Pesendorfer and Swinkels in the Continuum

Louis Makowski *

This version: June 4, 2002[†]

As emphasized by Dubey, Mas-Colell, and Shubik (1980), Vives (1988), and others, strategic games with a continuum of players constitute a useful technique in economics. The analysis tends to becomes relatively straightforward, and the conclusions help us understand what to expect in large finite economies. Here we use this technique to help us understand what drives the remarkable asymptotic results on information aggregation in common-value auctions derived by Pesendorfer and Swinkels (1997, 1998).

To set the stage, consider the following example. Suppose a new model car is introduced of unknown quality. Each individual takes the car for a test drive, resulting in a very coarse 0-1 signal about its true quality: the individual either has a bad or good driving experience. Since there are many potential buyers, the join of all individuals' test-driving experiences is much more informative about the car's true quality than any one person's experience. Each individual, after his test drive—*without any knowledge about others' experiences*—places a sealed bid for a unit of the car, there being less cars available than the number of potential buyers. Pesendorfer and Swinkels (1997) (henceforth P&S) show that in a large economy—in spite of each bidder's very coarse information,—the equilibrium price of a car will reflect the join of all individuals' information!

Like a child who first sees a magician pull a rabbit out of a hat, one is amazed by this conclusion. Since it is proved using formidable asymptotic arguments, one also is a little in the dark about what exactly drives the conclusion, in spite of P&S's nice exposition. How did they do it? Are we to take their conclusion seriously, in terms of our faith in the market's ability to aggregate information? Framing the P&S model in a continuum setting will relieve us of the need to do tough asymptotics, and thus will help us unravel some of their magic.

In an ambitious sequel, Pesendorfer and Swinkels (1998) (henceforth P&S2) extend the findings in their 1997 paper to a setting in which buyers' preferences have both a common-value (quality) and a private-value (taste) component. Many,

---

[*] Joe Ostroy motivated my interest in understanding the papers of Pesendorfer and Swinkels. Many conversations with Iraj Deilami, Giovanna Oettinger, and Mort Schwartz about auctions kept me focused and grounded during the writing of this manuscript. A presentation to the UC Davis theory group was also helpful.

[†] First version: April 1, 2002. Comments very welcome. You can reach me at lmakowski@ucdavis.edu.

perhaps even most, real-world markets have this feature. They show that in this more realistic setting, asymptotically, equilibria will be both fully-revealing *and* allocatively efficient! To clarify what drives this even more remarkable conclusion, we also will reformulate the Pesendorfer and Swinkels 2-dimensional model in the continuum.

The plan of the paper is as follows. We concentrate mainly on a pure common-value model. Using such a model, P&S show there exists a unique equilibrium that, asymptotically, fully aggregates information. Recasting their model into the continuum, we are able to construct simple examples that show the basic logic behind their fully-revealing equilibrium. Then we generalize, proving both an existence and uniqueness result: in the continuum, fully-revealing equilibria always exist (Theorem 1 in Section 1) and, under a regularity assumption, *all* equilibria will be fully revealing (Theorem 2 in Section 2). It is worth emphasizing that all the proofs are quite easy; notably, the proof of existence is entirely constructive. Section 3 extends our continuum model to a 2-dimensional setting and proves the existence of a fully revealing *and* allocatively efficient equilibrium; the proof is again constructive. It is worth noting, by way of contrast, that P&S2 are unable to prove existence in their 2-dimensional finite-agent model because of the considerable mathematical complexity.

The transparency that the continuum permits shows its value as a tool for analysis: it can serve as a complement to asymptotic analysis. But it is not a substitute. Hence Section 4 gives some indication of the relation between analysis at the limit (in the continuum) and P&S's asymptotic analysis, revealing some of the complications that the former allows us to suppress.

Most of the paper deals with our first question above — How did they do it? — and focuses mainly on a pure common-value setting. The concluding section, Section 5, briefly addresses our second question, how seriously to take their strong conclusions. This section also contains a brief discussion of some related papers in the literature.

1. A common-value model

There is a continuum of buyers with Lebesgue measure 1 and a continuum of units of a homogeneous, indivisible commodity with Lebesgue measure k, where $0 < k < 1$. The assumption of a continuum of bidders and objects is to be regarded as an idealization for a large economy with intense competition.

Each buyer puts a valuation v on a single unit, and only wants one unit. The value v is the same for all buyers and equals the quality of the commodity. While the true quality is initially unknown, it is common knowledge that Nature has picked v from a finite set $\mathcal{V}$ containing L possible qualities indexed by ℓ, where

$$v_1 > \ldots > v_\ell > \ldots > v_L \geq 0.$$

Each buyer receives a signal $s \in \{0, 1\}$ about the commodity's quality, with the good signal ($s = 1$) being more likely the higher the true quality. Let η_ℓ denote the fraction of agents receiving the good signal when the commodity's quality is v_ℓ. We will assume

no aggregate uncertainty: the fraction of buyers receiving a good signal always equals η_ℓ when the commodity's true quality is v_ℓ, where $\eta_\ell > 0$ for all ℓ and η_ℓ is strictly increasing with v_ℓ.

Regard this as an idealized expression of the law of large numbers, describing the limiting economy resulting when buyers' signals are independent Bernoulli trials with η_ℓ the probability of a good signal. Notice, even for the lowest quality commodity v_L, the probability of a good signal η_L can be high; thus an individual who has received a good signal may still be very much in the dark about the commodity's true quality.

Remark 1. P&S allow for an arbitrary number of possible signals, including the binary case we are focusing on. The binary case, with its coarse 0-1 signal, is the hardest for discovering the commodity's true quality; in this sense, it is the most remarkable. Further, at least insofar as existence goes, there is no effective loss of generality in focusing on this case. To illustrate, suppose there are many possible signals $s \in \mathcal{S}$ with higher signals being more informative (perhaps even a continuum of possible different signals). One can always partition $\mathcal{S}$ into two sets $\mathcal{S}_0$ and $\mathcal{S}_1$ consisting, respectively, of the "relatively bad" and "relatively good" signals. If each agent only acts on the basis of this coarsening of his full information s—whether $s \in \mathcal{S}_0$ or $s \in \mathcal{S}_1$,—we would return to the binary model. If full information aggregation is possible in the coarsened binary model—where individuals effectively are throwing away some of their information,—a fortiori it is possible when they act on the basis of their full information.

Following P&S, we will be interested in the *symmetric* equilibria of a sealed-bid Vickrey auction. A mixed (behavioral) strategy for a player in such an auction can be described by a pair of probability distributions f and g on $\mathbf{R}_+$, where $f(b)$ (respectively, $g(b)$) is interpreted as the probability that an individual who has received a bad (good) signal will bid b. For simplicity, we restrict our attention to mixed strategies (f, g) with finite supports.

We assume that when all buyers play (f, g), then for any possible bid $b \in B$, the measure of buyers bidding b when there are η good signals is given by

$$d(b, \eta) \equiv g(b)\eta + f(b)(1 - \eta).$$

Regard this as another idealized expression of the law of large numbers, describing the distribution of bids in the limiting economy when all buyers *independently* play the mixed strategy (f, g). It follows that the proportion of the population bidding P or more when there are η good signals is given by

$$D(P, \eta) \equiv \sum_{b \geq P} d(b, \eta).$$

Similarly, the proportion of the population bidding strictly more than P when there are η good signals is given by

$$D(P+, \eta) \equiv \sum_{b > P} d(b, \eta).$$

In a Vickrey auction with a finite number of bidders and objects, the k highest bidders get a unit and pay the $k + 1$st highest bid; in case of ties (less than k individuals bidding strictly more than the $k + 1$st highest bid), every individual bidding more than the $k+1$st bid gets a unit for sure, while the remaining units are divided randomly among the individuals who have tied at the $k + 1$st bid. To give the continuum analogue, suppose each buyer plays the mixed strategy (f, g). The analogue of the $k+1$st bid if there are η good signals is the largest bid P such that $D(P, \eta) > k$; let us denote it by $P(\eta)$. It is helpful to regard k as the economy's mean supply and

$D(\cdot,\eta)$ as the economy's mean demand function when there are η good signals. Graphically the mean demand curve is a downward sloping staircase; the Vickrey $k+1$st price when there are η good signals is the lowest price such that mean supply equals mean demand. Figure 1 illustrates for the case when the supports of f and g are contained in $\{b_1, b_2, b_3, b_4\}$.

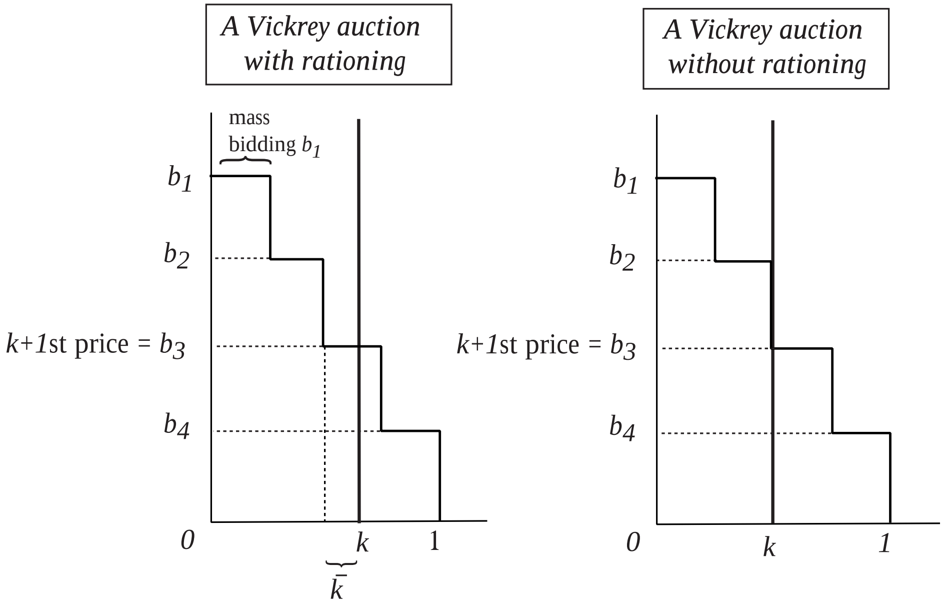

Figure 1

Let $x(b,\eta)$ be the probability that an individual will get a unit if he bids b and η buyers receive good signals. Then in a Vickrey auction, if there are η good signals, $x(b,\eta) = 0$ if $b < P(\eta)$, $x(b,\eta) = 1$ if $b > P(\eta)$, and

$$x(P(\eta),\eta) = \frac{\bar{k}(P(\eta),\eta)}{d(P(\eta),\eta)},$$

where $\bar{k}(P,\eta) \equiv k - D(P+,\eta)$ equals the measure of units left to allocate at the price of P when η buyers receive good signals. In particular, if the quality of the commodity turns out to be v_ℓ, there will be η_ℓ good signals, resulting in the equilibrium price $P(\eta_\ell) \equiv P_\ell$; everyone who bid more than P_ℓ will get a unit for sure, while everyone who bid exactly P_ℓ will get a unit with probability $x(P_\ell, \eta_\ell)$ = the number of units remaining to be rationed divided by the number of claimants.

In the continuum, no one infinitesimal individual can influence the (aggregate) distribution of bids. Hence his expected payoff if be bids b after receiving the signal s is given by

$$\Pi^s(b) \equiv \sum_\ell \mathrm{Prob}(v_\ell \mid s)(v_\ell - P_\ell)x(b,\eta_\ell),$$

where $\mathrm{Prob}(v_\ell \mid s)$ is his probability belief that the commodity is of quality v_ℓ given his signal s. The mixed strategy (f,g) is a *(symmetric) equilibrium* for the Vickrey auction if, for all $b \in \text{supp } f$,

$$\Pi^0(b) \geq \Pi^0(b') \quad \text{for all } b' \in \mathbb{R}_+,$$

and similarly for all $b \in \text{supp } g$,

$$\Pi^1(b) \geq \Pi^1(b') \quad \text{for all } b' \in \mathbb{R}_+.$$

So (f,g) is an equilibrium if no individual sees a profitable deviation from the mixed strategy (f,g). We call (f,g) a *fully revealing* equilibrium if it is an equilibrium such that

$$P_\ell = v_\ell \quad \text{for all } \ell.$$

We begin with two simple examples of fully revealing, symmetric equilibria. In the first, everyone plays a pure strategy;

in the second, which is more typical, mixed strategies are required for full information aggregation. We then generalize to show that, with mixed strategies, full information aggregation is always possible (Theorem 1 below).

Example 1 Suppose a new model car is introduced with unknown quality. There are half as many cars as potential buyers, so the mean supply is $k = .5$. The car is either of high quality $v_1 = 80$ or low quality $v_2 = 40$. Each individual takes the car for a test drive before submitting his bid, resulting in either a good or bad driving experience. The probability of a good experience is $\eta_1 = .8$ (respectively, $\eta_2 = .4$) if the car is of high (low) quality; so $v_\ell = 100\eta_\ell$ for all ℓ. Consider the symmetric, pure-strategy equilibrium in which each individual bids v_1 (resp. v_2) if he has a good (bad) experience; so $g(80) = 1$ and $f(40) = 1$. Then, as illustrated in Figure 2, the equilibrium price of a car will always reflect its true quality: the price will be 80 of $v = v_1$ and 40 if $v = v_2$.

Figure 2 illustrates the basic intuition why price tends to increases with quality in common value auctions: Since the proportion of good signals η_ℓ increases with the quality of the commodity v_ℓ, if people with good signals tend to bid more, the equilibrium price will tend to be higher for higher quality goods. But the path from this basic intuition to full information aggregation is far from obvious. Following the lead of P&S, we will build such a path in our continuum model.

It is interesting to observe that the equilibrium in Example 1 involves all buyers playing pure strategies. By contrast in the model of P&S, since there is a continuum of possible qualities, mixed strategies are essential for full information revelation when each individual only receives a coarse binary signal. In this regard, Example 1 is special; beyond the case of only 2 qualities, mixed strategies are also required in our continuum economy. Indeed, mixing may be required for full revelation even when there are only 2 qualities, as the next example illustrates.

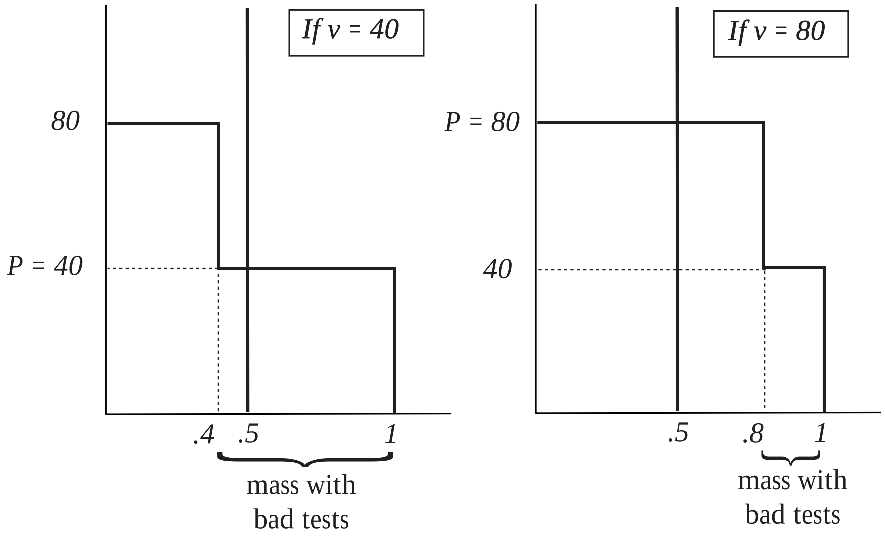

Figure 2

Example 2 (the significance of mixed strategies in P&S) This example is identical to Example 1 except now there are ten times as many potential buyers as cars, so the mean supply $k = .1$. The consequence is that on average more people will have good test drives than there are cars available—even if the car is of low quality. Now the pure strategies in Example 1 would lead to a price of 80 regardless of the car's quality. So these strategies are no longer equilibrium strategies: individuals would end up over-paying if $v = v_2$.

What will happen in equilibrium now? Can there possibly be a fully-revealing equilibrium?

Suppose that individuals with good test drives use a mixed strategy: each bids 80 with probability $g(80) = .2$ and 40 with probability $g(40) = .8$. By contrast, all individuals with bad test drives continue to bid 40, so $f(40) = 1$ (no mixing). Then, as illustrated in Figure 3, the equilibrium price of a car will again always reflect its true quality.

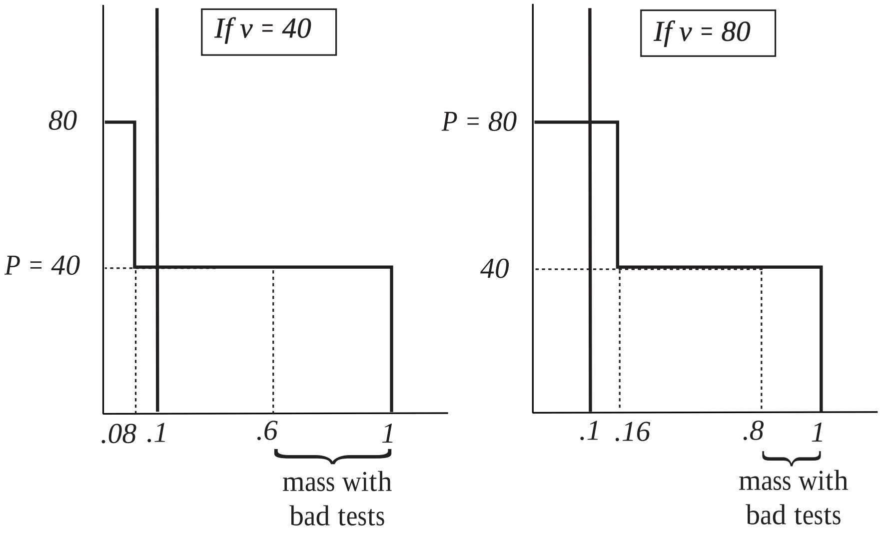

Figure 3

We immediately can generalize, to show that in our continuum model there always exists a symmetric equilibrium that is fully revealing.

Theorem 1 (existence) *There always exists a fully revealing symmetric equilibrium.*

Proof. We will construct an equilibrium mixed strategy (f, g) with supports in $\{v_1, \ldots, v_L\}$. Below $f_\ell \equiv f(v_\ell)$ and $g_\ell \equiv g(v_\ell)$.

First suppose $\eta_1 > k \geq \eta_L$. Let λ satisfy $\eta_\lambda > k \geq \eta_{\lambda+1}$. So for all qualities up to v_λ, the mass of buyers with good signals will be larger than the mean supply k. Define g_ℓ for $\ell < \lambda$ as follows. Choose g_1 to satisfy

$$g_1\eta_1 > k \geq g_1\eta_2;$$

then choose g_2 to satisfy

$$(g_1 + g_2)\eta_2 > k \geq (g_1 + g_2)\eta_3.$$

Continue in this way until choosing $g_{\lambda-1}$ to satisfy

$$(g_1 + g_2 + \ldots + g_{\lambda-1})\eta_{\lambda-1} > k \geq (g_1 + g_2 + \ldots + g_{\lambda-1})\eta_\lambda.$$

[Recall $\eta_1 > \eta_2 > \ldots$; so for all $\ell \leq \lambda - 1$, $g_\ell \in (0, 1)$. Further $g_1 + \ldots + g_{\lambda-1} < 1$.] To complete the distribution, set $g_\lambda = 1 - (g_1 + \ldots + g_{\lambda-1})$ and $g_\ell = 0$ for all $\ell > \lambda$. Also set $f_\ell = 0$ for all $\ell \leq \lambda$. So, thus far, by construction we have:

$$P_\ell = v_\ell \quad \text{for all } \ell \leq \lambda.$$

To complete the construction of f, choose $f_{\lambda+1}$ to satisfy

$$\eta_{\lambda+1} + f_{\lambda+1}(1 - \eta_{\lambda+1}) > k \geq \eta_{\lambda+2} + f_{\lambda+1}(1 - \eta_{\lambda+2});$$

then choose $f_{\lambda+2}$ to satisfy

$$\eta_{\lambda+2} + (f_{\lambda+1} + f_{\lambda+2})(1 - \eta_{\lambda+2}) > k \geq \eta_{\lambda+3} + (f_{\lambda+1} + f_{\lambda+2})(1 - \eta_{\lambda+3}).$$

Proceed in this fashion until choosing f_{L-1} to satisfy

$$\eta_{L-1} + (f_{\lambda+1} + \ldots + f_{L-1})(1 - \eta_{L-1}) > k \geq \eta_L + (f_{\lambda+1} + \ldots + f_{L-1})(1 - \eta_L).$$

To complete the probability distribution set f_L equal to $1 - (f_1 + \ldots + f_{L-1})$. Notice, by construction of f, we now also have ensured

$$P_\ell = v_\ell \quad \text{for all } \ell > \lambda.$$

Thus, if everyone plays (f, g), there will be full information aggregation. Further, it is an equilibrium choice for everyone to play (f, g) since any deviation would yield zero expected profit.

The other possible cases are similar. If even $\eta_L > k$, so there are always more buyers with good signals than units available (like in Example 2), set $\lambda = L$ and define g_ℓ for all ℓ as above; set $f_\ell = 0$ for all $\ell < L$ and set $f_L = 1$. In this case, (f, g) is fully revealing with all individuals who receive a bad signal always bidding as if the commodity were of the lowest quality.

Only one case remains. If $\eta_1 \leq k$, set $g_1 = 1$ and set $g_\ell = 0$ for all $\ell > 1$; so all individuals with a good signal always bid as if the commodity were of the highest quality, leaving $k - \eta_\ell \geq 0$ units to be allocated among individuals with bad signals (for every possible ℓ). Analogous to the first case, choose f_1 to satisfy $\eta_1 + f_1(1 - \eta_1) > k \geq \eta_2 + f_1(1 - \eta_2)$, and so on until f_{L-1}; then let f_L have all the remaining probability weight. Given our maintained assumption that $k < 1$, all f_ℓ will be positive in this case, and again (f, g) will be a fully revealing equilibrium. □

The proof of existence is simple and constructive, a payoff from reformulating the P&S model in a continuum setting. The construction shows the basic method for "pulling the rabbit out of the hat": Individuals with good signals tend to bid more in (f, g), hence the equilibrium price tends to increase with the commodity's true value. Choosing the probability weights in f and g with sufficient care—as in the proof of Theorem 1,—the equilibrium price increases *exactly* in tandem with the commodity's true value, leading to full information aggregation. Thus the equilibrium price reveals the join of everyone's information, although any one buyer when submitting his sealed bid—given his very coarse binary signal—is basically in the dark about the commodity's exact quality. This is the fundamental insight underlying the asymptotic equilibrium in P&S.

2. Uniqueness

Theorem 1 shows there exist equilibria that are fully revealing. In this section we address the uniqueness question:

Will all *equilibria be fully revealing?*

P&S rightfully emphasize the importance of uniqueness (in the sense used above). If there were a multiplicity of equilibria, some revealing and some not, then one would be tempted to conclude that full information aggregation is not a fundamental property of the market, but rather a consequence of a special choice of equilibrium. Uniqueness shows the invisible hand of the market (competition) will *always* ensure the probability weights in (f, g) are just right! So it is interesting to see what also drives this second conclusion, again in our simpler continuum setting.[1]

---

[1] Uniqueness in the above sense does not required there to be only one equilibrium (f, g). Indeed, the proof of Theorem 1 showed there are many fully revealing symmetric equilibria. This multiplicity is due in part to our assuming, for simplicity, that there are only a finite number of possible qualities. By contrast, P&S assume there is a continuum of possible qualities, and they show there is only one Bayesian equilibrium in their model.

In this section we prove that, under the regularity assumption that people with good signals tend to bid more, all equilibria will be fully revealing. What does it mean that people with good signals tend to bid more? To formalize, let $F(b) = \sum_{b' \geq b} f(b')$, so F equals one minus f's cumulative distribution function; similarly let $F(b+) = \sum_{b' > b} f(b')$. Define $G(b)$ and $G(b+)$ analogously. Recall $d(b, \eta) \equiv g(b)\eta + f(b)(1 - \eta) = f(b) + (g(b) - f(b))\eta$. Hence mean demand at P if there are η good signals is given by:

$$\begin{aligned} D(P, \eta) &\equiv \sum_{b \geq P} d(b, \eta) = \sum_{b \geq P} [f(b) + (g(b) - f(b))\eta] \\ &= \sum_{b \geq P} f(b) + (\sum_{b \geq P} g(b) - \sum_{b \geq P} f(b))\eta \\ &= F(P) + [G(P) - F(P)]\eta. \end{aligned}$$

Similarly, a little algebra shows:

$$D(P+, \eta) = F(P+) + [G(P+) - F(P+)]\eta.$$

Definition 1. *People with good signals tend to bid more* in (f, g) means that:

$$G(b) - F(b) \geq 0 \quad \text{for all } b \in \mathbb{R}_+.$$

A symmetric equilibrium (f, g) is *regular* if both f and g have finite supports, and people with good signals tend to bid more.

The restriction to mixed strategies with finite supports is purely for simplicity, the real bite of "regularity" is the restriction to strategies in which people with good signals tend to bid more. We say "tend to" because regularity does not rule out the possibility that $G(b) = F(b)$ for some bids b. In particular, it does not preclude the possibility of everyone bidding the same price P regardless of his signal, so $f(P) = g(P) = 1$. But, among other things, we will see that such a strategy will not be played in equilibrium; market forces will lead some buyers to deviate from P (Lemma 3 below).

The main result of this section is

Theorem 2 (uniqueness) *Any regular equilibrium will be fully revealing.*

Remark 2. Restricting our attention to regular equilibria is a compromise between generality and transparency. Theorem 2 leaves open the possibility that there may exist equilibria that are not fully revealing when people play non-regular strategies; we have not been able to rule out this possibility. The problem is that, if $G(P) < F(P)$, mean demand will be inversely related to quality since $D(P, \eta) = F(P) + [(G(P) - F(P)]\eta$. If one accepts that it is sensible to assume people with good signals will tend to bid more, then Theorem 2 tells us that all sensible equilibria will be fully revealing. From an economic point of view, this is satisfying enough. Mathematically, it will be seen that the main purpose of restricting ourselves to regular equilibria is to prove Lemma 4 below, that pooling leads to a winner's curse. If people with bad signals bid *more*, this conclusion is far from obvious. Strange possibilities arise, whose analysis would take us into complex and treacherous regions, under-cutting our desire to reveal in a simple manner the basic logic behind the P&S conclusions.

To prove uniqueness, some extra notation will be helpful. Throughout this section we assume (f, g) is a fixed equilibrium. Given everyone plays (f, g), let the function $\mathcal{P} : \mathcal{V} \to \mathbb{R}_+$ specify the equilibrium (that is, k + 1st) price for each possible quality. The range of this function, $\mathcal{P}^* \equiv \{\mathcal{P}(v_\ell) : \ell = 1, \ldots, L\}$, is the set of possible equilibrium prices. For any $P \in \mathcal{P}^*$, $T(P) \equiv \{\ell : v_\ell \in \mathcal{P}^{-1}(P)\}$ is the set of qualities (more precisely, quality-indices) that are possible if the equilibrium price turns out to be P. We will call $P \in \mathcal{P}^*$ *separating* if $T(P)$ is a singleton (no ties); if $T(P)$ contains more than one quality, we will say that P involves *pooling*.

A *separating equilibrium* is an equilibrium in which all prices $P \in \mathcal{P}^*$ are separating.

Lemma 1 *Any separating equilibrium is fully revealing.*

Proof. For any possible quality v_ℓ, recall $P_\ell = \mathcal{P}(v_\ell)$. There are two possibilities to rule out, that in equilibrium $P_\ell < v_\ell$ or $P_\ell > v_\ell$ for some ℓ; we will show that either situation leads to a profitable deviation.

In a Vickrey auction the probability of winning if one bids P_ℓ and the commodity is of quality v_ℓ, namely $x_\ell \equiv x(P_\ell, \eta_\ell)$, is strictly less than 1 since $x(b, \eta_\ell) = 1$ only if one bids $b > P_\ell$. Thus if $P_\ell < v_\ell$ for some ℓ, any individual who bids P_ℓ could benefit by bidding epsilon more, choosing ϵ sufficiently small not to affect his probability of winning any quality selling for more than P_ℓ. This deviation raises his probability of winning if the true quality is v_ℓ from $x_\ell < 1$ to 1, thus raising his expected payoff. So we can conclude $v_\ell \leq P_\ell$ for all ℓ, that is, the best any buyer can hope for in a separating equilibrium is a zero expected payoff.

On the other hand, if $P_\ell > v_\ell$ for some ℓ, then someone will win with a positive probability when $v = v_\ell$ and end up paying more than the commodity is worth. Such an individual could increase his expected payoff by bidding v_L, thus ensuring himself of a non-negative expected payoff rather than an expected loss. $\square$

Given Lemma 1, we need only worry about equilibria that involve pooling. A simple but useful fact is that if P is an equilibrium price and if someone bids P after receiving the signal s, his expected profit conditional on the true quality being in $T(P)$ must be nonnegative:

Lemma 2 *If $P \in \mathcal{P}^*$ and an individual bids P with positive probability after receiving the signal s:*

$$\pi^s(P) \equiv \sum_{\ell \in T(P)} \text{Prob}(v_\ell \mid s)(v_\ell - P)x(P, \eta_\ell) \geq 0.$$

Proof. If $\pi^s(P)$ were negative for some $P \in P^*$, the individual could increase his expected payoff $\Pi^s(P)$ by bidding ϵ less than P, choosing ϵ sufficiently small not to affect his probability of winning any quality selling for less than P. $\square$

Among other things, the following lemma rules out pure pooling equilibria in which everyone always bids P, so $f(P) = g(P) = 1$, hence $F(P+) = G(P+) = 0$ while $F(P) = G(P) = 1$.

Lemma 3 (people with good signals bid strictly more) *If (f, g) is a regular equilibrium and $P \in \mathcal{P}^*$, then $F(P+) = G(P+)$ implies $g(P) > f(P)$, hence $G(P) > F(P)$.*

Proof. Suppose the contrary, that there is a $P \in \mathcal{P}^*$ such that $F(P+) = G(P+)$ and $G(P) = F(P)$. Then $D(P, \eta) \equiv F(P) - (G(P) - F(P))\eta = F(P)$, so mean demand will be insensitive to quality at P. This implies *all* qualities will have P as their equilibrium ($k+1$st) price; in other words, $\mathcal{P} = \{P\}$, a singleton, and there is a pure pooling equilibrium at P. Similarly, since by assumption $D(P+,\eta) \equiv F(P+) - (G(P+) - F(P+))\eta$ = F(P+), demand above P will be invariant to quality; so the probability of winning $x(P,\eta) \equiv \dfrac{k - D(P+,\ \eta)}{d(P,\eta)}$ will be a constant for all η, say x, where $x \in (0,1)$.

Lemma 2 implies $\pi^s(P) \geq 0$ for both $s = 0$ and $s = 1$. Since the probability of winning is a constant, $\pi^s(P) = xE(v \mid s)$ for all s. But an individual with a good signal puts more probability weight on better qualities, hence $E(v \mid s = 1) > E(v \mid s = 0)$, so $\pi^1(P) > \pi^0(P) \geq 0$. Since $x < 1$, it follows that an individual with signal $s = 1$ could increase his expected payoff by bidding an epsilon more than P—ensuring himself of a unit by avoiding being rationed. □

Using the previous lemma, we can now show that the probability of winning $x(P,\eta)$ whenever there is pooling at P is strictly decreasing in η (quality). In other words, pooling leads to a winner's curse.

Lemma 4 (pooling leads to a winner's curse) *Let (f,g) be a regular equilibrium. If $P \in \mathcal{P}^*$ involves pooling, then*

$$\frac{\partial x(P,\eta_\ell)}{\partial \eta} < 0 \text{ for all } \ell \in T(P).$$

Proof. Recall

$$x(P,\eta) \equiv \frac{k - D(P+,\eta)}{d(P,\eta)},$$

where $d(P,\eta) = f(P) + (g(P) - f(P))\eta$ and $D(P+,\eta) = F(P+) + (G(P+) - F(P+))\eta$. Hence differentiating shows

$$\text{sign}\left(\frac{\partial x(P,\eta)}{\partial \eta}\right) = -\,\text{sign}\left(x(P,\eta)\big(g(P) - f(P)\big) + (G(P+) - F(P+)\right). \tag{a}$$

Since (f,g) is regular, both $G(P+) - F(P+) \geq 0$ and $G(P) - F(P) \geq 0$, where the latter can be re-written

$$(g(P) - f(P) + (G(P+) - F(P+)) \geq 0. \tag{b}$$

There are two cases to consider. First, if $G(P+) - F(P+) > 0$, then (a)–(b) imply there will always be a winner's curse, even if $g(P) - f(P) < 0$. (Remember $x(P,\eta_\ell) < 1$ for all $\ell \in T(P)$ since one only wins for sure in a Vickrey auction by bidding strictly more than P.)

On the other hand, if $G(P+) - F(P+) = 0$, Lemma 3 implies $g(P) - f(P) > 0$, hence (a) immediately implies $\dfrac{\partial x(P,\eta_\ell)}{\partial \eta} < 0$ *unless* $x(P,\eta_\ell) = 0$ as in the right panel of Figure 1. So we need only show $x(P,\eta_\ell) > 0$ for all $\ell \in T(P)$. If $G(P+) = F(P+) = 0$, then no one is bidding above P, so $x(P,\eta_\ell) > 0$ is immediate. If $G(P+) = F(P+) > 0$ and $x(P,\eta_\ell) = 0$ for some $\ell \in T(P)$, then $x(P,\eta_\ell) = 0$ for all $\ell \in T(P)$ since $G(P+) = F(P+)$ implies the mean demand above P is invariant with quality; furthermore, it also implies P must be the largest equilibrium price. But since some people with good and bad signals are bidding more than P:

$$\sum_{\ell \in T(P)} \text{Prob}(v_\ell \mid s)(v_\ell - P) \geq 0 \quad \text{for all } s.$$

(Otherwise an individual with signal s would bid an epsilon below P rather than an epsilon above P, and thus avoid a loss.) Furthermore, since $E(v \mid s = 1) > E(v \mid s = 0)$, the displayed inequality must be strict when $s = 1$. But then $g(P) > 0$ implies some people with $s = 1$ could increase their expected payoff by bidding more than P, increasing there probability from 0 to 1 of getting a unit when the quality turns out to be in $T(P)$. We conclude, $x(P, \eta_\ell) > 0$ for all $\ell \in T(P)$ when $G(P+) = F(P+)$. □

The winner's curse is the market force that tends to eliminate pooling. Before showing this in general (Lemma 5 below), we illustrate with an example.

Example 3 (The winner's curse in action) Consider the following variation on Examples 1 and 2, still with $L = 2$. As before $\eta_1 = .8$, $\eta_2 = .4$, $v_1 = 80$, and $v_2 = 40$. But now $k = .9$, so there are more units than individuals with good signals, even when $v = v_1$.

Suppose everyone's prior is that each signal is equally likely. Choose P to satisfy:

$$\text{Prob}(v = 80 \mid s = 0)(80 - P)(\frac{1}{2}) + \\ \text{Prob}(v = 40 \mid s = 0)(40 - P)(\frac{5}{6}) = 0.$$

Consider the mixed strategy in which everyone with a bad signal bids P and everyone with a good signal bids 80, so $f(P) = 1$ and $g(80) = 1$. As illustrated in Figure 4, this strategy, if played by everyone, leads to a price of P regardless of quality (a pooling outcome). Notice P is chosen to ensure that individuals with bad signals earn a zero expected profit by bidding P. Can this be a symmetric equilibrium? The individuals with good signals have no incentive to deviate. They get a unit with probability 1 regardless of v; and since they put a larger probability weight on $v = 80$ than do individuals with a bad signal, their expected payoff is strictly positive. What about individuals with bad signals? It might seem that they too have no incentive to deviate since their expected payoff is zero when bidding P. But this is forgetting the winner's curse! Notice the probability of winning is strictly *less* when $v = 80$ than when $v = 40$. Hence by bidding a bit more than P, an individual with a bad signal could ensure himself a positive expected payoff. That is, he would be better off leaving the pool.

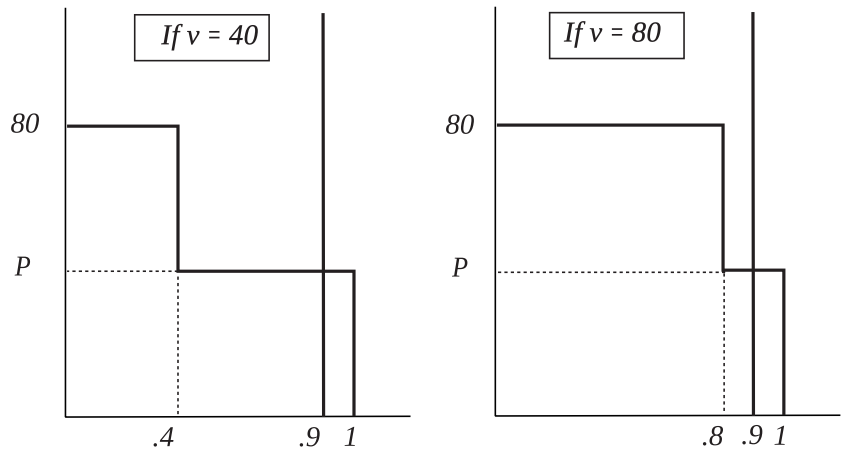

Figure 4

Lemma 5 (separating to avoid the winner's curse) *In any regular equilibrium (f, g), there is no pooling.*

Proof. Assume the contrary, that (f, g) leads to pooling at some price $P \in \mathcal{P}^*$. We will show a contradiction. Let $T = T(P)$, the qualities possible when the price is P. Suppose T

contains M elements ℓ_m, indexed by $m = 1, \ldots, M$, where $v_{\ell_1} > v_{\ell_2} > \ldots > v_{\ell_M}$.

By Lemma 2, for anyone who bids P after receiving the signal s:

$$\pi^s(P) \equiv \sum_{\ell \in T} \text{Prob}(v_\ell \mid s)(v_\ell - P)x(P, \eta_\ell)] \geq 0,$$

where $x(P, \eta_\ell) > 0$ for all $\ell \in T$ except possibly $\ell = \ell_1$ (using Lemma 4). Notice $v_\ell < P$ for some $\ell \in T$. (Otherwise, $v_\ell \geq P$ for all $\ell \in T$ with strict inequality for at least one member; since $x(P, \eta_\ell) < 1$ for all $\ell \in T(P)$, an individual with signal s could increase his expected payoff by bidding an epsilon more than P, avoiding being rationed.) Similarly, $v_\ell > P$ for some $\ell \in T$ since $x(P, \eta_{\ell_M}) > 0$. Hence there must be a cutoff m, say m^*, where $1 < m^* < M$, such that $v_{\ell_m} \geq P$ iff $m \leq m^*$. The above displayed inequality can be re-written

$$\pi^s(P) \equiv \left[\sum_{m \leq m^*} \text{Prob}(v_{\ell_m} \mid s)(v_{\ell_m} - P)x(P, \eta_{\ell_m}) \right] + \left[\sum_{m > m^*} \text{Prob}(v_{\ell_m} \mid s)(v_{\ell_m} - P)x(P, \eta_{\ell_m}) \right] \geq 0.$$

The first bracketed term must be strictly positive, otherwise we would have a contradiction; hence $x(P, \eta_{\ell_{m^*}}) > 0$. Dividing by $\alpha \equiv x(P, \eta_{\ell_{m^*}})$ shows

$$\pi^s(P) \leq \frac{\pi^s(P)}{\alpha} < \sum_{\ell \in T} \text{Prob}(v_\ell \mid s)(v_\ell - P),$$

where the last inequality follows from the fact that the probabilities $x(P, \eta_{\ell_m})$ are increasing in m (Lemma 4), hence the weights $x(P, \eta_{\ell_m})/\alpha$ are all less than or equal to 1 for $m \leq m^*$, while they are all greater than 1 for $m > m^*$. That is, an individual with signal s would be better off bidding an epsilon more than P, a contradiction. $\square$

Theorem 2 now follows since Lemma 5 tells us that all regular equilibria are separating, while Lemma 1 tells us that any separating equilibrium must be fully revealing.

3. An extension: Two-dimensional signals

Pesendorfer and Swinkels in their 1998 sequel (P&S2 for short) extend their 1997 model to one in which each individual receives both a common-value and a private-value signal. P&S2 gives conditions under which, asymptotically, there will be both (i) full information aggregation *and* (ii) an efficient allocation of resources. By contrast, in a pure common-value setting it does not matter which buyers get units of the commodity; hence the question of allocative efficiency does not arise. What drives this even more remarkable conclusion? Is it not likely that some high private-value buyers will get bad common-value signals, hence bid without vigor because their signals mislead them, and consequently (inefficiently) end up with none of the commodity?

We can use the continuum to help reveal some more of the Pesendorfer and Swinkels magic. Formally, suppose now that each buyer's utility depends on both the common-value parameter v and a private-value parameter t. In particular, if a buyer must pay P to get a unit, his final utility from a commodity with common-value v when he is of type t is given by

$$V(v,t) - P,$$

where the function V is strictly increasing in both its arguments. A simple example is the additive function $V(v,t) = v + t$. We will continue to assume that there are L possible common values indexed by ℓ; and now we also will assume there are J possible private values indexed by j, with

$$t_1 > \ldots > t_j > \ldots > t_J > 0.$$

Each buyer knows his private value t_j and receives a signal s—as before—about the common value.

The key assumptions P&S2 makes are that (i) buyers' private values are independently drawn from a commonly-known distribution, and (ii) the probability of receiving a good or bad signal is independent of an individual's private value. Let μ_j be the probability that an individual is assigned private value t_j, where $\sum_j \mu_j = 1$. To give an idealized continuum expression to these assumptions, notice (i) plus the law of large numbers leads to

no aggregate uncertainty about private values: the Lebesgue measure of buyers with private value t_j is commonly-known to be μ_j, for $j = 1, \ldots, J$;

and (ii) plus the law of large numbers leads to

disintegrability: if the commodity's common value is v_ℓ, a fraction η_ℓ of buyers with private value t_j will receive a good signal, for any ℓ and all j.

Below, disintegrability will play the same role as "no aggregate uncertainty" played in our analysis of the pure common-value case. Disintegrability implies that, to determine the commodity's true quality, it is sufficient to know the number of good signals that the individuals of any *one* type j receive.

Extend the definition of a symmetric equilibrium in an obvious way: let (f_j, g_j) be a mixed strategy for each buyer of type j, and call $(f_j, g_j)_{j=1\ldots J}$ an equilibrium if no buyer has a profitable deviation.

We assume the distribution of private values $(\mu_j)_{j=1\ldots J}$ is such that, in a pure private-values setting, mean demand and supply would intersect at a unique price (that is, they would not intersect over an interval). This implies there is a type j^* such that

$$\sum_{j<j^*} \mu_j < k < \sum_{j \le j^*} \mu_j.$$

Call individuals of type j^* the "marginal buyers" since, in a pure private-values setting, the Walrasian-equilibrium price would equal t_{j^*}. There is *allocative efficiency* if all k units of the commodity are distributed, each individual with private value $t_j > t_{j^*}$ gets a unit (with probability one), and each individual with private value $t_j < t_{j^*}$ does not get a unit.

Let $v_\ell{}^* \equiv V(v_\ell, t_{j^*})$, the value to a marginal buyer of a commodity with common-value v_ℓ. We will say the equilibrium is *fully revealing* if the price always equals the value to a marginal buyer:

$$P_\ell = v_\ell{}^* \quad \text{for all } \ell.$$

Theorem 3 (existence) *In the 2-dimensional model, there exists an equilibrium* $(f_j, g_j)_{j=1\ldots J}$ *that is both fully revealing and allocatively efficient.*

Like the proof of Theorem 1, the proof of this theorem is constructive and simple—another demonstration of the usefulness of the continuum. The proof follows closely the intuition given in P&S2. Let $\bar{b} \equiv V(v_1, t_1)$, that is, an individual's willingness to pay for a unit when his private value is highest and the commodity's common value is also highest; similarly, let $\underline{b} \equiv V(v_L, t_J)$, the willingness to pay when both the private and common values are lowest. In the constructed equilibrium, the supports of all individuals' strategies will be contained in $\{\bar{b}, \underline{b}, v_1{}^*, v_2{}^*, \ldots, v_L{}^*\}$ and only individuals of type j^* will randomize. In particular, let all individuals with private values $t_j > t_{j^*}$ bid $\bar{b}$ regardless of their signals, so

$$f_j(\bar{b}) = g_j(\bar{b}) = 1 \quad \text{for all } j < j^*. \tag{†}$$

Similarly, let all individuals with private values $t_j < t_{j^*}$ always bid $\underline{b}$, so

$$f_j(\underline{b}) = g_j(\underline{b}) = 1 \quad \text{for all } j > j^*. \tag{‡}$$

Finally, let each individual of type j^* play a mixed strategy very similar to the one constructed in the proof of Theorem 1 except that (i) the economy has only $k^* \equiv k - \sum_{j<j^*} \mu_j$ rather than k units (reflecting the fact that individuals of types $j < j^*$ have locked into some of the units already), (ii) the mass of buyers equals μ_{j^*} instead of unity (reflecting the fact that only the bidding of individuals of type j^* will determine the equilibrium price), and (iii) the value of quality ℓ to active bidders is $v_\ell{}^*$ rather than v_ℓ (incorporating the private values of the active bidders).

Figure 5 illustrates the basic idea behind Theorem 3. The construction of the mixed strategy of individuals of type j^* ensures the equilibrium price will always be fully revealing, just as in the proof of Theorem 1. As illustrated in the figure, only the bidding of type j^* individuals determines the commodity's equilibrium price; but disintegrability implies *their* bidding is sufficient for full information aggregation. The equilibrium is also allocatively efficient because (†) ensures all individuals with private values greater than t_{j^*} will get an object. (The highest bid that an individual of type j^* will make is $V(v_1 + t_{j^*}) < \bar{b}$.) Similarly, (‡) ensures all individuals with private values less than t_{j^*} will not get an object. (The lowest bid that an individual of type j^* will make is $V(v_L + t_{j^*}) > \underline{b}$.)

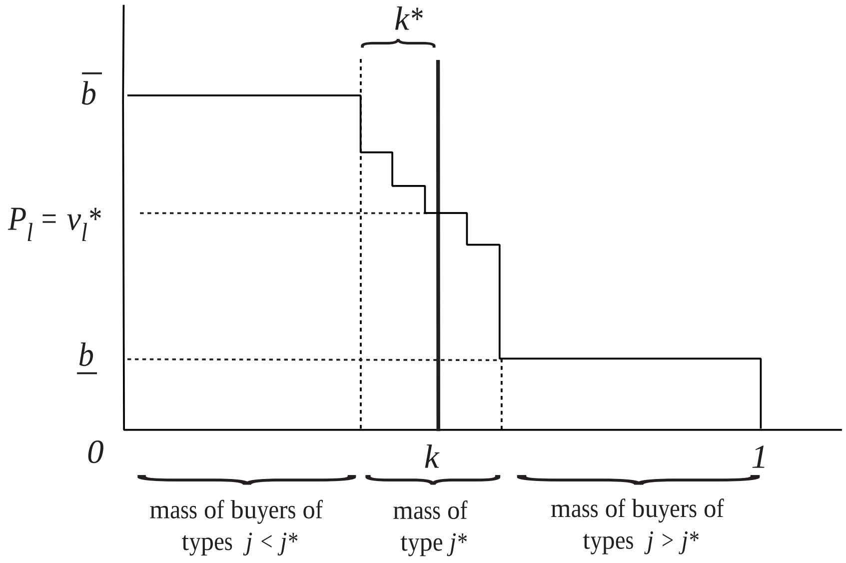

Figure 5

The complete proof of Theorem 3, the 2-dimensional extension of Theorem 1 above, is given in the appendix. It spells out the details of type j^* buyers' mixed strategy. We leave open the question whether uniqueness (Theorem 2) also extends to the 2-dimensional case. Based on the results in P&S2, we conjecture that it does.

Remark 3. It is of some interest to observe that in the continuum there is no difficulty in proving existence, indeed the proof of Theorem 3 shows it is easy. By contrast, Pesendorfer and Swinkels are unable to prove existence for the 2-dimensional version of their finite-agent model, there being substantial technical difficulties. What accounts for the difference? It may be that our restriction to a finite number of possible private values is important for showing existence even in the continuum: P&S2 assumes a continuum of possible private values.

Remark 4. How seriously should we take Theorem 3? How successful are Pesendorfer and Swinkels in extending their results to the 2-dimensional case? P&S2's assumption that individuals' private values are independently drawn from a *commonly-known* distribution—which is reflected in our assumption that in the continuum there is no aggregate uncertainty about private values—is a very strong assumption. It goes against the grain of Hayek's idea that the world is full of decentralized knowledge. In this spirit, it seems much more realistic to assume the opposite, that each individual knows his own tastes but not the tastes of others, not even the distribution of others' tastes. In particular, suppose Nature *first* picks a distribution for selecting individuals' private values from a large set of possible distributions—a choice that is not common knowledge,—and then Nature independently picks individuals' private values from her chosen distribution. In this more realistic model, there could remain a huge amount of aggregate uncertainty about the distribution of private values, even in the continuum. In this revised setting, since no one buyer knows the entire distribution of private values $(\mu_j)_{j=1\dots J}$, how is he to know whether to bid actively or not, that is, whether his private value t_j happens to be the economy's marginal private value t_{j^*}? Since Pesendorfer and Swinkels's 2-dimensional extension depends crucially on the assumption of no asymptotic aggregate uncertainty about private values, we view their 2-dimensional extension as more a negative than a positive result.

4. Some asymptotics

In this section we return to the pure common-value model. Our goal is to clarify the connection between analysis at the limit (our continuum model) and P&S's asymptotic analysis. The basic lesson is that analysis at the limit is a complement, not a substitute, for asymptotics. For example, knowing (f, g) is a symmetric equilibrium in our continuum economy does not imply it will continue to be a symmetric Bayesian equilibrium for a sufficiently large but finite economy. Until one reaches the limit, one must deal with aggregate uncertainty—annoying variance—arising from two sources, one exogenous and the other endogenous:

(i) buyers receiving *independent* signals, and

(ii) buyers playing *independent* mixed strategies.

Indeed, the simplicity of our analysis at the limit arises precisely from avoiding these two sources of aggregate uncertainty.

While an equilibrium (f, g) in the continuum may not be an equilibrium for a large but finite economy, we will show that, for any $\epsilon > 0$, all buyers playing (f, g) is at least an ϵ-equilibrium for a sufficiently large finite economy (Theorem 4 below). This may be of some interest given the fact that an exact Bayesian equilibrium for a finite economy can be very delicate and hard to find (one of the lessons from P&S). Following the path leading to Theorem 4 also has value in itself: It will allow the reader to see concretely the difference between analysis at the limit and asymptotic analysis, revealing some of the complications that our continuum analysis has allowed us to suppress.

To proceed, consider a fixed *finite* economy with n individuals indexed by i and kn units of the indivisible commodity; so the mean supply is $kn/n = k$, just as in the continuum. As in our continuum economy, in the finite economy there are L possible qualities indexed by ℓ; and if the true quality is v_ℓ, each individual's signal s_i is a Bernoullian random variable with η_ℓ the probability of a good signal. Unlike our continuum economy, this finite economy faces the above two sources of aggregate uncertainty. In particular, the commodity's true quality no longer uniquely determines the distribution of signals, hence we will have to pay attention to the actual profile of signals received by individuals, denoted by $\mathbf{s} = (s_1, \ldots, s_i, \ldots, s_n)$. Further, even when all buyers play the same mixed strategy (f, g), the distribution of bids is no longer perfectly correlated with the distribution of signals $\mathbf{s}$; so we also will have to pay attention to the actual profile of bids chosen by individuals, denoted by $\mathbf{b} = (b_1, \ldots, b_i, \ldots, b_n)$. Putting it together, the logic of the finite model is:

$$v_\ell \Longrightarrow \mathbf{s} \Longrightarrow \mathbf{b},$$

where each "implication" involves some uncertainty (variance). But once the profile of bids $\mathbf{b}$ has been chosen, the mean demand is completely determined, hence the $k+1$st price is completely determined.

After receiving his signal s_i, each buyer i submits a sealed bid $b_i \in \mathbb{R}_+$, resulting in a profile of bids $\mathbf{b}$. Let $\hat{P}(\mathbf{b})$ denote the $k+1$st highest bid in the profile $\mathbf{b}$. [This is the finite analogue of $P(\eta)$.] In case of ties every individual bidding more than $\hat{P}(\mathbf{b})$ gets a unit for sure, and the remaining units are divided randomly among the individuals bidding $\hat{P}(\mathbf{b})$. Let $\hat{x}_i(\mathbf{b})$ be the probability that i will get a unit if individuals bid $\mathbf{b}$. [This is the finite analogue of $x(b, \eta)$.] Individual i's expected utility payoff in the auction if individuals bid $\mathbf{b}$ and the object is of quality v is:

$$(v - \hat{P}(\mathbf{b}))\hat{x}_i(\mathbf{b}).$$

To remain as close as possible to the continuum model, we will continue to focus on mixed strategies with finite supports. A mixed (behavioral) strategy for i in a finite Vickrey auction remains a pair of probability distributions f and g on $\mathbb{R}_+$, with $f(b)$ (respectively $g(b)$) interpreted as the probability that i will bid b if he receives a bad (good) signal. Let $\hat{\Pi}^s(b)$ denote the expected payoff to a typical individual in the n-buyer economy if he receives the signal s, bids b, and everyone else plays (f, g). Thus, taking the typical individual to be individual 1:

$$\hat{\Pi}^s(b) = \sum_\ell \text{Prob}(v_\ell \mid s)\left[\sum_{\mathbf{b}} \text{Prob}(\mathbf{b} \mid b,\ v_\ell)(v_\ell - \hat{P}(\mathbf{b}))\hat{x}_1(\mathbf{b})\right],$$

where $\text{Prob}(\mathbf{b} \mid b,\ v_\ell)$ is the probability of the profile of bids $\mathbf{b}$ given the commodity is of quality v_ℓ and individual 1 bids b while all others play the mixed strategy (f, g). This is the finite analogue of $\Pi^s(b)$, whose formula we copy here for easy comparison:

$$\Pi^s(b) = \sum_\ell \text{Prob}(v_\ell \mid s)(v_\ell - P_\ell)x(b, \eta_\ell).$$

A *(symmetric) Bayesian equilibrium* for a finite Vickrey auction is a mixed strategy (f, g) such that, for all b in the support of f:

$$\hat{\Pi}^0(b) \geq \hat{\Pi}^0(b') \quad \text{for all } b' \in \mathbb{R}_+,$$

and similarly for all b in the support of g:

$$\hat{\Pi}^1(b) \geq \hat{\Pi}^1(b') \quad \text{for all } b' \in \mathbb{R}_+.$$

So no one individual has a profitable deviation.

Suppose the commodity's true quality is v_ℓ. Define the random variable

$$\hat{\eta}_\ell(\mathbf{s}) \equiv \frac{\#\{i : s_i = 1\}}{n},$$

the mean number of good signals in the profile $\mathbf{s}$. While $\hat{\eta}_\ell$ is a random variable since $\mathbf{s}$ is one, the law of large numbers implies that the mean number of good signals will almost always converge to η_ℓ in the sense that

$$\text{Prob}(\lim_{n\to\infty} \hat{\eta}_\ell = \eta_\ell) = 1.$$

Continue to suppose the commodity's true quality is v_ℓ, and now also suppose everyone plays (f, g). To address the uncertainty of individuals' bidding in the finite case—even given the commodity's true quality is v_ℓ and given everyone plays (f, g),—it will be convenient to define some more random variables resulting from the joint randomness of $\mathbf{s}$ and $\mathbf{b}$:

$$\begin{aligned}
\hat{f}(b)(\mathbf{s}, \mathbf{b}) &\equiv \frac{\#\{i : s_i = 0 \text{ and } b_i = b\}}{\#\{i : s_i = 0\}}, \\
\hat{g}(b)(\mathbf{s}, \mathbf{b}) &\equiv \frac{\#\{i : s_i = 1 \text{ and } b_i = b\}}{\#\{i : s_i = 1\}}, \\
\hat{d}_\ell(b)(\mathbf{s}, \mathbf{b}) &\equiv \hat{g}(b)(\mathbf{s}, \mathbf{b})\hat{\eta}_\ell(\mathbf{s}) + \hat{f}(b)(\mathbf{s}, \mathbf{b})(1 - \hat{\eta}_\ell(\mathbf{s}) \\
&= \frac{\#\{i : b_i = b\}}{n}, \\
\hat{D}_\ell(P)(\mathbf{s}, \mathbf{b}) &\equiv \sum_{b \geq P} \hat{d}_\ell(b)(\mathbf{s}, \mathbf{b}).^2
\end{aligned}$$

The random variables $\hat{f}(b)$ and $\hat{g}(b)$ are, of course, the finite analogues of the non-random quantities in the continuum $f(b)$ (the mass of individuals with bad signals bidding b) and $g(b)$ (the mass with good signals bidding b). Similarly, the random variables $\hat{d}_\ell$ and $\hat{D}_\ell$ are the finite analogues of the non-random continuum quantities $d(b, \eta_\ell) \equiv d_\ell(b)$ (the total mass bidding b if the commodity is of quality ℓ) and $D(P, \eta_\ell) \equiv D_\ell(P)$ (the total mass bidding P or more).

The law of large numbers implies that each of these random variables will converge in probability to its corresponding deterministic (continuum) value as $n \to \infty$. We also will need to know something about the "speed of convergence." Focusing on individuals' mean demand, a standard application of Chebyshev's inequality[3] leads to the conclusion that there is a constant M such that for all $c > 0$ and any P:

$$\text{Prob}(\mid \hat{D}_\ell(P) - D_\ell(P) \mid \geq c) \leq \frac{M}{c^2 n} \quad \text{for all } n.$$

---

[2] If the denominator of $\hat{f}(b)(\mathbf{s}, \mathbf{b})$ above happens to equal zero, define $\hat{f}(b)(\mathbf{s}, \mathbf{b})$ equal to zero. Similarly for $\hat{g}(b)(\mathbf{s}, \mathbf{b})$.

[3] See, for example, Chung, Chapter 7, Theorem 12.

Since the mean demand in the finite economy $\hat{D}_\ell(P)$ converges in probability to the mean demand in the continuum $D_\ell(P)$, it follows that — except in the knife-edge case when the mean demand and supply intersect over an interval as illustrated in the right panel of Figure 1 — the $k+1$st price in the finite economy when everyone plays (f, g) and $v = v_\ell$, denoted by $\hat{P}_\ell$ (another random variable), converges in probability to the $k+1$st price in the continuum, P_ℓ. Further, the probability of winning in the finite model if the $k+1$st price is P_ℓ, one has bid P_ℓ, and $v = v_\ell$,—denoted by $\hat{x}_\ell(P_\ell)$, yet another random variable,—converges in probability to $x_\ell(P) \equiv x(P, \eta_\ell)$, the corresponding probability in the continuum which is not a random variable.[4] These observations motivate looking more closely at the exceptional, knife-edge case; we will see it involves "imperfect competition."

Let us call (f, g) *asymptotically perfectly competitive* if, regardless of the commodity's true quality v_ℓ, mean demand and supply intersect at a *unique* price if all individuals play (f, g) in the limiting continuum economy. In other words, (f, g) is asymptotically perfectly competitive if

$$\bar{k}(P_\ell, \eta_\ell) > 0 \quad \text{for all } \ell.$$

To motivate this terminology, compare the left and right panels of Figure 1. Notice when $\bar{k} > 0$ (left panel), an arbitrarily small positive measure of agents cannot affect the equilibrium price by changing its bid. By contrast, when $\bar{k} = 0$ (right panel), the Vickrey price would jump to b_2 if even an arbitrarily-small positive measure of agents increases its bid from b_3 to b_2. We identify "perfect competition" with the inability of an arbitrarily-small mass of agents to influence prices. Accepting this, the left but not the right panel in Figure 1 involves perfect competition.

Lemma 6 *Let (f, g) be any mixed strategy that is asymptotically perfectly competitive. Then there is a constant M such that for any quality $v = v_\ell$ and all $c > 0$:*

$$\text{Prob}(\hat{P}_\ell \neq P_\ell \textit{ or } \mid \hat{x}_\ell(P_\ell) - x_\ell(P_\ell) \mid \geq c) < \frac{M}{c^2 n} \quad \textit{for all } n.$$

It follows that for all $\epsilon > 0$ there is an N such that, in any economy with more than N buyers:

$$\mid \hat{\Pi}^s(b) - \Pi^s(b) \mid < \epsilon \quad \textit{for all } b.$$

Proof. Comparing the formula for $\hat{\Pi}^s(b)$, that is, the expected payoff from bidding b after receiving signal s in the finite model, with the formula for $\Pi^s(b)$, the corresponding expected payoff in the continuum, we see that there are two possible sources of divergence when the true quality is v_ℓ:

1. $\hat{P}(\mathbf{b})$ may be different from P_ℓ for some possible profiles of bids $\mathbf{b}$.

2. Even if $\hat{P}(\mathbf{b}) = P_\ell$, the probability of winning $\hat{x}_\ell(\mathbf{b})$ may be different from $x_\ell(b)$.

Hence, to prove the theorem, it will suffice to show that for any quality v_ℓ, the probability of profiles $\mathbf{b}$ in which the random variable $\hat{P}_\ell \neq P_\ell$ becomes arbitrarily small as $n \to \infty$; and further, among profiles such that $\hat{P}_\ell(\mathbf{b}) = P_\ell$, for any $\delta > 0$, the divergence $\mid \hat{x}_\ell(P_\ell) - x_\ell(P_\ell) \mid < \delta$ for all n sufficiently. That is, the probability bound in the first statement of the theorem implies the second statement.

---

[4] Formally, $\hat{x}_\ell(P)(\mathbf{s}, \mathbf{b}) \equiv \dfrac{k - \sum_{P' > P} \hat{d}_\ell(P')(\mathbf{s}, \mathbf{b})}{\hat{d}_\ell(P)(\mathbf{s}, \mathbf{b})}$.

To establish the first statement, suppose everyone plays (f, g) and the commodity's true quality is v_ℓ. We have already observed that the mean demand in the finite model converges in probability to the mean demand in the continuum. Hence, since (f, g) is asymptotically perfectly competitive, the $k+1$st price in the finite model will converge in probability to P_ℓ. Further, when the random variable $\hat{P}_\ell = P_\ell$, the law of large numbers implies $\hat{x}_\ell(P_\ell)$, the probability of winning if one bids P_ℓ, converges in probability to $x_\ell(P_\ell)$. Indeed, a standard application of Chebyshev's inequality (see the reference in Footnote 3) leads to the probability bound displayed in the theorem. □

The construction in the proof of Theorem 1 showed there exist mixed strategies (f, g) that are asymptotically perfectly competitive *and* fully revealing; furthermore, everyone playing such an (f, g) is a zero-profit equilibrium in the continuum. Now Lemma 6 implies that, for any $\epsilon > 0$, everyone playing such an (f, g) in a *finite* economy is an ϵ-equilibrium, provided the number of buyers n is sufficiently large. That is, any one buyer deviating from playing (f, g) can lead to no more than an expected payoff of ϵ if everyone else also is playing (f, g). Putting it together, we have shown:

Theorem 4 (existence of an ϵ-equilibrium that is typically fully-revealing) *For any $\epsilon > 0$, there exists a mixed strategy (f, g) and an integer N such that (f, g) is a symmetric ϵ-equilibrium for any economy with $n > N$ buyers. Further, if all buyers play (f, g), there exists a constant M such that the $k+1$st price random variable $\hat{P}_\ell$ satisfies*

$$\text{Prob}(\hat{P}_\ell \neq v_\ell) < \frac{M}{n} \quad \textit{for all } \ell \textit{ and } n.$$

5. Conclusion

We began with two questions about Pesendorfer and Swinkels's remarkable conclusions on information aggregation in large economies: "How did they do it?" and "How seriously should we take their conclusions, in terms of our faith in the market's ability to aggregate information?" Now that we have a sense of how they did it, let's address the second question. In Remark 4 in Section 3 we have already expressed our view that P&S2 depends crucially on a very strong common knowledge assumption about the distribution of private values. Hence we view the extension to two dimensions as a negative result regarding the ability of sealed-bid auctions to aggregate information. Here we confine ourselves to the pure common-value model.

Pesendorfer and Swinkels' 1997 results are especially remarkable when compared to the previous literature, notably the results on information aggregation by Wilson (1977) and Milgrom (1981). Working in a model with a fixed finite number of units—hence only letting the number of buyers increase,—Wilson and Milgrom find that full information aggregation requires very strong assumptions: for every possible quality v there must be a signal s such that no matter what information a buyer might infer from the behavior of others, he still puts arbitrarily small probability on the true quality being less than v after receiving s. So there must be signals s that are *very informative*. This leads P&S to conclude, "We thus view the Wilson and Milgrom results as essentially negative: only under extreme informational assumptions is information fully aggregated." (1977, p. 1248)

By contrast, P&S consider what happens when *both* the number of buyers and the number of available objects grow in tandem. Given this not unnatural departure, P&S find information will be fully aggregated even when each buyer receives a signal that is only minimally informative. They conclude: "Thus nonnegligible supply can be a substitute for strong signals. This provides a much more encouraging foundation for a belief in the information aggregation properties of markets." (1977, p. 1249)

How encouraged should we be? From an economic point of view, there is a weak link in the P&S argument for information aggregation: the crucial role played by mixed strategies. Mixed strategies are often criticized as being unrealistic, e.g., "decision makers do not toss coins." But sometimes this criticism is ill-founded. Notably, under the Aumann-Brandenburger reinterpretation of mixed strategies, it isn't that any player really randomizes; rather, any player's "mixed strategy" summarizes others' beliefs about what the player might do, reflecting *their* uncertainty about the player (see Brandenburger's 1992 survey). But in the P&S model buyers *really* have to randomize according to the probabilities in their equilibrium mixed strategies for the equilibrium price to be fully revealing. An "as if" story won't do the trick at all. That buyers actually randomize so finely seems quite incredible, which diminishes one's confidence in the Pesendorfer and Swinkels' route for explaining how markets can aggregate information.

Certainly the P&S findings are mathematically remarkable. Who would have guessed that one could produce such a rabbit out of the hat? In a sense, the boldness of their approach is related to its weakness. In the P&S model, individuals bid without learning anything about others' signals. That there should be full information aggregation with such minimal learning is incredible indeed. Our guess is a model that allows for learning, e.g., an English auction in which one can observe the enthusiasm with which others bid, and hence infer something about their signals, will in the long run lead to a more robust basis for our faith in the market's ability to aggregate information.[5]

To give a second suggestion, in the world one observes many "consumer reports" on the qualities of new commodities. Notably, there are several magazines that specialize in evaluating new model cars, giving prospective buyers the benefit of many experts' test-driving experiences. People considering buying a new car often spend long hours enthusiastically reading such magazines. Thus the road to information aggregation is often more concrete than the wonderful, magical road discovered by Pesendorfer and Swinkels. The public-good nature of information is not to be downplayed.

We conclude with a note of appreciation. While the above remarks have a critical edge, they should not overshadow our admiration for Pesendorfer and Swinkels' two remarkable papers. The ability of the market to aggregate information is a vital topic for economists, a largely unexplored and virgin territory. As explorers, Pesendorfer and Swinkels have done great service by making some of the issues clearer for those of us standing on their shoulders.

A Proof of Theorem 3

As discussed in the text and illustrated in Figure 5, all buyers of types $j < j^*$ will bid $\bar{b}$ in the constructed equilibrium, while all buyers of types $j > j^*$ will bid $\underline{b}$. Below we specify the

---

[5] After completing the current paper, I was motivated to pursue this hunch. See Makowski (2002).

mixed strategy (f_{j^*}, g_{j^*}) of buyers of type j^*. As discussed in the text, it closely resembles the mixed strategy constructed in the proof of Theorem 1, with the supports of f_{j^*} and g_{j^*} in $\{v_1{}^*, \ldots, v_L{}^*\}$. Below $f_\ell \equiv f_{j^*}(v_\ell{}^*)$, $g_\ell \equiv g_{j^*}(v_\ell{}^*)$, and $\eta_\ell{}^* \equiv \mu_{j^*}\eta_\ell$.

First suppose $\eta_1{}^* > k^* \geq \eta_L{}^*$. Let λ satisfy $\eta_\lambda{}^* > k^* \geq \eta_{\lambda+1}{}^*$. So for all values up to $v_\lambda{}^*$, the mass of type j^* buyers with good signals will be larger than the remaining mean supply k^*. Define g_ℓ for $\ell < \lambda$ as follows. Choose g_1 to satisfy

$$g_1\eta_1{}^* > k^* \geq g_1\eta_2{}^*;$$

then choose g_2 to satisfy

$$(g_1 + g_2)\eta_2{}^* > k^* \geq (g_1 + g_2)\eta_3{}^*.$$

Continue in this way until choosing $g_{\lambda-1}$ to satisfy

$$(g_1 + g_2 + \ldots + g_{\lambda-1})\eta_{\lambda-1}{}^* > k^* \geq (g_1 + g_2 + \ldots + g_{\lambda-1})\eta_\lambda{}^*.$$

[Notice $\eta_1{}^* > \eta_2{}^* > \ldots$; so for all $\ell \leq \lambda - 1$, $g_\ell \in (0, 1)$. Further $g_1 + \ldots + g_{\lambda-1} < 1$.] To complete the distribution, set $g_\lambda = 1 - (g_1 + \ldots + g_{\lambda-1})$ and $g_\ell = 0$ for all $\ell > \lambda$. Also set $f_\ell = 0$ for all $\ell \leq \lambda$. So, thus far, by construction we have:

$$P_\ell = v_\ell{}^* \quad \text{for all } \ell \leq \lambda.$$

To complete the construction of f, choose $f_{\lambda+1}$ to satisfy

$$\eta_{\lambda+1}{}^* + f_{\lambda+1}(1 - \eta_{\lambda+1}{}^*) > k^* \geq \eta_{\lambda+2}{}^* + f_{\lambda+1}(1 - \eta_{\lambda+2}{}^*);$$

then choose $f_{\lambda+2}$ to satisfy

$$\eta_{\lambda+2}{}^* + (f_{\lambda+1} + f_{\lambda+2})(1 - \eta_{\lambda+2}{}^*) > k^* \geq \eta_{\lambda+3}{}^* + (f_{\lambda+1} + f_{\lambda+2})(1 - \eta_{\lambda+3}{}^*).$$

Proceed in this fashion until choosing f_{L-1} to satisfy

$$\eta_{L-1}{}^* + (f_{\lambda+1} + \ldots + f_{L-1})(1 - \eta_{L-1}{}^*) > k^* \geq \eta_L{}^* + (f_{\lambda+1} + \ldots + f_{L-1})(1 - \eta_L{}^*).$$

To complete the probability distribution set f_L equal to $1 - (f_1 + \ldots + f_{L-1})$. Notice, by construction of f, we now also have ensured

$$P_\ell = v_\ell{}^* \quad \text{for all } \ell > \lambda.$$

Thus, if everyone of type j^* plays $(f, g) \equiv (f_{j^*}, g_{j^*})$, there will be full information aggregation. Further, it is an equilibrium choice for all these individuals to play (f, g) since any deviation would yield zero expected profit. Similarly, it is an equilibrium choice for all buyers with private values t_j larger than t_{j^*} to bid $\bar{b}$ since $P_\ell{}^* < V(v_\ell, t_j)$ for all such buyers (regardless of ℓ); and it is optimal for all buyers with private values less than t_{j^*} to bid $\underline{b}$. So our constructed strategies form an equilibrium in the 2-dimensional model.

The other possible cases are similar. If even $\eta_L{}^* > k^*$, so there are always more buyers of type j^* with good signals than units remaining, set $\lambda = L$ and define g_ℓ for all ℓ as above; set $f_\ell = 0$ for all $\ell < L$ and set $f_L = 1$. In this case, all individuals of type j^* playing (f, g) leads to a fully revealing equilibrium in which all j^* individuals with bad signals bid as if the commodity were of lowest quality.

Only one case remains. If $\eta_1{}^* \leq k^*$, set $g_1 = 1$ and set $g_\ell = 0$ for all $\ell > 1$; so all type j^* individuals with a good signal bid as if the commodity were of the highest quality, leaving $k^* - \eta_\ell{}^* \geq 0$ units to be allocated among individuals of type j^* with bad signals (for every possible ℓ). Analogous to the first case, choose f_1 to satisfy $\eta_1{}^* + f_1(1 - \eta_1{}^*) > k^* \geq \eta_2{}^* + f_1(1 - \eta_2{}^*)$, and so on until f_{L-1}; then let f_L have all the remaining probability weight. Given the fact that $k^* < \mu_{j^*}$, all f_ℓ will be positive in this case, and again all individuals of type j^* playing $(f, g) \equiv (f_{j^*}, g_{j^*})$ leads to a fully revealing equilibrium.

References

[1] Adam Brandenburger, *Knowledge and equilibrium in games*, Journal of Economic Perspectives **6** (1992), 83–101.

[2] Kai Lai Chung, *Elementary probability theory with stochastic processes*, Springer-Verlag, New York, 1974.

[3] P. Dubey, A. Mas-Colell, and M. Shubik, *Efficiency properties of strategic market games: An axiomatic approach*, Journal of Economic Theory **22** (1980), 339–362.

[4] M. O. Jackson, *Efficiency and information aggregation in auctions with costly information*, Discussion Paper, Caltech, 1999.

[5] Louis Makowski, *Learning in an open common-value auction*, Discussion Paper, UC Davis, 2002.

[6] Paul Milgrom, *A convergence theorem for competitive bidding with differential information*, Econometrica **47** (1979), 670–688.

[7] ______, *Rational expectations, information aggregation, and competitive bidding*, Econometrica **49** (1981), 923–943.

[8] Wolfgang Pesendorfer and Jeroen M. Swinkels, *The loser's curse and information aggregation in common value auctions*, Econometrica **65** (1997), 1247–1281.

[9] ______, *Efficiency and information aggregation in auctions*, Discussion paper, Northwestern University, 1998.

[10] Xavier Vives, *Aggregation of information in large Cournot markets*, Econometrica **56** (1988).

[11] Robert Wilson, *A bidding model of perfect competition*, Review of Economic Studies **44** (1977), 511–518.