Tradeoff Consistency restricted to cases in which $x = y$, $x' = y'$ , $A + B = \Omega$ and $A \equiv B$, with $A' + B'$ ranging over all events $E \in \Sigma$. It is for the purpose of enabling this implication that we have not required the condition $y' \succsim y$ in the definition of Tradeoff Consistency.

**Example 3. (A 3-Color Urn)** Let $\{\texttt{Red},\texttt{Blue},\texttt{Green}\}$ denote a partition of $\Omega$, such as the draw of a red/blue/green ball. The DM is told that the urn contains an equal number of blue and green balls (i.e. $\texttt{Blue}\equiv\texttt{Green}$); in addition, half of the balls have a black dot, whence $\texttt{Dot}\equiv \texttt{Dot}^c$. The decision-maker may be ambiguity averse in the sense that $\rho(\texttt{Red}) + \rho(\texttt{Red}^c) < 1$. For specificity, let consequences be given in terms of money, with utility linear, implying risk-neutrality; as demonstrated by Rabin (2000), linearity is an excellent approximation of smooth, convex utility for small and even moderated monetary stakes. (The risk-neutrality assumption is immaterial but may add common sense). At issue is the evaluation the act $h$ given as

$$[1 \text{ on } \texttt{Red}, 2 \text{ on } \texttt{Blue}, 0 \text{ on } \texttt{Green}]$$

compared to the constant act $1_\Omega$, i.e. receipt of \$1 for sure. Conditional on the event "Not Red", the act $h$ yields payoffs of \$2 and \$0 with equal probability, hence an expected payoff of \$1. Since $h$ yields a payoff of \$1 in the event "Red" as well, $h$'s unconditional expected payoff is \$1, without room for ambiguity, even though the events Red, Blue, and Green may all be ambiguous themselves. Thus, the Bernoulli Principle can be applied to evaluate $h$, yielding a certainty equivalent of \$1 irrespective of the DM's ambiguity attitudes. And, indeed, since $1_\Omega \sim [2 \text{ on } \texttt{Dot}, 0 \text{ on } \texttt{Dot}]$ by assumption, Trade-off Consistency implies

$$h \sim 1_\Omega.$$

The argument just given exemplifies the following more general "Deliberative Sure-Thing Principle". Think of ambiguity as reflecting the difficulty of deciding which subjective probability judgment $\pi \in \Pi$ to adopt among those that are consistent with the given probabilistic information[11]. Suppose that You as the DM would prefer act $f$ to act $g$ no matter which subjective probability You might end up settling on. Then You should prefer $f$ to $g$ prior to having decided on Your preferred $\pi$, and indeed even if you are unable to make up your mind at all. By this reasoning, in Example 3, you should be indifferent between $h$ and $1_\Omega$. Utility Sophistication as in Definition 1 combines probabilistic risk-neutrality (EU maximization over risky acts) with the Deliberative Sure-Thing Principle, the "utility" with the "sophistication".[12]

[11] This hypothetical judgment $\pi$ viewed as a "deliberative contingency" is the counterpart to the "uncertain contingency (conditioning event) in Savage's original sure-thing principle

[12] It should be interesting to study models that maintain the Deliberative Sure-Thing Principle but abandon Probabilistic Risk-Neutrality in order to accommodate empirical departures from EU maximization under risk such as the Allais paradox. However, these might be difficult to axiomatize since it is not clear what would take the role of the Trade-Off Consistency since that axiom involves probabilistic risk-neutrality in an essential way.

Trade-off Consistency becomes particularly powerful if the underlying likelihood ordering is minimally precise. For in this case not only does it entail utility sophistication, utility sophistication itself becomes particularly powerful, as it implies that a DM's multi-act preferences are determined by his preferences over risky acts together with his preferences over bets. Mathematically, this is the consequence of the existence of a non-linear expectation operator that reflects the DM's ambiguity attitudes.

The key to deriving this built-in expectation operator is the mixture-space structure induced by minimally precise likelihood orderings as introduced in Nehring (2007). With each random variable $Z \in \mathcal{Z}$, one can associate an equivalence class $[Z]$ of events $A \in \Sigma$ as follows. Let $A \in [Z]$ if there exists a partition $\{E_i\}$ of $\Omega$ such that $Z = \sum z_i 1_{E_i}$, and such that, for all $i \in I$ and $\pi \in \Pi : \pi(A \cap E_i) = z_i \pi(E_i)$. Note that $[Z]$ is non-empty by the convex-rangedness of $\Pi$. Moreover, it is easily seen that for any two $A, B \in [Z] : \pi(A) = \pi(B)$ for all $\pi \in \Pi$, and thus $A \equiv B$. Hence by Compatibility also $A \sim_{bet} B$. One therefore arrives at a well-defined ordering of random variables $\widehat{\succsim_{bet}}$ on $\mathcal{Z}$ by setting

$$Y \widehat{\succsim_{bet}} Z \text{ if } A \succsim_{bet} B, \text{ for any } A \in [Y] \text{ and } B \in [Z].$$

Let $\widehat{\rho}$ denote the associated unique extension of $\rho$ to $\mathcal{Z}$ given by

$$\widehat{\rho}(Z) = \rho(A) \text{ for any } A \in [Z]. \tag{2}$$

Again, by the construction of the mixture-space, this is well-defined, and one has

$$Y \widehat{\succsim_{bet}} Z \text{ if and only if } \widehat{\rho}(Y) \geq \widehat{\rho}(Z).$$

Clearly, by Compatibility, $\widehat{\rho}$ is a monotone, normalized evaluation functional on $\mathcal{Z}$; $\widehat{\rho}$ is sup-norm continuous if and only if $\rho$ is event-continuous (Lemma 3 in the Appendix). We shall call $\widehat{\rho}(Z)$ the "Bernoulli integral" of $Z$.[13]

We are now in a position to state the main result of the paper.

**Theorem 1** *Let $\trianglerighteq$ be an minimally precise likelihood ordering. The following three statements are equivalent:*

1. *The preference ordering $\succsim$ is regular, trade-off consistent and compatible with $\trianglerighteq$.*

[13] The operator $\widehat{\rho}$ merits the appellation "integral" on conceptual rather than narrowly mathematical grounds. In contrast to the Choquet integral, it does not come with an explicit formula for $I$.

*2. The preference ordering $\succsim$ is Archimedean and utility-sophisticated with respect to $\trianglerighteq$, for some onto function $u : X \rightarrow [0,1]$.*

*3. There exists an onto function $u : X \rightarrow [0,1]$ and an event-continuous set-function $\rho$ compatible with $\trianglerighteq$ with associated Bernoulli integral $\widehat{\rho}$ defined by (2) such that, for all $f, g \in \mathcal{F}$:*

$$f \succsim g \text{ iff } \widehat{\rho}(u \circ f) \geq \widehat{\rho}(u \circ g).$$

Theorem 1 achieves two things. First of all, it delivers an axiomatic foundation for utility-sophisticated preferences when the underlying likelihood ordering is minimally precise and when the set of consequences is rich; both of these assumptions are essential for the result. Second, it shows that multi-act preferences are uniquely determined by cardinal consequence utilities (captured by preferences over risky acts and represented by $u$), event attitudes (captured by betting preferences and represented by $\rho$) and Bernoullian rationality (captured by Trade-off Consistency and formally represented by the Bernoulli integral, i.e. the mapping $\rho \mapsto \widehat{\rho}$). [14]

**Utility Sophistication De-Relativized.—**

In Definition 1, utility sophistication has been defined relative to a given likelihood relation $\trianglerighteq$. As already suggested, alternative interpretations of the likelihood relation are possible. Conceptually, to ascertain whether the DM "in fact" satisfies the Bernoulli principle, it appears most satisfactory to do this by checking for utility sophistication (respectively the axioms characterizing it) relative to exhaustively specified beliefs, $\trianglerighteq^{jud}$ or $\trianglerighteq^{rev}$ . The former has the obvious problem that it may not be observable, and the latter simply has not been defined yet in the literature[15], and any proposed definition is likely to be controversial.

There is thus obvious appeal in trying to sidestep these issues by reference to an operationally accessible subset of the DM's full beliefs $\trianglerighteq^{\text{inf}}$ that represents (parts of) the probabilistic information at his disposal such as the relation $\trianglerighteq^{\text{rand}}$ reflecting the existence of an independent random device.

[14] Remarkably, to achieve this unique determination, no assumption needs to be made as to how the decision maker takes account of and 'integrates' the ambiguity of the various consequences of a multi-valued acts. This contrasts with the Choquet integral, which also achieves a unique determination, but on the basis of a rank-dependence principle rather than Bernoullian rationality. (The Cumulative Dominance axiom in Sarin-Wakker (1992) desribes especially transparently how the evaluation of ambiguous multi-valued acts is determined by betting and risk-preferences within the CEU model. Since the rank-dependence principle is very different from the Bernoulli principle, it is not surprising that the two may lead to different results, as we shall demonstrate further in section 5.4 below.

[15] For an attempt in this direction, see Nehring (2001).

But it seems that one has to pay a big price for this: first, replacing an exhaustive relation $\trianglerighteq^{jud}$ by a subrelation $\trianglerighteq^{\inf}$ may make the DM utility sophisticated where before he was not; for a trivial but telling example, consider the case of the "vacuous relation" $\trianglerighteq^{\inf}=\trianglerighteq_{\varnothing}$ given by $A \trianglerighteq_{\varnothing} B$ iff $A \supseteq B$. However, it turns out that this indeterminacy vanishes as soon as $\trianglerighteq^{\inf}$ is minimally precise.

Suppose, for example, that preferences have been verified to satisfy Trade-off Consistency /utility sophistication with respect to $\trianglerighteq^{\mathrm{rand}}$, but we are really interested in satisfaction of these conditions relative to the DM's beliefs $\trianglerighteq^{jud}$ which, however, are unobservable. First, as already stated, since $\trianglerighteq^{\inf}=\trianglerighteq^{ran}$ describes *some* of his beliefs, $\trianglerighteq^{jud}$ must contain $\trianglerighteq^{\inf}$ . Second, preferences arguably must reflect these beliefs at least in the minimal sense of satisfying Compatibility with respect to them as defined in section 2.2. Rather remarkably, it follows that preferences must in fact be trade-off consistent / utility sophisticated with respect to judged beliefs, as shown by the following result.

**Proposition 2** *Suppose that the preference ordering $\succsim$ is utility-sophisticated and regular with respect to the minimally precise $\trianglerighteq$ . Then $\succsim$ is utility-sophisticated and regular with respect to $\trianglerighteq' \supseteq \trianglerighteq$ if and only if it is compatible with $\trianglerighteq'$ .*

Appealing to Theorem 1, the key step of the proof is to verify that $\widehat{\rho'} = \widehat{\rho}$, which suffices by Theorem 1.[16]

## 4. SEPARATING EVENT ATTITUDES FROM CONSEQUENCE ATTITUDES

As an important dimension of its generality, Theorem 1 does not assume Stake Invariance (Savage's axiom P4). While in the context of probabilistically sophisticated preferences P4 is typically viewed as a rationality axiom expressing consistency of revealed likelihood judgements, this interpretation is no longer viable under ambiguity, since in this more general context betting preferences may reflect

[16]One may wonder whether, more generally, the expectation operator $\widehat{\rho}$ is invariant to the reference-relation $\trianglerighteq$, for arbitrary minimally precise $\trianglerighteq$ with which $\succsim$ is compatible. In that case, whether or not an agent was utility sophisticated would be entirely independent of which minimally precise $\trianglerighteq$ is used as a reference belief to ascertain the utility sophistication of preferences. Indeed, we conjecture that such invariance holds with significant generality. Note in particular that, in view of Proposition 2, the following property is sufficient for this invariance:

If $\succsim$ is compatible with the miminimally precise likelihood orderings $\trianglerighteq$ and $\trianglerighteq'$, then their intersection is also minimally precise.

Since a general result that delivers this property (under suitable regularity conditions) would in effect involve a substantial generalization of Lyapunov's (1940) convexity theorem, a demonstration of this conjecture is not attempted here.

not just likelihood judgments but also ambiguity attitudes.[17]

In the absence of P4, betting preferences over extreme stakes represented by $\succsim_{bet}$ fail to describe preferences over bets with intermediate stakes. However, if preferences are utility-sophisticated, $\succsim_{bet}$ determines the Bernoulli integral $\widehat{\rho}$, and thus all preferences (in particular: all betting preferences) are determined once consequence / risk attitudes captured by $u$ are given. As a result, preferences over bets with intermediate stakes will partly depend on these attitudes. By modus tollens, Stake Invariance P4 is therefore necessary for a clean separation of consequence and pure event attitudes (beliefs and ambiguity attitudes). In this section, we will show that Stake Invariance is also sufficient for such a separation and characterize the restrictions on stake-invariant betting preferences imposed by utility sophistication.

P4 turns out to be equivalent to the following invariance properties of betting preferences.

**Axiom 8 (Union Invariance)** *For any $T \in \Lambda$ and any $A, B \in \Sigma$ disjoint from $T$: $A \succsim_{bet} B$ if and only if $A + T \succsim_{bet} B + T$.*

**Axiom 9 (Splitting Invariance)** *For any $A, B \in \Sigma$ and any partitions of $A$ and $B$ into equally likely subevents $\{A_1, ..., A_n\}$ and $\{B_1, ..., B_n\}$, with $A_i \equiv A_j$ and $B_i \equiv B_j$ for all $i, j \leq n$, $A \succsim_{bet} B$ if and only if $A_1 \succsim_{bet} B_1$.*

The two invariance axioms are intuitive and have intrinsic appeal even in the absence of utility sophistication. In view of their appeal, it is not surprising that both conditions have some *incognito* precedents in the literature. On the one hand, Epstein-Zhang (2001) effectively build Union Invariance into their very definition of an event $T$ as "revealed unambiguous".[18] Splitting Invariance as well is not entirely new, as it can be reformulated as a restriction on betting preferences over independent events. Say that events $A$ and $B$ are *independent* if $\pi(B/A) = \pi(B/A^c)$ for all $\pi \in \Pi$. If preferences are compatible with the minimally precise likelihood ordering $\trianglerighteq$ as maintained, then it can be shown easily that they satisfy Splitting Invariance if and only if

$$\rho(A \cap B) = \rho(A)\rho(B) \tag{3}$$

for all $A \in \Sigma$ and $B \in \Lambda$ such that $A$ and $B$ are independent. In defining "product capacities" for

[17] Note that restricted to bets on unambiguous events, P4 still obtains as an implication of weak compatibility with the underlying belief context, and does not need to be assumed independently.

[18] That is to say, Epstein-Zhang's definition of revealed unambiguous events is such that Union Invariance (applied to revealed unambiguous events instead of $\Lambda$) holds by definition.

independent events, authors such as Ghirardato (1997) and Hendon et al. (1996) have appealed to generalizations of (3) that allow both events $A$ and $B$ to be ambiguous.

Alternatively, P4 can be characterized in terms of "constant-linearity" of the evaluation functional $I$. An evaluation functional $I$ (in particular $\widehat{\rho}$) is *constant-additive* if $I(Y + c1_\Omega) = I(Y) + c$; $I$ is *positively homogeneous* if $I(\alpha Y) = \alpha I(Y)$ for any $\alpha \in [0,1]$; $I$ is *constant-linear* if it is constant-additive and positively homogeneous. Again, this condition is of independent interest and has been studied in the literature, especially by Ghirardato et al. (2004) . Note that, for two-outcome acts $[x$ on $A; y$ on $A^c]$ with $x \succsim y$, a constant-linear Bernoulli integral has the following simple "biseparable" representation (Ghirardato-Marinacci (2001))

$$\widehat{\rho}\,(u \circ f) = u(x)\rho(A) + u(y)\,(1 - \rho(A))\,. \tag{4}$$

Constant Linearity can be viewed as a cardinal stake-invariance property of multi-act preferences. The following result derives this property from the weaker and arguably more primitive ordinal P4 property, assuming utility sophistication.

**Theorem 2** *Suppose $\succsim$ is regular, trade-off consistent and compatible with the minimally precise likelihood ordering $\trianglerighteq$. Then the following three statements are equivalent.*

1. *$\succsim$ satisfies P4.*
2. *$I$ is constant-linear.*
3. *$\succsim$ satisfies Union and Splitting Invariance.*

In the Appendix, we demonstrate the implications $2) \Longrightarrow 1)$, $1) \Longrightarrow 3)$, and $3) \Longrightarrow 2)$. The first implication $2) \Longrightarrow 1)$ is valid for any constant-linear evaluation functional $I$, without reference to a minimally precise likelihood ordering. The second implication $1) \Longrightarrow 3)$ relates two different properties of betting preferences, making essential use of utility sophistication. Finally, the implication $3) \Longrightarrow 2)$ mirrors the invariance properties of betting preferences in corresponding properties of the Bernoulli integral $\widehat{\rho}$; utility sophistication closes the circle via the identity $I = \widehat{\rho}$.[19]

Theorem 2 entails the desired separation of event attitudes from consequence valuations, as formalized by the following result. Note that while Theorem 2 shows that utility sophistication imposes

[19]It may seem a bit surprising that utility sophistication entails non-trivial restrictions on betting preferences given stake-independence. To see how this is possible, note that while utility sophistication by iteslf does not restrict betting preferences *for given stakes* $x$ and $y$, it does constrain betting preferences across stakes, even in the absence of P4. The existence of such restrictions explains how the imposition of *further* restrictions on betting preferences across stakes such as P4 can entail restrictions on betting preferences for given stakes.

Union- and Splitting Invariance on stake-invariant betting preferences, the following Proposition 3 adds that these are in fact the *only* restrictions on betting preferences imposed by utility sophistication. In this result, $\succsim_{risk}$ represent given (EU maximizing) risk-preferences while $\succsim_{\mathcal{B}}$ represents given betting preferences; the two must agree on the set of bets on risky events. The result asserts that these are jointly consistent with utility sophistication if and only if $\succsim_{\mathcal{B}}$ satisfies Union- and Splitting-Invariance, and that in this case they determine the overall preference ordering uniquely.

**Proposition 3** *Let $\trianglerighteq$ be an minimally precise likelihood ordering. Let $\succsim_{risk}$ be a preference ordering on risky acts $\mathcal{F}^{risk}$ that is trade-off consistent, regular, and compatible with $\trianglerighteq$ restricted to $\Lambda$. Furthermore, let $\succsim_{\mathcal{B}}$ be a complete and transitive relation on $\Sigma$ that is Archimedean and compatible with $\trianglerighteq$ such that $(\succsim_{risk})_{bet}$ agrees with the restriction of $\succsim_{\mathcal{B}}$ to $\Lambda \times \Lambda$. Then the following two statements are equivalent:*

*1. $\succsim_{\mathcal{B}}$ satisfies Union and Splitting Invariance with respect to $\trianglerighteq$.*

*2. There exists a preference ordering $\succsim$ on all of $\mathcal{F}$ that is stake-invariant, Archimedean and tradeoff-consistent with respect to $\trianglerighteq$ and whose restrictions to $\mathcal{F}^{risk}$ and $\succsim_{bet}$ agree with $\succsim_{risk}$ and $\succsim_{\mathcal{B}}$, respectively.*

*The preference ordering specified in (2) is unique.*

## 5. APPLICATION TO SPECIFIC MODELS

### 5.1 Minimum Expected Utility

Theorems 1 and 2 reduce the task of developing more specific models of decision making under ambiguity to one of modelling betting preferences. In particular, Theorem 2 is just one step away from characterizing the classical Gilboa-Schmeidler (1989) model in the present framework. One simply needs to add an appropriate condition of ambiguity aversion. For present purposes, the following counterpart to their Uncertainty Aversion axiom (originally introduced in Schmeidler (1989)) for betting preferences suffices. This condition captures the intuition that the ambiguities of disjoint events can never reinforce each other, but that they can cancel each other out.

**Axiom 10 (Preference for Randomization over Bets)**

*For any $A, B \in \Sigma$ such that $A \succsim_{bet} B$ and $T \in \Lambda$ such that $T \cap D \equiv T^c \cap D$ for any $D \in \{A \backslash B, A + B, B \backslash A, (A + B)^c\}$ : $(T \cap A) + (T^c \cap B) \succsim_{bet} B$.*

Here the event $T$ is specified to have conditional probability $\frac{1}{2}$ irrespective of the joint realization of $A$ and $B$; thus the event $(T \cap A) + (T^c \cap B)$ can be viewed as describing a random bet that is paid out in the event $A$ or in the event $B$, contingent on the outcome of the "fair coin toss" $T$.

By comparison, Schmeidler's original definition which applies to general multi-valued acts it can be reformulated here as follows.

**Axiom 11 (Preference for Randomization over Multi-Valued Acts)**

*For any $f, g \in \mathcal{F}$ such that $f \succsim g$ and any $T \in \Lambda$ such that $T \cap D \equiv T^c \cap D$ for all $D$ contained in the algebra generated by $f$ and $g$ : $[f, T; g, T^c] \succsim g$.*

For utility-sophisticated preferences, the two axioms are equivalent.[20] Otherwise, the first is substantially weaker. The second is, in general, conceptually unsatisfactory as a definition of *ambiguity aversion* proper since it may easily be violated by probabilistically sophisticated decision makers that are not expected utility maximizers.

From Theorems 1 and 2, we obtain the following characterization of the classical Minimum Expected Utility (MEU) model given by the following representation:

$$f \succsim g \text{ if and only if } \min_{\pi \in \Psi} E_\pi (u \circ f) \geq \min_{\pi \in \Psi} E_\pi (u \circ g),$$

for appropriate utility functions $u$ and belief sets $\Psi$.

**Proposition 4** *Let $\trianglerighteq$ be an minimally precise likelihood ordering. Then the following two conditions are equivalent:*

1. *$\succsim$ is trade-off consistent and compatible with $\trianglerighteq$, satisfies preference for randomization over bets and is stake-invariant and regular.*

2. *$\succsim$ has a Minimum Expected Utility representation with $u(X) = [0, 1]$ and a (unique) closed convex set $\Psi \subseteq \Pi$.*

We provide only a sketch of the sufficiency part of the proof. By Theorems 1 and 2, $\succsim$ has a Bernoulli integral representation with $\widehat{\rho}$ constant-linear. By Preference for Randomization over Bets, $\widehat{\rho}$ is quasi-concave (see Lemma 5 in the Appendix). By the Archimedean assumption, it is

[20]Klibanoff (2001b), for example, is explicit about the utility-sophisticated character of Schmeidler's notion by saying that "one may interpret this requirement as saying that the individual likes smoothing expected utility across states" (p. 290).

continuous in the sup-norm. Hence by the argument of Gilboa-Schmeidler (1989), there exists a unique closed convex set $\Psi$ such that, for all $Z \in \mathcal{Z}$:

$$\widehat{\rho}(Z) = \min_{\pi \in \Psi} E_\pi Z.$$

Since $I = \widehat{\rho}$ Theorem 1, this yields the desired representation of preferences; the inclusion $\Psi \subseteq \Pi$ follows from the compatibility of betting preferences with $\trianglerighteq$ .

The main contribution of Proposition 4 is the transparency and conceptual appeal of the central axioms: Trade-off Consistency as an expression of Bernoullian rationality, Preference for Randomization over Bets as an expression of ambiguity aversion, and Stake Invariance. By contrast, the content of the substantive axioms in Gilboa and Schmeidler's characterization appears much less transparent and conceptually primitive. We hope that Proposition 4 makes intelligible at the level of the axiomatization why the MEU model plays the distinguished role in the literature that it does.

Proposition 4 naturally provokes the question whether more generally, there is a general 'method' to translate axiomatizations in the Anscombe-Aumann framework into the present one, with the hope of shedding new light on them. And, indeed, as we show in an accompanying note (Nehring 2007b), there is such a translation; in fact, it turns out that in that framework, utility sophistication is equivalent to Monotonicity and Independence over Roulette Lotteries.

### 5.2. Variational Preferences

Recently, Maccheroni et al. (2006) have proposed an interesting generalization of the MEU model to a representation of the form (adapted to the present model)

$$f \succsim g \text{ if and only if } \min_{\pi \in \Pi} E_\pi \left( u \circ f + c(\pi) \right) \geq \min_{\pi \in \Pi} E_\pi \left( u \circ g + c(\pi) \right), \tag{5}$$

where $c : \Pi \to [0, \infty]$ is convex and has $\min_{\pi \in \Pi} c(\pi) = 0$.[21] The MEU model corresponds to the limiting case of $c(\pi) = 0$ for $\pi \in \Psi \subseteq \Pi$, and $c(\pi) = \infty$ otherwise. In their model, the evaluation functional $I = \widehat{\rho}$ is quasi-concave and constant-additive rather than quasi-concave and constant-linear. Again, a translation of their axiomatization into the present framework is illuminating. Analogous to their weakening of Gilboa-Schmeidler's Certainty Independence axiom to a "Weak

[21] Formally, Maccheroni et al. (2006) assume consequences to belong to a convex vector space. As suggested by them, we will take this space in classical Anscombe-Aumann manner to be the of lotteries over final outcomes, modeled here via a an exogeneous random device. Note in particular that the utility-based mixture operation proposed in Ghirardato et al. (2003) cannot be applied here, since variational preferences do not satisfy P4.

Certainty Independence" axiom, the axiom to weakened here in Proposition 4 is Stake-Independence. Specifically, one can show that one can characterize variational preferences by substituting Union Invariance for Stake-Independence in Proposition 4, provided that utility is unbounded below or above, or that $\trianglerighteq$ is equal to $\trianglerighteq^{\text{rand}}$ .[22] In view of Theorem 2 which has demonstrated the basic character of the Union Invariance axiom, this confirms that the variational preference model is indeed a natural generalization of the MEU model at the axiomatic level.

It follows immediately from the functional form (5) that variational preferences satisfy the following one-sided form of stake-independence:

**Axiom 12 (Non-Decreasing Aversion to Ambiguity)** *For any consequences $x, x', y, y' \in X$ such that $x \succsim x' \succ y' \succsim y$ and any $A \in \Sigma$ and $T \in \Lambda$ :*

$$[x' \text{ on } T, y' \text{ on } T^c] \succsim [x' \text{ on } A, y' \text{ on } A^c] \text{ implies } [x \text{ on } T, y \text{ on } T^c] \succsim [x \text{ on } A, y \text{ on } A^c]. \quad (6)$$

In a nutshell, an increase in the stakes involved can only exacerbate, never dampen the Ellsberg paradox. Variational preferences, in contrast to MEU preferences, allow the following converse of (6) to fail, reflecting greater ambiguity aversion at greater stakes.

**Axiom 13 (Non-Increasing Aversion to Ambiguity)** *For any consequences $x, x', y, y' \in X$ such that $x \succsim x' \succ y' \succsim y$ and any $A \in \Sigma$ and $T \in \Lambda$ :*

$$[x \text{ on } T, y \text{ on } T^c] \succsim [x \text{ on } A, y \text{ on } A^c] \text{ implies } [x' \text{ on } T, y' \text{ on } T^c] \succsim [x' \text{ on } A, y' \text{ on } A^c]. \quad (7)$$

For example, a decision maker may well prefer a bet of \$1 on a draw from an urn with unknown composition (getting \$0 otherwise) over a bet of \$1 on an event with an objective probability of 40%, and exhibit at the same time the opposite preference once the stakes are raised to \$10,000 (versus \$0).

[22] The unboundedness assumption plays a significant role in Maccheroni et al. (2006), as does the assumption that $\trianglerighteq$ is equal to $\trianglerighteq^{\text{rand}}$ via the Anscombe-Aumann framework.

In the absence of these assumptions, Union Invariance is sufficient but not quite necessary. The characterizing condition is the following slightly weaker and a bit more complicated condition that requires Union Invariance conditional on some randomization:

For any $S \in \Lambda$, and any partition of $\Omega$ into equally likely events $\{T_1, ..., T_n\}$, with $T_i \equiv T_j$ such that $T_i \cap D \equiv T_j \cap D$ for all $i, j \leq n$ and any $D \in \{A, B, S\}$, and any $m \leq n$ :

$$A \succsim_{bet} B \text{ if and only if } \left(\left(\sum_{i \leq m} T_i\right) \cap A\right) + \left(\left(\sum_{m}^{n} T_i\right) \cap S\right) \succsim_{bet} \left(\left(\sum_{i \leq m} T_i\right) \cap B\right) + \left(\left(\sum_{m}^{n} T_i\right) \cap S\right).$$

In view of Proposition 4, this provides an easy way to test the variational model vis-a-vis the MEU model.

**Proposition 5** *A variational preference ordering as given by (5) is MEU if and only if satisfies Non-Increasing Aversion to Ambiguity.*

Conceptually, testing for Non-Increasing Aversion to Ambiguity is attractive due to its close connection to the Ellsberg style experiments. From a practical operational point of view, this test is meaningful (and captures an essential difference between the variational and MEU models) even if the other key assumptions, in particular utility sophistication, are false – and, of course, it is well-known that empirically decision-makers often violate the independence axiom under risk. Furthermore, Non-Increasing Aversion to Ambiguity relies only on ordinal information about consequences; by contrast, the experimental strategy sketched in Maccheroni et al. (2006) relies on the elicitation of cardinal utilities.[23]

### 5.3 Choquet Expected Utility

The Choquet Expected Utility (CEU) model ranks acts according to the Choquet integral of utilities $\int u \circ f d\nu$; it is the main alternative model in the literature in which preferences over general multi-valued acts are determined by preferences over risky acts and preferences over bets.[24] In contrast to utility sophistication, the CEU model is designed to also allow for departures from expected utility in the absence of ambiguity, accommodating for example the Allais (1953) paradox. If one writes the non-normalized capacity $\nu$ as $\phi \circ \rho$, such departures are reflected in the non-linearity of $\phi$.

When are Choquet preferences utility-sophisticated? The following Lemma implies that CEU preferences can be utility sophisticated only in very limited circumstances.

**Lemma 1** *Suppose that a CEU preference ordering $\succsim$ is utility-sophisticated relative to $\trianglerighteq$ with $\#u(X) \geq 3$, and that $B_1 \equiv B_2$ with $B_1 \cap B_2 = \varnothing$. Then, for any $A$ disjoint from $B_1 + B_2$,*

$$\nu(A + B_1 + B_2) = \nu(A) + \nu(B_1 + B_2).$$

[23] Application of (**??**) requires identification of unambiguous events, either directly or from behavior. But since variational preferences are not constant linear, it is not clear how cardinal utilities can be practially identified without identifying unambiguous events as well.

[24] This property comes out especially clearly in Sarin-Wakker's (1992) axiomatization based on a Cumulative Dominance axiom which explicitly constructs multi-act preferences from preferences over bets.

The Lemma immediately implies the following corollary.

**Proposition 6** *Suppose that a CEU preference ordering $\succsim$ is utility-sophisticated relative to the minimally precise likelihood ordering $\trianglerighteq$ with $\#u(X) \geq 3$; then $\succsim$ is in fact SEU.*

It may be illuminating to restate this result[25] as a comparison between the Choquet and the Bernoulli integrals.

**Proposition 7** *Suppose that the set-function $\rho : \Sigma \to [0,1]$ is compatible with the minimally precise likelihood ordering $\trianglerighteq$, additive on risky events and constant-linear. Then the Choquet and the Bernoulli integrals agree, that is: $\int Z d\rho = \widehat{\rho}(Z)$ for all $Z \in \mathcal{Z}$, if and only if $\rho$ is additive.*

But the Lemma evidently has strong implications without any richness assumption on $\trianglerighteq$. In particular, it precludes a DM's Ellsbergian ambiguity aversion with respect to the events $B_1 + B_2$ and $(B_1 + B_2)^c$, as stated by the following Corollary.

**Corollary 1** *Suppose that, in addition to the assumptions of Lemma 1, there exists an event $T$ such that $T \equiv T^c$. Then not $T \succ_{bet} B_1 + B_2$ and $T^c \succ_{bet} B_1 + B_2$.*

To understand better why a CEU maximizer fails to be utility sophisticated, consider the following variation of Example 3, in which the DM has CEU preferences with linear weighting function $\phi = id$. This DM is probabilistically risk-neutral but not utility sophisticated. Note that the example relies on the existence of only a single pair of events that are equally likely to each other yet ambiguous.

**Example 3 (ctd.).**

Assume that betting preferences are based on lower probabilities (i.e. $\rho(E) = \min_{\pi \in \Psi} \pi(E)$ for all $E$, for some $\Psi \subseteq \Pi$), and that the event Red is treated as ambiguous, i.e. that $\max_{\pi \in \Psi} \pi(\texttt{Red}) > \min_{\pi \in \Psi} \pi(\texttt{Red})$. One easily computes[26] that

$$\int u \circ h d\nu = 1 - \frac{1}{2}\left(\max_{\pi \in \Psi} \pi(\texttt{Red}) - \min_{\pi \in \Psi} \pi(\texttt{Red})\right), \tag{8}$$

[25] At first glance, Proposition 6 might seem to conflict with a well-known result of Schmeidler (1989) who showed that the CEU and MEU models coincide for convex capacities. Proposition 6 thus implies that capacities that are compatible with an equidivisible context cannot be convex; this can also be easily verified directly.

[26] One computes that $\int u \circ f d\nu = 2\rho(\texttt{Blue}) + 1[\rho(\texttt{Red}+\texttt{Blue}) - \rho(\texttt{Blue})] = \rho(\texttt{Red}+\texttt{Blue}) + \rho(\texttt{Blue})$. Plugging in $\rho(\texttt{Red}+\texttt{Blue}) = \frac{1}{2}(1 + \min_{\pi \in \Xi} \pi(\texttt{Red}))$ and $\rho(\texttt{Blue}) = \frac{1}{2} \min_{\pi \in \Xi} \pi(\texttt{Red}^c)$, one obtains (8).

and therefore

$$h \prec 1_{\Omega}. \tag{9}$$

Thus, in violation of the Deliberative Sure-Thing Principle, the ambiguity of the individual outcomes leads the ambiguity-averse CEU decision maker to treat the act $h$ as if its valuation (certainty equivalent) was ambiguous, and thus deserved an 'ambiguity discount' relative to its precise expected utility. This is especially clear in the extreme case of $\min_{\pi \in \Psi} \pi(\texttt{Red}) = 0$ and $\max_{\pi \in \Psi} \pi(\texttt{Red}) = 1$. In this case, the certainty equivalent of $h$ is $\frac{1}{2}$ rather than 1 since the CEU maximizer evaluates the act $h$ based lower probabilities $\rho(\texttt{Blue}) = 0$ and $\rho(\texttt{Red+Blue}) = \frac{1}{2}$. Note that these lower probabilities are the same that he might have had, had he only been given the less informative piece of information that at most half of the balls are blue, and at most half of them green. While the ambiguity discount of $\frac{1}{2}$ would make perfect sense with respect to this weaker information, it does not make sense with respect to the original information. Thus, the CEU based evaluation of the act $h$ can be viewed as ignoring some of the relevant probabilistic information (the equilikelihood of the events Blue and Green); this explains its violation of Deliberative Sophistication.[27]

## 6. A FULLY BEHAVIORAL CRITERION OF BERNOULLIAN RATIONALITY

Utility sophistication as an expression of Bernoullian rationality has been defined relative to separately specified "beliefs". Importantly, Proposition 2 showed that in order to ascertain whether preferences are utility sophisticated relative to the DM's 'true' beliefs $\trianglerighteq^{jud}$, it suffices to check this on any sufficiently rich subset of beliefs $\trianglerighteq \subseteq \trianglerighteq^{jud}$; this greatly facilitates the operationalization of the concept, since $\trianglerighteq$ may represent observable probabilistic information that is available to the DM.

Nonetheless, in line with the revealed preference tradition that prevails in much of economics and decision theory, it is natural to ask whether anything meaningful can be said about a DM's Bernoullian rationality in the absence of *any* non-behavioral information whatsoever. What we are looking for is an appropriately defined behavioral criterion of Bernoullian rationality. Qua definition, it cannot be valid or not, only 'useful' and 'intuitively sound'. Our main standard for intuitive soundness will be the plausibility (generic likelihood, as it were) that a positive or negative

[27]Note that the Choquet integral is forced to ignore this additional information because it evaluates multi-valued acts directly in terms of the capacities of the event-partition generated by the act, while the Bernoulli integral depends exploits a richer set of capacities in its evaluation. The limitation of the Choquet integral thus does not stem from its rank-dependent character per se.

ascription of utility sophistication would be borne out by the (non-behavioral) knowledge of the DM's beliefs.

Consider a situation in which, by investigation of a DM's preferences $\succsim$, it is found that there exists a minimally precise likelihood ordering $\trianglerighteq$ relative to which preferences are utility-sophisticated. Note that the *existence* of such a likelihood ordering is itself a logically well-defined property of preferences only. The ordering $\trianglerighteq$ might have been 'found' in different ways: by accident, by construction from preferences, or from conjectural derived from (partial) knowledge of the DM's situation, as in the AA scenario. Even the latter is consistent with a fully behavioral viewpoint, as long as it is not invested with any authority (on its own) to entitle inferences regarding the DM's beliefs.

We would submit that, in this situation, it is compelling to explain the observed preferences by attributing to the DM $\trianglerighteq$ as his beliefs (at least) as well as Bernoullian rationality. Moreover, by Theorem 1), this 'hypothesis' fully explains preferences *without gap.* Moreover, it is hard to see how the very rich and specific structure of preferences could have come about by accident. Moreover, the attribution of Bernoullian rationality is *robust* with respect to the inference of specific beliefs, since, in view of Proposition 2, utility sophistication with respect to $\trianglerighteq$ implies utility sophistication with respect to any attributable superrelation $\trianglerighteq'$ . Thus we propose the following definition.

**Definition 2 (Revealed Utility Sophistication)** *The preference ordering $\succsim$ is* **revealed utility-sophisticated** *if it is utility-sophisticated relative to some minimally precise likelihood ordering $\trianglerighteq$.*

Two remarks on the scope of definition 2 are in order. First, minimal precision serves here as a sufficient condition for verifying utility sophistication behaviorally. It has been adopted due to the validity of Theorem 1 and Proposition 2 for such relations. Weaker sufficient conditions may be defensible. Indeed, from a more applied viewpoint which asks whether a given information about the DM's preferences is probabilistic evidence indicating his Bernoullian rationality, minimal precision appears to be far stronger than necessary.

**(Example 3, ctd.)** Return, for example, to draw from a three-color urn in Example 3 represented by the partition {`Red`,`Blue`,`Green`}.. Suppose that preferences are found to be utility-sophisticated relative to the likelihood relation $\trianglerighteq_{BG}$ generated by the single judgment `Blue`$\equiv$`Green`, but that preferences over acts measurable with respect to the partition {`Red`,`Not-Red`} depart from SEU. [28] Utility sophistication with respect to $\trianglerighteq_{BG}$ is equivalent the condition that, given any $x_{\mathtt{R}}$,

$$(x_{\mathtt{R}}, x_{\mathtt{B}}, x_{\mathtt{G}}) \succsim (x_{\mathtt{R}}, y_{\mathtt{B}}, y_{\mathtt{G}}) \text{ if and only if } \frac{1}{2}u(x_{\mathtt{B}}) + \frac{1}{2}u(x_{\mathtt{G}}) \geq \frac{1}{2}u(y_{\mathtt{B}}) + \frac{1}{2}u(y_{\mathtt{G}}).$$

[28]**reflect ambiguity aversion** in that, in its biseparable representation (4), $\nu\,(\mathtt{Red}) + \nu\,(\mathtt{Red}) < 1$.

This feature of the DM's preferences constitutes strong evidence for the DM's Bernoullian rationality, as it is naturally attributed to a Bernoullian-rational response to a belief that `Blue` is as likely as `Green`, together with ambiguity about the events `Red` and `Not-Red`, but would appear hard to explain otherwise.[29] Note in particular that if the choice is between Bernoullian rationality and rank-dependence, which appears to be the only genuine alternative available in the literature that accommodates ambiguity, it follows from Lemma 1 that the DM *must* be Bernoulli rational, since there is no room to explain the described preference pattern within the CEU model.

The example illustrates that Bernoullian rationality has powerful implications already with very sparse beliefs in small finite state spaces; their more detailed analysis would clearly be a worthwhile project for future research. In particular, it may be of interest to worthwhile to determine under what conditions and in what sense the implications of minimal precision hold approximately in finite state spaces.

Secondly, Definition 2 states a condition that positively verifies Bernoullian rationality; clearly, its negation cannot be taken as a falsification of Bernoullian rationality, since it may be due to a lack of minimal precision of beliefs instead. After all, any monotone preference relation is utility-sophisticated if beliefs are vacuous. We have not attempted here to provide a falsification criterion since, in order to motivate such a criterion convincingly, it seems necessary to rely on a behaviorally general notion of revealed beliefs which raises substantial conceptual and mathematical difficulties of its own.[30] Consider, for example, probabilistically sophisticated preferences that are not SEU. While a strong case can be made to attributing precise probabilistic beliefs in this situation[31], there is room for doubt since it is frequently possible to argue that the DM may instead be ambiguity averse but Bernoulli rational.[32]

To make Definition 2 systematically applicable, an operational criterion of its satisfaction is desirable. We will now provide such a criterion for the special case of stake-invariant preferences. This assumption is helpful since it can be shown to imply the existence of a *unique* maximal likelihood

In contrast to (4) proper, we are allowing the spate space to be arbitrary, in particular: finite, hence cannot assume the representing capacity to be normalized.

[29]Of course, it is very easy to formally construct preference relations $\succsim$ that are compatible with some likelihood ordering $\trianglerighteq$ strictly containing $\trianglerighteq_{BG}$ such that $\succsim$ is compatible but not utility-sophisticated with respect to $\trianglerighteq$ . The relevant issue is whether such pairs $(\succsim, \trianglerighteq)$ are 'plausible', 'likely', 'natural'.

[30]An attempt is made in Nehring (2001).

[31]Cf. Epstein (1999) and Epstein-Zhang (2001).

[32]This is essentially the line taken in Ghirardato-Marinacci (2002) if not quite in those terms.

relation $(\succsim^*)_{bet}$ relative to which a given preference ordering is utility-sophisticated;[33] from this one immediately infers the equivalence of revealed utility sophistication to equidivisibility of the relation $(\succsim^*)_{bet}$.

For preparation, a bit of further background is needed. If preferences are P4, then in order to be utility-sophisticated relative to an minimally precise likelihood ordering, they must be constant-linear in view of Theorem 2 . If they are indeed constant-linear, there exists a cardinal utility function $u$ over consequences that is unique up to positive affine transformations. Thus, the following utility-based mixture-operation $\alpha f \oplus (1-\alpha) g$ on the space of acts is well-defined: for $\alpha \in [0,1]$, $\alpha f \oplus (1-\alpha) g$ denotes any act $h$ such that, for all $\omega \in \Omega$, $u(h_\omega) = \alpha u(f_\omega) + (1-\alpha) u(g_\omega)$; note that by Eventwise Monotonicity the choice of the act $h$ is immaterial. Ghirardato et al. (2003) provide a direct behavioral definition of the mixture operation.

A (possibly incomplete) relation $\succsim'$ is *independent* if, for all $f, g, h$ and $\alpha \in (0,1]$, $f \succsim' g$ if and only if $\alpha f \oplus (1-\alpha) h \succsim' \alpha g \oplus (1-\alpha) h$. In Nehring (2001), we have obtained (a version of) the following result, a version of which can also be found in Ghirardato et al. (2004, Propositions 4 and 5).[34] The step from i) to ii) follows from versions of well-known results due to Bewley (1986, for finite state spaces) and Walley (1991).

**Proposition 8** *Suppose that the preference ordering $\succsim$ has a constant-linear representation $I \circ u$ such that $u(X)$ is convex. Then*

*i) there exists a unique maximal independent subrelation $\succsim^*$, with*

$$f \succsim^* g \text{ if and only for all } h \text{ and all } \alpha \in (0,1],\ \alpha f \oplus (1-\alpha) h \succsim \alpha g \oplus (1-\alpha) h.$$

*ii) There exists a unique closed convex set of priors $\Pi^*$ such that*

$$f \succsim^* g \text{ if and only } E_\pi u \circ f \geq E_\pi u \circ g \text{ for all } \pi \in \Pi^*. \tag{10}$$

*In particular, $\Pi^*$ is the unique minimal set of closed, convex of priors $\Pi$ such that $\succsim$ is utility-sophisticated with respect to $\Pi$ and $u$.*

*Furthermore, $(\succsim^*)_{bet}$ is the unique maximal coherent likelihood relation $\trianglerighteq$ such that $\succsim$ is utility-sophisticated with respect to $\trianglerighteq$ and $u$.*

[33] While we believe the provided criterion to be applicable also in the stake-dependent case, this needs to be verified in future research.

[34] A first version of this result was presented in the talk Nehring (1996) which made use of a different version of condition i); the exact version of the characterization of $\succsim^*$ in i) was arrived at independently by Ghirardato et al. (2004).

Proposition 8 entails the following operational characterization of revealed utility sophistication.

**Proposition 9** *Suppose that the preference ordering $\succsim$ has a constant-linear representation $I \circ u$ such that $u(X)$ is convex. Then the following three conditions are equivalent:*

1. *$\succsim$ is revealed utility-sophisticated;*
2. *$\Pi^*$ is convex-ranged;*
3. *$(\succsim^*)_{bet}$ is minimally precise.*

## 7. REVEALED UNAMBIGUOUS BELIEFS

With a behavioral criterion for utility sophistication in place, Proposition 9 suggests a natural definition of "revealed probabilistic beliefs", namely as $(\succsim^*)_{bet}$. By construction, $(\succsim^*)_{bet}$ encompasses any likelihood ordering that can be attributed to the DM, assuming a utility sophisticated response to it, and is the only likelihood relation with this property. The assumption is crucial for this unique maximality property. Without it, that is: if compatibility is the only restriction relating preferences and beliefs, unique maximality is lost almost always, and the behavioral identification of "revealed probabilistic beliefs" is likely to require subtler and more contestable considerations.

**Definition 3 (Revealed Probabilistic Beliefs)** *Suppose that the preference ordering $\succsim$ is revealed utility-sophisticated with constant-linear representation $I \circ u$ such that $u(X)$ is convex. Then $(\succsim^*)_{bet}$ defines the decision maker's* revealed probabilistic beliefs.

In earlier work (Nehring (1996), see also Nehring (1999) and Nehring (2001)) as well as in the rich contribution by Ghirardato et al. (2004), analogous definitions have been put forward without restriction to revealed utility-sophisticated preferences.[35] These earlier definitions are subject to the valid criticism that they sometimes arbitrarily attribute to ambiguity what could be attributed with equal legitimacy to failures of utility sophistication. For example, consider a DM who is probabilistically sophisticated in the sense of Machina-Schmeidler (1992) but not SEU. Following Machina-Schmeidler (1992) and Epstein (1999) and Epstein-Zhang (2001), a strong case can be made for attributing the likelihood ordering $\succsim_{bet}$ as the agent's revealed likelihood ordering $\trianglerighteq^{rev}$; in particular, $\trianglerighteq^{rev}$ is the unique maximal likelihood ordering with which preferences are compatible.

[35]These definitions have been given in terms of $\succsim^*$ instead of $(\succsim^*)_{bet}$, . Since the latter is (isomorphic to) a likelihood relation, but not the former is an incomplete preference relation, only the latter is interpretable *as* a belief.

This position is consistent with Definition 3, since such preferences $\succsim$ fails to be revealed utility-sophisticated, precluding the interpretation of $(\succsim^*)_{bet}$ as identifying the DM's probabilistic beliefs.

Restricting the domain of the definition of revealed probabilistic beliefs as proposed has significant implications for the understanding of some of the major models of decision making under ambiguity. For example, for MEU preferences with set of priors $\Psi$, the set of revealed priors given in Proposition 8 is $\Pi^* = \Psi$.[36] However, it is well-known that MEU preferences may be utility-sophisticated without being SEU.[37] Such examples show that the set $\Psi$ cannot, in general, be convincingly interpreted as representing the decision maker's beliefs. Yet such an interpretation constitutes a large part of the intuitive appeal of the MEU model in the first place. The proposed domain restriction comes to the rescue, by salvaging this interpretation for the case of convex-ranged $\Psi$. In particular, it salvages this interpretation for the original MEU model axiomatized by Gilboa-Schmeidler (1989) as reformulated here along the lines of section 5.1.

As an application, the concepts of revealed utility sophistication and revealed probabilistic beliefs can be combined to yield a fully behavioral characterization of MEU preferences in a Savage framework based on an assumption of 'pure' ambiguity aversion– the first such characterization in the literature.

**Proposition 10** *Then the following two statements are equivalent:*

1. $\succsim$ *has a Minimum Expected Utility representation with convex-ranged set of priors* $\Psi$ *and convex* $u(X)$

2. $\succsim$ *has a constant-linear representation* $I \circ u$ *such that* $u(X)$ *is convex and satisfies*

   *i) Revealed Utility Sophistication, and*

   *ii) Preference for Randomization over Bets with respect to* $\trianglerighteq = (\succsim^*)_{bet}$.

Proposition 10 provides a fully behavioral counterpart to Proposition 4, and thus indirectly to Gilboa-Schmeidler's (1989) classical result. The advance of Proposition 10 over Proposition 4 is, evidently, its non-reliance on an independently given likelihood ordering in any form. The main price paid is the condition of Revealed Utility Sophistication, which combines an assumption of Bernoullian rationality with an assumption of minimally precise beliefs. By contrast, the significant

[36] This has been first observed in Nehring (1996); see Ghirardato et al. (2004, Proposition 16) for a published proof.

[37] See, in particular, Marinacci (2002).

advantage of Proposition 4 is to separate these two into assumptions of tradeoff-consistency on the one hand and minimal precision of the belief context on the other.

Proposition 10 can also fruitfully compared with other characterizations of the MEU in a pure Savage framework by Casadesus-Masanell et al. (2000) and Ghirardato et al. (2003). In contrast to Proposition 10, these are more widely applicable since they impose no substantive restrictions on the set of priors $\Psi$. Yet the added generality had a substantial cost at the level of interpretation. Indeed, as just argued, if $\Psi$ is not convex-ranged, it may not be legitimate to interpret the set as representing the DM's beliefs; indeed, there may not be any solid grounds for attributing departures from expected utility within this model to ambiguity at all, as illustrated by the special case of probabilistically sophisticated MEU preferences.[38]

This is mirrored in the axioms. In Ghirardato et al. (2003), there is no counterpart to Revealed Utility Sophistication, and Preference for Randomization over Bets is replaced by a substantially stronger axiom of Utility Hedging.[39] The first difference explains both the gain in generality and the loss in interpretation. That loss extends to the Utility Hedging axiom, which is not an axiom of ambiguity aversion per se, in contrast to Preference for Randomization over Bets. For example, in the special case of probabilistically sophisticated MEU preferences, Utility Hedging can be interpreted as an assumption of probabilistic risk-aversion.

[38] See in particular Marinacci (2002). Probabilistically sophisticated MEU preferences are non-degenerate as they include the rank-dependent preferences with convex probability transform $\phi$.

[39] Utilty Hedging requires that, for any acts $f, g$ and any $\alpha \in [0,1]$, $f \succsim g$ implies $\alpha f \oplus (1-\alpha) g \succsim g$, using the utiltiy-based mixture operation $\oplus$ defined in section 6.

## APPENDIX 1: REPRESENTATION OF EQUIDIVISIBLE LIKELIHOOD ORDERINGS

The following is an extremely brief summary of the representation of equidivisible likelihood orderings obtained in Nehring (2007).

**Axiom 14 (Partial Order)** *$\trianglerighteq$ is transitive and reflexive.*

**Axiom 15 (Nondegeneracy)** $\Omega \vartriangleright \emptyset$.

**Axiom 16 (Positivity)** *$A \trianglerighteq \emptyset$ for all $A \in \Sigma$.*

**Axiom 17 (Additivity)** *$A \trianglerighteq B$ if and only if $A + C \trianglerighteq B + C$ , for any $C$ such that $A \cap C = B \cap C = \emptyset$.*

the event $A$ is *non-null* if $A \vartriangleright \emptyset$.

**Axiom 18 (Splitting)** *If $A_1 + A_2 \trianglerighteq B_1 + B_2$, $A_1 \trianglerighteq A_2$ and $B_1 \trianglerighteq B_2$, then $A_1 \trianglerighteq B_2$.*

**Axiom 19 (Equidivisibility)** *For any $A \in \Sigma$, there exists $B \subseteq A$ such that $B \equiv A \backslash B$.*

To obtain a real-valued representation, a condition expressing the notion of "continuity in probability" is needed. It relies on the following notion of a "small", "$\frac{1}{K}-$"event: $A$ is a $\frac{1}{K}-$*event* if there exist $K$ mutually disjoint events $A_i$ such that $A \trianglelefteq A_i$ for all $i$. A sequence of events $\{A_n\}_{n=1,..,\infty}$ is *converging in probability* to the event $A$ if, for all $K \in \mathbf{N}$ there exists $n_K \in \mathbf{N}$ such that for all $n \geq n_K$ the symmetric difference $A_n \triangle A$ is a $\frac{1}{K}-$event.

**Axiom 20 (Continuity)** *For any sequences $\{A_n\}_{n=1,..,\infty}$ and $\{B_n\}_{n=1,..,\infty}$ converging in probability to $A$ and $B$ respectively,*

$$A_n \trianglerighteq B_n \text{ for all } n \text{ implies } A \trianglerighteq B.$$

These axioms ensure the existence of a representation in terms of the a unique closed convex set of priors $\Pi$. In addition, Equidivisibility entails that this set of prior be "dyadically convex-ranged". As in section 2, A set of priors $\Pi$ is *convex-ranged* if, for any event $A \in \Sigma$ and any $\alpha \in (0,1)$, there exists an event $B \in \Sigma$, $B \subseteq A$ such that $\pi(B) = \alpha\pi(A)$ for all $\pi \in \Pi$. The set $\Pi$ is *dyadically convex-ranged* if this holds for all dyadic $\alpha \in (0,1)$, i.e. of numbers of the form $\alpha = \frac{\ell}{2^k}$, where $k$ and $\ell$ are non-negative integers such that $\ell$ does not exceed $2^k$. As shown in Nehring (2007), dyadic range-convexity coincides with range-convexity if $\Sigma$ is a $\sigma$-algebra.

**Theorem 3** *A relation $\trianglerighteq$ on an event algebra $\Sigma$ has a multi-prior representation with a (unique) dyadically convex-ranged closed convex set of priors $\Pi$ if and only if it satisfies Partial Order, Positivity, Nondegeneracy, Additivity, Splitting, Equidivisibility and Continuity.*

## APPENDIX 2: PROOFS

**Proof of Proposition 1.** That ii) implies i) is straightforward; as to the Archimedean property, merely note that $I-$continuity implies an analogous property for decreasing sequences $\{A_n\}$ by switching the roles of $x$ and $y$.

For the converse, take any $g \in \mathcal{F}$. By Eventwise Monotonicity and boundedness, $x^- \precsim g \precsim x^+$. By the convex-rangedness of $\trianglerighteq$, there exists a totally ordered chain of risky events $\mathcal{T} \subseteq \Lambda$ such that, for any $T \in \Lambda$, there exists $T' \in \mathcal{T}$ such that $T' \equiv T$. Hence one can infer from the Archimedeanicity of $\succsim$ (applied to the case $A + B = \Omega$, i.e. $A \in \Lambda$) the existence of an event $T_g \in \Lambda$ such that $g \sim [x^+, T_g; x^-, T_g^c]$. By compatibility, all such events $T_g$ have the same precise probability $\overline{\pi}(T_g)$. Hence the mapping $V : g \to \overline{\pi}(T_g)$ is well-defined and represents $\succsim$ by construction. For any consequence/constant act $z$, set $u(z) := \overline{\pi}(T_z)$. By Eventwise Monotonicity, $V$ can be written as $I \circ u$, with $I$ monotone and compatible with $\trianglerighteq$; note that $I$ is normalized by construction; moreover, the uniqueness claim is straightforward from Solvability which implies that $u$ is onto.

It remains to verify that $I$ is event-continuous. To do so, consider a sequence $\{A_n\} \subseteq \Lambda_E$ and $A \in \Lambda_E$ such that $\overline{\pi}(A_n/E)$ converges to $\overline{\pi}(A/E)$ and such that the family is $\{A_n\} \cup A$ is ordered by set-inclusion. Take any $x, y \in X$ and $Z \in \mathcal{Z}$. W.l.o.g. $x \geq y$. It clearly suffices to show convergence of $I(x1_{A_n} + y1_{E \setminus A_n} + Z1_{E^c})$ for the case of $\{A_n\}$ being an increasing or decreasing sequence. The proof for both cases is analogous; assume the former, and suppose that the claim is false. I.e., in view of the monotonicity of $I$, suppose that $\sup_{n \in N} I(x1_{A_n} + y1_{E \setminus A_n} + Z1_{E^c}) < I(x1_A + y1_{E \setminus A} + Z1_{E^c})$. By normalization, there exist an event $T \in \Lambda$ such that $\sup_{n \in N} I(x1_{A_n} + y1_{E \setminus A_n} + Z1_{E^c}) < I(1_T) < I(x1_A + y1_{E \setminus A} + Z1_{E^c})$. Hence, by Archimedeanicity, there exist $A' \in \Lambda_E$ and $A' \lhd A$ such that $I(1_T) < I(x1_{A'} + y1_{E \setminus A'} + Z1_{E^c})$. But by the convergence assumption, $A' \trianglelefteq A_n$ for some $n$, hence $I(x1_{A'} + y1_{E \setminus A'} + Z1_{E^c}) \leq \sup_{n \in N} I(x1_{A_n} + y1_{E \setminus A_n} + Z1_{E^c}) < I(1_T)$, a contradiction. $\square$

In the following Lemma, we state a key mathematical property of the Bernoulli integral $\widehat{\rho}$ that will be used repeatedly in the sequel. Let $\mathcal{S}$ denote any finite partition of $\Omega$ into events $S_i \in \Sigma$. Say that $Z \in \mathcal{Z}$ is $\trianglerighteq$-risky conditional on the finite partition $\mathcal{S}$ if, for all $S_i \in \mathcal{S}$, $Z1_{S_i}$ is $\Lambda_{S_i}$-measurable; let $\mathcal{Z}_{\mathcal{S}}$ denote their class. For $Z \in \mathcal{Z}_{\mathcal{S}}$, an expectation conditional on $\mathcal{S}$ is any random variable $\zeta$

such that

$$\begin{aligned}\zeta(\omega) &= \sum_{z\in[\mathbf{0},\mathbf{1}]} z\pi(\{\omega'\in S_i \mid Z(\omega')=z\}/S_i) \text{ if } \omega\in S_i \text{ and } S_i \text{ is non-null, and} \\ \zeta(\omega) &= \text{arbitrary if } \omega\in S_i \text{ and } S_i \text{ is null;}\end{aligned}$$

let the set of such $\zeta$ be denoted by $E(Z/\mathcal{S})$.

**Lemma 2 (Characterization of Intrinsic Integral)** *$\widehat{\rho}$ is the unique mapping $r:\mathcal{Z}\to[0,1]$ such that*

*i) For any event $A\in\Sigma$, $r(1_A)=\rho(A)$, and*

*ii)* **(Conditional Linearity)** *For any partition $\mathcal{S}$ and any $Z\in\mathcal{Z}_{\mathcal{S}}$, $r(Z)=r(\zeta)$ for any $\zeta\in E(Z/\mathcal{S})$.*

Note that Conditional Linearity implies in particular that $\widehat{\rho}$ restricted to risky random variables is the ordinary expectation with respect to $\overline{\pi}$ or equivalently $\rho$.

**Proof of Lemma 2.**

It is immediate from its definition that $\widehat{\rho}$ satisfies i). To verify the Conditional Linearity of $\widehat{\rho}$, write $Z$ as $\sum_{i,j} z_{ij}1_{A_{ij}}$ with $S_i=\sum_{j\le n_j} A_{ij}$ for all $i$. Consider any $C$ such that $\pi(C\cap A_{ij})=z_{ij}\pi(A_{ij})$ for all $i,j$ and all $\pi\in\Pi$; such $C$ exist by the convex-rangedness of $\Pi$. Then $C\in[Z]$ by construction and fact, for all non-null $S_i$ and all $\pi\in\Pi$,

$$\pi(C\cap S_i)=\sum_j \pi(C\cap A_{ij})=\sum_j z_{ij}\left(\overline{\pi}(A_{ij}/S_i)\,\pi(S_i)\right)=\left(\sum_j z_{ij}\overline{\pi}(A_{ij}/S_i)\right)\pi(S_i).$$

¿From this evidently $C\in[\zeta]$ for any $\zeta\in E(Z/\mathcal{S})$. Thus indeed $C\in[Z]\cap[\zeta]$, and therefore

$$\widehat{\rho}(Z)=\rho(C)=\widehat{\rho}(\zeta).$$

Conversely, assume that $r$ satisfies i) and ii). Consider any $Z{=}\sum_i z_i 1_{S_i}$ and any $C\in[Z]$ such that $\pi(C\cap S_i)=z_i\pi(S_i)$ for all $i,j$ and all $\pi\in\Pi$; such $C$ exist by the convex-rangedness of $\Pi$. By construction of $C$, $1_C\in\mathcal{Z}_{\mathcal{S}}$ with $Z\in E(1_C/\mathcal{S})$. Hence

$$r(Z)=r(1_C)\ \text{(by ii)}=\rho(C)\ \text{(by i)}\ =\widehat{\rho}(Z),$$

which establishes that $r=\widehat{\rho}$. $\square$

**Proof of Theorem 1.**

iii) implies ii) To show that $\succsim$ is utility-sophisticated with respect to $\trianglerighteq$, take any $f, g$ such that $E_\pi u \circ f \geq E_\pi u \circ g$ for all $\pi \in \Pi$, and take $A \in [u \circ f]$ and $B \in [u \circ g]$. By construction, $\pi(A) \geq \pi(B)$ for all $\pi \in \Pi$, and therefore by the compatibility of $\rho$

$$\widehat{\rho}\,(u \circ f) = \rho\,(A) \geq \rho\,(B) = \widehat{\rho}\,(u \circ g)\,,$$

i.e. $f \succsim g$. By the same token, if $E_\pi u \circ f > E_\pi u \circ g$ for all $\pi \in \Pi$, then $f \succ g$.

To verify that $\succsim$ is Archimedean, in view of Proposition 1 we need to verify that $\widehat{\rho}$ is event-continuous exploiting the event-continuity of $\rho$. Thus, take some $x, y \in X$ , $Z \in \mathcal{Z}$, $E \in \Sigma$, $A \in \Lambda_E$ and some increasing sequence $\{A_n\}$ of events contained in $A$ such that $\overline{\pi}\,(A_n/E)$ converges to $\overline{\pi}\,(A/E)$ ; we need to show that $\widehat{\rho}(x1_{A_n} + y1_{E \setminus A_n} + Z1_{E^c})$ converges to $\widehat{\rho}(x1_A + y1_{E \setminus A} + Z1_{E^c})$. By conditional linearity (Lemma 2),

$$\widehat{\rho}(x1_A + y1_{E \setminus A} + Z1_{E^c}) = \widehat{\rho}((\overline{\pi}\,(A/E)\,x + (1 - \overline{\pi}\,(A/E))\,y)\,1_E + Z1_{E^c})$$

and likewise

$$\widehat{\rho}(x1_{A_n} + y1_{E \setminus A_n} + Z1_{E^c}) = \widehat{\rho}((\overline{\pi}\,(A_n/E)\,x + (1 - \overline{\pi}\,(A_n/E))\,y)\,1_E + Z1_{E^c}).$$

Convergence of $\widehat{\rho}(x1_{A_n} + y1_{E \setminus A_n} + Z1_{E^c})$ to $\widehat{\rho}(x1_A + y1_{E \setminus A} + Z1_{E^c})$ thus follows from the sup-norm continuity of $\widehat{\rho}$ established by the following Lemma.

**Lemma 3** *$\widehat{\rho}$ is sup-norm continuous if and only if $\rho$ is event-continuous.*

**Proof.**

To demonstrate the "only-if" part, take any disjoint $B, E \in \Sigma$, any $A \in \Lambda_E$ with $A \subseteq E$ and any increasing (respectively decreasing) sequence $\{A_n\}$ of events contained in (resp. containing) $A$ such that $\overline{\pi}\,(A_n/E)$ converges to $\overline{\pi}\,(A/E)$. By the sup-norm continuity of $\widehat{\rho}$ and conditional linearity (Lemma 2), we have

$$\lim_{n \to \infty} \rho\,(A_n + B) = \lim_{n \to \infty} \widehat{\rho}\,(1_{A_n + B}) = \lim_{n \to \infty} \widehat{\rho}\,(\overline{\pi}\,(A_n/E)\,1_E + 1_B) = \widehat{\rho}\,(\overline{\pi}\,(A/E)\,1_E + 1_B) = \rho\,(A + B)\,.$$

Conversely, take any sequence $\{Z_n\}$ in $\mathcal{Z}$ converging to $Z$ in sup-norm. Clearly, there exists an increasing sequence $\{\alpha_n\}$ converging to 1 such that $\alpha_n Z \leq Z$. Take any event $E \in [Z]$ and events

$A_n \in \Lambda_E$ such that $\overline{\pi}(A_n/E) = \alpha_n$. By construction, $A_n \in [\alpha_n Z]$, hence $\widehat{\rho}(\alpha_n Z) = \rho(A_n)$. Hence by event-continuity of $\rho$,

$$\lim_{n\to\infty} \inf \widehat{\rho}(Z_n) \geq \lim_{n\to\infty} \inf \widehat{\rho}(\alpha_n Z) = \lim_{n\to\infty} \inf \rho(A_n) \geq \rho(E) = \widehat{\rho}(Z).$$

By the same token, $\lim_{n\to\infty} \sup \widehat{\rho}(Z_n) \leq \widehat{\rho}(Z)$, and thus $\lim_{n\to\infty} \widehat{\rho}(Z_n) = \widehat{\rho}(Z)$ as desired. $\square$

ii) implies i) It is clear that Utility Sophistication implies Compatibility. To verify Trade-off Consistency, take any $x, y, x', y' \in X$, $f, g \in \mathcal{F}$ and events $A$ disjoint from $B$ and $A'$ disjoint from $B'$ such that $A \equiv B \rhd\!\rhd \emptyset$ and $A' \equiv B'$ and such that $[x \text{ on } A; y \text{ on } B; f(\omega) \text{ elsewhere}] \succsim [x' \text{ on } A; y' \text{ on } B; f(\omega) \text{ elsewhere}]$. By the assumption on $A$ and $B$, for all $\pi \in \Pi$, $\pi(A) = \pi(B) > 0$; therefore, if it was the case that $u(x) + u(y) < u(x') + u(y')$, then the strict part of Utility Sophistication would imply that $[x \text{ on } A; y \text{ on } B; f(\omega) \text{ elsewhere}] \prec [x' \text{ on } A; y' \text{ on } B; f(\omega) \text{ elsewhere}]$, which is false by assumption. Thus $u(x) + u(y) \geq u(x') + u(y')$, which implies by the non-strict part of Utility Sophistication that $[x \text{ on } A'; y \text{ on } B'; g(\omega) \text{ elsewhere}] \succsim [x' \text{ on } A'; y' \text{ on } B'; g(\omega) \text{ elsewhere}]$, as needed to be shown.

i) implies iii)

Since Trade-off Consistency implies Eventwise Monotonicity for minimally precise likelihood orderings as remarked in the text, by Proposition 1 there exist an onto function $u : X \to [0, 1]$ and a normalized functional $I : \mathcal{Z} \to [0, 1]$ that is monotone, event-continuous and compatible with $\trianglerighteq$ such that $f \succsim g$ if and only if $I(u \circ f) \geq I(u \circ g)$, for all $f, g \in \mathcal{F}$. In particular, $\rho$ is event-continuous as the restriction of $I$ to indicator functions. It remains to show that $I = \widehat{\rho}$.

Step 1. We shall first consider the case of dyadic-valued utilities; a number is *dyadic* if $\alpha = \frac{\ell}{2^m}$, where $m$ is natural or zero, and $\ell$ is an odd integer or zero; $m$ will be referred to as the (dyadic) order of $\alpha$ denoted by $|\alpha|$. Let $\mathbf{D}$ denote the set of dyadic numbers in $(0, 1]$.

**Lemma 4** *For any $\alpha \in \mathbf{D}, w, x, y \in X, B \in \Sigma$, $A \in \Lambda_B$ with $A \subseteq B$ and $T \in \Lambda$ such that $\overline{\pi}(T) = \overline{\pi}(A/B) = \alpha$ : if $w \sim [x, T; y, T^c]$, then $[w, B; f(\omega)$ elsewhere$] \sim [x, A; y, B \backslash A; f(\omega)$ elsewhere$]$.*

The Lemma is proved by induction on the order of $\alpha$. If the order of $\alpha$ is 1, i.e. if $\alpha = \frac{1}{2}$, the assertion follows directly from Trade-off Consistency. Suppose thus that the Lemma has been shown for all dyadic coefficient $\alpha'$ with $|\alpha'| < |\alpha|$. Assume that $\alpha \geq \frac{1}{2}$; the case of $\alpha < \frac{1}{2}$ can be proved essentially identically. Then $\alpha = \frac{1}{2} + \frac{1}{2}\beta$, where $\beta$ is dyadic with $|\beta| = |\alpha| - 1$.

Now define risky events $T_1, T_2, T_3$ such that $T_1 + T_2 + T_3 = \Omega, T_2 + T_3 = T$, and $\overline{\pi}(T_2) = \frac{1}{2}\beta$. Since $\overline{\pi}(T) = \alpha$, one has also $\overline{\pi}(T_3) = \frac{1}{2}$ and $\overline{\pi}(T_2/T_1 + T_2) = \beta$. In parallel, define events $A_1, A_2, A_3 \in \Lambda_B$ such that $A_1 + A_2 + A_3 = B, A_2 + A_3 = A$, and $\overline{\pi}(A_2/B) = \frac{1}{2}\beta$. Since $\overline{\pi}(A/B) = \alpha$, one has also $\overline{\pi}(A_3/B) = \frac{1}{2}$ and $\overline{\pi}(A_2/A_1 + A_2) = \beta$. Such events exist by the convex-rangedness of $\Pi$.

Take any $D \in \Lambda$ such that $\overline{\pi}(D) = \beta$, and $z \in X$ such that $z \sim [x, D; y, D^c]$; such $z$ exists by Solvability. Since $\overline{\pi}(T_2/T_1 + T_2) = \beta$, by the induction assumption this implies that

$$[z, T_1 + T_2; x, T_3] \sim [y, T_1; x, T_2; x, T_3],$$

hence by the assumption that $w \sim [x, T; y, T^c]$ and transitivity also that

$$[z, T_1 + T_2; x, T_3] \sim [w, T_1 + T_2; w, T_3]. \tag{11}$$

Writing $[x, A; y, B\backslash A; f(\omega) \text{ elsewhere}] = [y, A_1; x, A_2; x, A_3; f(\omega) \text{ elsewhere}]$, by the induction assumption one also has

$$[x, A; y, B\backslash A; f(\omega) \text{ elsewhere}] \sim [z, A_1 + A_2; x, A_3; f(\omega) \text{ elsewhere}].$$

By Trade-off Consistency and (11), in turn

$$[z, A_1 + A_2; x, A_3; f(\omega) \text{ elsewhere}] \sim [w, A_1 + A_2; w, A_3; f(\omega) \text{ elsewhere}].$$

Since $B = A_1 + A_2 + A_3$, we get by transitivity

$$[x, A; y, B\backslash A; f(\omega) \text{ elsewhere}] \sim [w, B; f(\omega) \text{ elsewhere}]$$

as desired.

Step 2. We shall next obtain the desired conclusion for the subset dyadic-valued functions $Y \in \mathcal{Z}$, which we shall abbreviate to $\mathcal{Z}_{\mathbf{D}}$. Thus, take any $Y = \sum_{i \leq n} y_i 1_{E_i} \in \mathcal{Z}_{\mathbf{D}}$; by solvability, there exists $f = [w_i, E_i]_{i \leq n} \in \mathcal{F}$ such that $u(w_i) = y_i$ for all $i$, so that $Y = u \circ f$. For each $i \leq n$, pick $A_i \subseteq E_i$ such that $\overline{\pi}(A_i/E_i) = u(w_i)$. By $n-$fold application of Lemma 4, $f \sim \left[x^+, \sum_{i \leq n} A_i; x^-, \left(\sum_{i \leq n} A_i\right)^c\right]_{i \leq n}$. Since $\sum_{i \leq n} A_i \in [Y]$ by construction, one obtains

$$I(Y) = I(u \circ f) = \rho(\sum_{i \leq n} A_i) = \widehat{\rho}(Y),$$

demonstrating that $I = \widehat{\rho}$ on $\mathcal{Z}_{\mathbf{D}}$.

Step 3.

This conclusion is extended to all of $\mathcal{Z}$ by an inductive continuity argument. Let $\mathcal{Z}_k$ the set of random variables $Y \in \mathcal{Z}$ such that in their canonical representation $Y = \sum_{i \leq n} y_i 1_{E_i}$ no more than $k$ $y_i$'s are not dyadic. Step 2 has established that $I = \widehat{\rho}$ on $\mathcal{Z}_{\mathbf{D}} = \mathcal{Z}_0$. Suppose therefore that $I = \widehat{\rho}$ on $\mathcal{Z}_k$; we need to show that $I = \widehat{\rho}$ on $\mathcal{Z}_{k+1}$. Take $Y = \sum_{i \leq n} y_i 1_{E_i} \in \mathcal{Z}_{k+1}$, and assume w.l.o.g. that $y_1 \in (0,1] \backslash \mathbf{D}$.

Take an increasing sequence $\{v_j\}$ in $\mathbf{D}$ converging to $y_1$, and take $B \in \left[\sum_{2 \leq i \leq n} y_i 1_{E_i}\right]$, $A \in [y_1 1_{E_1}]$ and an increasing sequence $\{A_j\}$ contained in $A$ such that $A_j \in [v_j 1_{E_1}]$; such events exist by repeated applications of equidivisibility. Denote $Y_j := v_j 1_{E_1} + \sum_{2 \leq i \leq n} y_i 1_{E_i}$. Note that by construction, $A_j + B \in [Y_j]$ and $A + B \in [Y]$. By the event-continuity of $\rho$, $\lim_{j \to \infty} \rho(A_j + B) = \rho(A + B)$, and therefore

$$\lim_{j \to \infty} \widehat{\rho}\,(Y_j) = \lim_{j \to \infty} \rho(A_j + B) = \rho(A + B) = \widehat{\rho}\,(Y)\,.$$

Likewise, take a decreasing sequence $\{v_j'\}$ in $\mathbf{D}$ converging to $y_1$, and denote $Y_j' := v_j' 1_{E_1} + \sum_{2 \leq i \leq n} y_i 1_{E_i}$. The same argument establishes that

$$\lim_{j \to \infty} \widehat{\rho}\left(Y_j'\right) = \widehat{\rho}\,(Y)\,.$$

By the induction assumption, for all $j$,

$$\widehat{\rho}\,(Y_j) = I\,(Y_j) \text{ and } \widehat{\rho}\left(Y_j'\right) = I\left(Y_j'\right).$$

Hence, by the monotonicity of $I$,

$$\widehat{\rho}\,(Y) = \lim_{j \to \infty} \widehat{\rho}\,(Y_j) = \lim_{j \to \infty} I\,(Y_j) \leq I(Y) \leq \lim_{j \to \infty} I\left(Y_j'\right) = \lim_{j \to \infty} \widehat{\rho}\left(Y_j'\right) = \widehat{\rho}\,(Y)\,,$$

which yields

$$\widehat{\rho}\,(Y) = I(Y)$$

as desired. $\square$

**Proof of Proposition 2.**

Necessity is trivial; for sufficiency, we first verify regularity. Boundedness does not depend on the context $\trianglerighteq$.

$\succsim$ is Archimedean with respect to $\trianglerighteq'$, since $\rhd\rhd'$ contains $\rhd\rhd$; finally, $\succsim$ extends trivially to $\trianglerighteq'$ as a superrelation of $\trianglerighteq$ .

Let $\Lambda', \rho', [.]'$, and $\widehat{\rho'}$ denote the family of unambiguous events, normalized capacity, equivalence classes and Bernoulli integral associated with $\trianglerighteq'$ obtained from Proposition 1 and Theorem 1. In view of Theorem 1, in order to show that $\succsim$ is utility-sophisticated and regular with respect to $\trianglerighteq'$, it suffices to show that $\widehat{\rho'} = \widehat{\rho}$.

From $\trianglerighteq' \supseteq \trianglerighteq$, it is immediate that $\Lambda' \supseteq \Lambda$. Hence by the additivity of the representing capacity on unambiguous events, $\rho'(A) = \rho(A)$ for all $A \in \Lambda$, hence by the ordinal uniqueness of $\rho$, in fact $\rho'(A) = \rho(A)$ for all $A \in \Sigma$.

Again from $\trianglerighteq' \supseteq \trianglerighteq$, it is immediate that $[Z]' \supseteq [Z]$ for all $Z \in \mathcal{Z}$. Thus, for any $Z \in \mathcal{Z}$, taking $A \in [Z]$, one has

$$\widehat{\rho'}(Z) = \rho'(A) = \rho(A) = \widehat{\rho}(Z),$$

as needed to be shown. $\square$

**Proof of Theorem 2.**

Step 1. Constant-Linearity of $\widehat{\rho}$ implies P4.

Take any $A, B \in \Sigma$ such that $\rho(A) \geq \rho(B)$, and any $x, y \in X$ with $u(y) < u(x)$. In view of Theorem 1, it suffices to show that $\widehat{\rho}(u \circ [x, A; y, A^c]) \geq \widehat{\rho}(u \circ [x, B; y, B^c])$. Indeed, this follows easily from the equalities

$$u \circ [x, A; y, A^c] = u(x)1_A + u(y)1_{A^c} = (u(x) - u(y))\, 1_A + u(y)1_\Omega,$$

whence by constant-linearity

$$\widehat{\rho}(u \circ [x, A; y, A^c]) = (u(x) - u(y))\, \rho(A) + u(y),$$

and similarly

$$\widehat{\rho}(u \circ [x, B; y, B^c]) = (u(x) - u(y))\, \rho(B) + u(y),$$

from which the desired conclusion follows immediately.

Step 2. P4 implies Union and Splitting Invariance.

Consider any $A \in \Sigma$ , $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$, and $A' \in \Lambda_A$ as well as $B_1 \in \Lambda_A$ and $B_2 \in \Lambda_{A^c}$ (both disjoint from $A'$) such that $\overline{\pi}(A'/A) = \alpha$ and $\overline{\pi}(B_1/A) = \overline{\pi}(B_2/A^c) = \beta$, and let $B = B_1 + B_2$.

*Claim:* $\rho(A' + B) = \alpha\rho(A) + \beta$.

Pick consequences $y, x$ such that $u(y) = \beta$ and $u(x) = \alpha + \beta$. By utility sophistication and the conditional linearity property of $\widehat{\rho}$ (Lemma 2),

$$[x, A; y, A^c] \sim [x^+, A' + B_1; x^-, A \backslash (A' + B_1); x^+, B_2; x^-, A^c \backslash B_2] = [x^+, A' + B; x^-, (A' + B)^c].$$

Moreover, taking any $T \in \Lambda$ with $\overline{\pi}(T) = \rho(A)$, by P4,

$$[x, A; y, A^c] \sim [x, T; y, T^c],$$

and thus by transitivity

$$[x^+, A' + B; x^-, (A' + B)^c] \sim [x, T; y, T^c].$$

One computes $\widehat{\rho}(u \circ [x, T; y, T^c]) = E_{\overline{\pi}}(u \circ [x, T; y, T^c]) = (\alpha + \beta)\overline{\pi}(T) + \beta\pi^0(T^c) = \alpha\rho(A) + \beta$, whence

$$\rho(A' + B) = \widehat{\rho}(1_{A'+B}) = \widehat{\rho}(u \circ [x, T; y, T^c]) = \alpha\rho(A) + \beta,$$

verifying the claim.

Specialized to the case $\beta = 0$, the Claim clearly entails Splitting Invariance.

To obtain Union Invariance, choose any $A \in \Sigma$ and $C \in \Lambda$ disjoint from $A$. It clearly suffices to show that $\rho(A + C) = \rho(A) + \rho(C)$.

Take any $A' \in \Lambda_A$ such that $\overline{\pi}(A'/A) = \frac{1}{2}$ and any $C' \in \Lambda_C$ such that $\overline{\pi}(C'/C) = \frac{1}{2}$. Clearly, $C' \in \Lambda$ and $A' + C' \in \Lambda_{A+C}$ with $\overline{\pi}(A' + C'/A + C) = \frac{1}{2}$. Hence by Splitting Invariance,

$$\rho(A' + C') = \frac{1}{2}(\rho(A + C)). \tag{12}$$

Now choose $B_1 \in \Lambda_A$ and $B_2 \in \Lambda_{A^c}$ (both disjoint from $A'$) such that $\overline{\pi}(B_1/A) = \overline{\pi}(B_2/A^c) = \frac{1}{2}\rho(C)$. Evidently, $B = B_1 + B_2 \in \Lambda$ with $\overline{\pi}(B) = \overline{\pi}(C') = \frac{1}{2}\rho(C)$. It is easily verified that therefore $A' + C' \equiv A' + B$, whence by Compatibility,

$$\rho(A' + C') = \rho(A' + B). \tag{13}$$

Since $\frac{1}{2} + \frac{1}{2}\rho(C) \leq 1$, the Claim can be applied, yielding

$$\rho(A' + B) = \frac{1}{2}\rho(A) + \rho(B) = \frac{1}{2}(\rho(A) + \rho(C)). \tag{14}$$

Combining equations (12), (13), and (14) yields the desired result.

Step 3a) Union Invariance implies Constant-Additivity.

Take any $Y = \sum_{i \in I} y_i 1_{E_i}$ and $c \in [0,1]$ such that $Y + c1_\Omega \in \mathcal{Z}$. Since $Y \leq (1-c)1_\Omega$, there exist $A \in [Y]$ and $S, T \in \Lambda$ such that $\rho(S) = \rho(A) \leq 1-c$, $\rho(T) = c$, and $T$ is disjoint from both $A$ and $S$. To see this, take $A = \sum_{i \in I} A_i$ with $A_i \in \Lambda_{E_i}$ and $\overline{\pi}(A_i/E_i) = y_i$, $S = \sum_{i \in I} S_i$ with $S_i \in \Lambda_{E_i}$ and $\overline{\pi}(S_i/E_i) = \rho(A)$, and $T = \sum_{i \in I} T_i$ with $T_i \in \Lambda_{E_i}$ and $\overline{\pi}(T_i/E_i) = c$ such that $T_i$ is disjoint from both $A_i$ and $S_i$, for all $i \in I$; such $A_i, S_i$, and $T_i$ exist by the convex-rangedness of $\Pi$. Clearly, $A + T \in [Y + c1_\Omega]$. Since $A \sim_{bet} S$ by assumption, $A + T \sim_{bet} S + T$ by Union Invariance which in turn is tantamount to

$$\rho(A+T) = \rho(S+T) = \rho(S) + \rho(T) = \rho(A) + c.$$

Hence

$$\widehat{\rho}(Y + c1_\Omega) = \rho(A+T) = \rho(A) + c = \widehat{\rho}(Y) + c.$$

Step 3b) Splitting Invariance implies Positive Homogeneity

Take $Y \in \mathcal{Z}$ and rational $c = \frac{m}{n} \leq 1$, where $m$ and $n$ are natural numbers. Take $A \in [Y]$ and $T \in \Lambda$ such that $\overline{\pi}(T) = \widehat{\rho}(Y)$. By equidivisibility of $\trianglerighteq$ /convex-rangedness of $\Pi$, there exist partitions of $A$ and $T$ can be split into $n$ equally likely subevents $\{A_1, ..., A_n\}$ and $\{T_1, ..., T_n\}$; by an argument paralleling that in i), the $A_i$ can be chosen to belong to $[\frac{1}{n}Y]$, whence $\sum_{i \leq m} A_i \in [\frac{m}{n}Y]$. Since by construction $A \sim_{bet} T$, by Splitting Invariance $A_1 \sim_{bet} T_1$, and therefore by Splitting Invariance again $\sum_{i \leq m} A_i \sim_{bet} \sum_{i \leq m} T_i$. It follows that

$$\widehat{\rho}\left(\frac{m}{n}Y\right) = \rho\left(\sum_{i \leq m} A_i\right) = \overline{\pi}\left(\sum_{i \leq m} T_i\right) = \frac{m}{n}\overline{\pi}(T) = \frac{m}{n}\widehat{\rho}(Y),$$

which establishes positive homogeneity for rational $\alpha$. This implies positive homogeneity for arbitrary $\alpha$, since by monotonicity of $\widehat{\rho}$,

$$\begin{aligned} \alpha\widehat{\rho}(Y) &= \sup\{\beta\widehat{\rho}(Y) \mid \beta \leq \alpha, \beta \in \mathbf{Q}\} = \sup\{\widehat{\rho}(\beta Y) \mid \beta \leq \alpha, \beta \in \mathbf{Q}\} \\ &\leq \widehat{\rho}(\alpha Y) \leq \inf\{\widehat{\rho}(\beta Y) \mid \beta \geq \alpha, \beta \in \mathbf{Q}\} = \alpha\widehat{\rho}(Y), \end{aligned}$$

and thus $\widehat{\rho}(\alpha Y) = \alpha\widehat{\rho}(Y)$. $\square$

**Proof of Proposition 3.** The necessity of Union and Splitting Invariance follows from Theorem 2. The validity of the converse can be seen as follows. First, applying the proof of Theorem 1 to preferences over risky acts $\succsim_{ua}$, one infers that these preferences have a SEU representation with utility function $u$, unique up to positive affine transformations. Likewise, applying the proof of Proposition 1, there exists a unique event-continuous $\rho$ representing $\succsim_{\mathcal{B}}$ such that $\rho(T) = \overline{\pi}(T)$ for $T \in \Lambda$. Let $\widehat{\rho}$ denote the associated expectation operator given by (2). By the proof of the implication 3)$\Longrightarrow$2) of Theorem 2, $\widehat{\rho}$ is constant-linear. Define $\succsim$ by setting for all $f, g \in \mathcal{F}$:

$$f \succsim g \text{ iff } \widehat{\rho}(u \circ f) \geq \widehat{\rho}(u \circ g). \tag{15}$$

Clearly, by the implication 1)$\Longrightarrow$3) of Theorem 1, if an extension with the desired properties exists, it must be given by (15). Conversely, this preference ordering $\succsim$ is Archimedean and tradeoff-consistent by the implication 3)$\Longrightarrow$1) of Theorem 1. Since $\rho$ agrees with $\overline{\pi}$ on $\Lambda$, the restriction of $\succsim$ to $\mathcal{F}^{risk}$ agrees with $\succsim_{ua}$ . Furthermore, by construction $\succsim_{bet}=\succsim_{\mathcal{B}}$. Since $\widehat{\rho}$ is constant-linear, $\succsim$ satisfies P4 by the implication 2)$\Longrightarrow$1) of Theorem 2.

Finally, we need to show that the ordering $\succsim$ given in (15) does not depend on the likelihood ordering $\trianglerighteq$ . That is, take two minimally precise likelihood orderings $\trianglerighteq^1$ and $\trianglerighteq^2$ with associated $\Lambda^1$ and $\Lambda^2$ relative to which $\succsim_{\mathcal{B}}$ is Archimedean, compatible and satisfies Union- and Splitting-Invariance, and take preference relations over risky acts $\succsim_{ua}^1$ and $\succsim_{ua}^1$ with the same associated preferences over lotteries, hence with the same representing Bernoulli utility function $u$. Let $\succsim^1$ and $\succsim^2$ denote the extensions to all Savage acts given by (15). Then we claim that in fact $\succsim^1=\succsim^2$ .

To see this, by Theorem 2, each $\succsim^i$ has a constant-linear representation $I \circ u$ with $u(X) = [0,1]$, ensuring applicability of Proposition 8 in section 6 below. For $i = 1, 2$, let $\succsim_{bet}^{*i}$ denote associated revealed likelihood relations. Since $\succsim^i$ is utility-sophisticated with respect to the minimally precise likelihood ordering $\trianglerighteq^i$ by construction and since $\succsim_{bet}^1=\succsim_{bet}^2=\succsim_{\mathcal{B}}$, by Proposition 8 evidently $\succsim_{bet}^{*1}=\succsim_{bet}^{*2}$ . For $i = 1, 2, 3$ let $\Lambda^i, [.]^i, \rho^i, \widehat{\rho}^i$ denote the families of risky events, equivalence class operators, normalized capacities and Bernoulli integrals associated with $(\succsim_{\mathcal{B}}, \trianglerighteq^1)$, $(\succsim_{\mathcal{B}}, \trianglerighteq^2)$, and $(\succsim_{\mathcal{B}}, \succsim_{bet}^{*1}=\succsim_{bet}^{*2})$, respectively. By maximality of $\succsim_{bet}^{*1}$ and $\succsim_{bet}^{*2}$, evidently $\Lambda^3 \supseteq \Lambda^1 \cup \Lambda^2$ and $[Z]^3 \supseteq [Z]^1 \cup [Z]^2$ for all $Z \in \mathcal{Z}$. Thus, clearly $\rho^1 = \rho^3 = \rho^2$ and $\widehat{\rho}^1 = \widehat{\rho}^3 = \widehat{\rho}^2$. By Theorem 1 therefore $\succsim^1=\succsim^2$ . $\square$

**Lemma 5** *$\widehat{\rho}$ is quasi-concave if and only if $\succsim$ satisfies Preference for Randomization over bets.*

**Proof.** Take any $Y, Z \in \mathcal{Z}$, $A \in [Y]$, $B \in [Z]$ and $\alpha \in (0,1)$. By continuity, it suffices to consider the case of $\alpha = 1$. By minimal precision, there exists $T \in \Lambda$ such that $T \cap D \equiv T^c \cap D$ for any $D \in \{A \backslash B, A + B, B \backslash A, (A + B)^c\}$. From the construction it is clear that $(T \cap A) + (T^c \cap B) \in$

$[\frac{1}{2}Y + \frac{1}{2}Z]$. Thus

$$\begin{aligned}\widehat{\rho}\left(\frac{1}{2}Y + \frac{1}{2}Z\right) &\geq \widehat{\rho}(Z) \text{ iff} \\ \rho\left((T \cap A) + (T^c \cap B)\right) &\geq \rho(B), \text{ i.e. iff} \\ (T \cap A) + (T^c \cap B) &\succsim {}_{bet} B,\end{aligned}$$

which establishes the asserted equivalence. $\square$

**Proof of Proposition 5.**

Let us call the conjunction of Non-Decreasing Aversion to Ambiguity and Non-Increasing Aversion to Ambiguity "Weak P4". In view of Proposition 4, and the observation that variational preferences satisfy Non-Decreasing Aversion to Ambiguity, it suffices to establish that whenever preferences satisfy the assumptions of Proposition 1, Weak P4 implies P4 proper.

Thus, consider any $x, y, x', y' \in X$ such that $x \succ y$ and $x' \succ y'$ and any $A, B \in \Sigma$ such that $[x \text{ on } A; y \text{ on } A^c] \succsim [x \text{ on } B; y \text{ on } B^c]$; we need to establish that $[x' \text{ on } A; y' \text{ on } A^c] \succsim [x' \text{ on } B; y' \text{ on } B^c]$.

In view of Proposition 1, there exist $T, T' \in \Lambda$ such that

$$[x \text{ on } A, y \text{ on } A^c] \sim [x \text{ on } T, y \text{ on } T^c] \text{ and } [x \text{ on } B, y \text{ on } B^c] \sim [x \text{ on } T', y \text{ on } T'^c]. \tag{16}$$

Since $[x \text{ on } A; y \text{ on } A^c] \succsim [x \text{ on } B; y \text{ on } B^c]$, by transitivity and compatibility,

$$T \trianglerighteq T'.$$

By Weak P4 and Boundedness, we infer that $[x^- \text{ on } A, x^+ \text{ on } A^c] \sim [x^- \text{ on } T, x^+ \text{ on } T^c]$, and, applying Weak P4 again, therefore also

$$[x' \text{ on } A, y' \text{ on } A^c] \sim [x' \text{ on } T, y' \text{ on } T^c],$$

as well as

$$[x' \text{ on } B, y' \text{ on } B^c] \sim [x' \text{ on } T', y' \text{ on } T'^c]$$

by the same token. Since $T \trianglerighteq T'$, compatibility and transitivity of yield

$$[x' \text{ on } A, y' \text{ on } A^c] \succsim [x' \text{ on } B, y' \text{ on } B^c]$$

as desired. $\square$

**Proof of Lemma 1.**

W.l.o.g. $u(X) \supseteq \{0, y, 1\}$, with $0 < y < 1$; we consider the case $y \geq \frac{1}{2}$; the case $y \leq \frac{1}{2}$ is similar. Specify consequences in utiles, and, for $z \in \{0, y, 1\}$ let

$$f_z := [1 \text{ on } B_1, z \text{ on } A, 0 \text{ elsewhere}],$$

and

$$g_z := [\frac{1}{2} \text{ on } B_1 + B_2, z \text{ on } A, 0 \text{ elsewhere}].$$

By construction, for all $\pi \in \Pi$, $E_\pi f_z = E_\pi g_z$, hence for all $z \in \{0, y, 1\}$,

$$f_z \sim g_z \tag{17}$$

by utility sophistication.

Now

$$\int u \circ f_z d\nu = \nu(B_1) + z[\nu(A + B_1) - \nu(B_1)],$$

while, for $z \geq \frac{1}{2}$

$$\int u \circ g_z d\nu = z\rho(A) + \frac{1}{2}[\nu(A + B_1 + B_2) - \nu(A)],$$

and for $z \leq \frac{1}{2}$

$$\int u \circ g_z d\nu = \frac{1}{2}\nu(B_1 + B_2) + z[\nu(A + B_1 + B_2) - \nu(B_1 + B_2)];$$

in particular,

$$\int u \circ g_0 d\nu = \frac{1}{2}\nu(B_1 + B_2).$$

By (17), $\int u \circ f_z d\nu = \int u \circ g_z d\nu$ for $z \in \{0, y, 1\}$; by straightforward computation, one verifies that hence

$$\begin{aligned} 0 &= \\ y\int u \circ f_1 d\nu + (1-y)\int u \circ f_0 d\nu - \int u \circ f_y d\nu &= \\ y\int u \circ g_1 d\nu + (1-y)\int u \circ g_0 d\nu - \int u \circ g_y d\nu &= \frac{1}{2}(1-y)(\nu(A) + \nu(B_1 + B_2) - \nu(A + B_1 + B_2)), \end{aligned}$$

hence $\nu(A) + \nu(B_1 + B_2) = \nu(A + B_1 + B_2)$, as needed to be shown. $\square$.

**Proof of Proposition 9.**

The equivalence of (1) and (2) is immediate from Proposition 8. The equivalence of (2) and (3) follows from the fact that $\Pi_{(\succsim^*_{bet})} = \Pi^*$, which in turn follows from the uniqueness of the multi-representation of minimally precise likelihood orderings shown in Nehring (2007, Theorem 2).

**Proof of Proposition 10.**

1) implies 2). As remarked in the text, for MEU preferences, $\Pi^* = \Psi$, hence $\succsim$ is revealed utility-sophisticated; hence $\succsim$ satisfies Preference for Randomization over Bets with respect to $\trianglerighteq = (\succsim^*)_{bet}$ as in Proposition 4.

2 implies 1). Set $\trianglerighteq = (\succsim^*)_{bet}$ with associated expectation operator $\widehat{\rho}$. Since constant-linearity combined with convexity of $u(X)$ implies regularity (boundedness is not really needed here), by Revealed Utility Sophistication and Theorem 1 (2 implies 3), $I = \widehat{\rho}$. Furthermore, by Preference for Randomization over Bets with respect to $(\succsim^*)_{bet}$, Lemma 5, $\widehat{\rho}$ and thus $I$ is quasi-concave. Hence, by the central argument of Gilboa-Schmeidler (1989), $I$ has an MEU representation. Finally, by Revealed Utility Sophistication and the fact that $\Psi = \Pi^*$ for MEU preferences, $\Psi$ is convex-ranged. $\square$

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