

Learning in an Open Common-Value Auction

Louis Makowski*

This version: June 9, 2002[†]

1. Introduction

Where do prices come from?

The question has a long history, dating back at least to Walras's tâtonnement (groping) story and Edgeworth's core bargaining story. Modern economists have added further depth to the question by pointing out the importance of asymmetric information, and the possibility of prices aggregating information. Thus, for example, Paul Milgrom (1981) criticizes fully-revealing rational expectations equilibria (REE) as being incomplete, giving no clue how an equilibrium price that aggregates agents' widely-dispersed information comes to be. Although surprising to most non-economists, we are still far from a satisfactory answer to the above fundamental question. But one promising route involves the analysis of auc-

tions. Auctions are an ancient and still widely-used institution for finding equilibrium prices, and they are sufficiently simple to permit formal game-theoretic analysis. Here we will report on some findings along this path. Our interest in auctions is as a beginning toward understanding market-processes more generally.

In this paper we focus on markets with intense competition, rather than price-formation when agents have monopoly power. There are two complementary routes to model such competition: (i) doing asymptotics (seeing what happens as the number of market participants increases) and (ii) working in an idealized economy with a continuum of infinitesimal agents (so seeing what happens in the limiting economy, rather than as one approaches the limit). Here we take the latter approach, just because it turns out to be a lot simpler than asymptotics, and also because it is hopefully revealing of conclusions that also would hold asymptotically. As emphasized by Dubey, Mas-Colell, and Shubik (1980), Vives (1988), and others, strategic games with a continuum of players constitute a useful, simplifying tool of analysis. Here we will use this tool to analyze a dynamic game, an English auction with a continuum of privately-informed bidders.

*Joe Ostroy motivated my interest in understanding the papers of Pendorfer and Swinkels; he also introduced me to the paper of Izmalkov. Many conversations with Iraj Deilami, Giovanna Oettinger, and Mort Schwartz about auctions kept me focused and grounded during the writing of this manuscript. A presentation to the UC Davis theory group was also helpful.

[†]First version: May 23, 2002. Comments very welcome. You can reach me at lmakowski@ucdavis.edu.

1.1 The seminal papers by Pesendorfer and Swinkels

The strand of literature that is closest to this paper involves asymptotic analysis of common-value sealed-bid auctions. Working in a model with a fixed number of units of an indivisible commodity—hence only letting the number of buyers increase,—Wilson (1977) and Milgrom (1981) find that full information aggregation requires very strong assumptions: for every possible quality v there must be a signal s such that no matter what information a buyer might infer from the behavior of others, he still puts arbitrarily small probability on the true quality being less than v after receiving s . So there must be signals s that are *very informative*.

Pesendorfer and Swinkels (1997) (henceforth P&S) achieve a remarkable breakthrough. They consider what happens when *both* the number of buyers and the number of available objects grow in tandem. Given this not unnatural departure, P&S find information will be fully aggregated even when each buyer receives a signal that is only minimally informative.

To illustrate, consider the following example. Suppose a new model car is introduced of unknown quality. Each individual takes the car for a test drive, resulting in a very coarse 0-1 signal about its true quality: the individual either has a bad or good driving experience. Since there are many potential buyers, the join of all individuals' test-driving experiences is much more informative about the car's true quality than any one person's experience. Each individual, after his test drive—*without any knowledge about others' experiences*—places a sealed bid for a unit of the car, there being less cars available than the number of potential buyers. Pesendorfer and Swinkels show that in a large economy, in spite of each

bidder's very coarse information about the car's true quality, the equilibrium price of a car will reflect the join of all individuals' information!

P&S is a difficult paper to understand, in spite of the authors' nice exposition; the results are proved using formidable asymptotic arguments. Thus one is left a little in the dark about what exactly drives their conclusions. How did they do it? Are we to take their conclusions seriously, in terms of our faith in the market's ability to aggregate information? To answer these questions, in Makowski (2002) I reformulated the P&S model in a continuum setting. This setting relieves us of the need to do tough asymptotics, and thus helps us to unravel some of the P&S magic. Once the reformulation was completed it became clear that, from an economic point of view, there is one very weak link in the P&S argument for information aggregation: *it depends crucially on bidders using mixed strategies*.

Mixed strategies are often criticized as being unrealistic, e.g., “decision makers do not toss coins.” But sometimes this criticism is ill-founded. Notably, under the Aumann-Brandenburger reinterpretation of mixed strategies, it isn't that any player really randomizes; rather, any player's “mixed strategy” summarizes others' beliefs about what the player might do, reflecting *their* uncertainty about the player (see Brandenburger's 1992 survey). But in the P&S model buyers *really* have to randomize according to the probabilities in their equilibrium mixed strategies for the equilibrium price to be fully revealing. An “as if” story won't do the trick at all. That buyers actually randomize so finely seems quite incredible, which diminishes one's confidence in the Pesendorfer and Swinkels' route for explaining how markets can aggregate information.

In an ambitious sequel, Pesendorfer and Swinkels (1998) (henceforth P&S2) extend the findings in their 1997 paper to a setting in which buyers' preferences have both a common-value (quality) and a private-value (taste) component. Many, perhaps even most, real-world markets have this feature. They show that in this more realistic setting, asymptotically, equilibria will be both fully-revealing *and* allocatively efficient!

To clarify what drives this even more remarkable conclusion, in Makowski (2002) I also reformulate the Pesendorfer and Swinkels 2-dimensional model in a continuum setting. The central role of mixed strategies carries over to P&S2, but there is also a second weakest link added: P&S2 assumes that individuals' private values are independently drawn from a commonly-known distribution; hence, by a law of large numbers, asymptotically there is *no aggregate uncertainty* about the distribution of buyers' private values. This is a very strong assumption, going against the grain of Hayek's idea that the world is full of decentralized knowledge. It seems much more realistic to assume the opposite, that each individual knows his own tastes but is *completely ignorant* about the tastes of others, not knowing even the distribution of others' tastes. P&S2 gives no clue how far one can move from no aggregate uncertainty to complete ignorance before losing either full information aggregation or efficiency.

In a sense, the boldness of Pesendorfer and Swinkels' approach is related to its weakness. In their model, like in Wilson (1977) and Milgrom (1981), individuals bid without learning anything about others' signals. (In terms of the example, each buyer places a final sealed bid for a car after his own test drive, but before learning anything about others' test driving experiences.) That there should be full information aggregation with such minimal learning is incredible indeed. This thought has

motivated my interest in studying information aggregation in an English—hence open rather than sealed-bid—auction. In such an auction one can observe the enthusiasm with which others bid, and hence infer something about their signals. The hope is that, by allowing for learning, we will in the long run have a more robust basis for our faith in the market's ability to aggregate information.

I now begin to summarize the contents of the current paper. Like P&S, we begin with a pure common-value setting, but consider an open English rather than a sealed-bid auction. The auction will feature a continuum of bidders of mass 1 and a corresponding continuum of objects of mass $k < 1$. The assumption of a continuum of bidders and objects is to be regarded as an idealization for a large economy with intense competition. Restricting our attention to symmetric pure-strategy equilibria, we prove there exists an equilibrium in which the price is fully revealing of the commodity's true quality. We also give conditions under which there is uniqueness, that is, any symmetric pure-strategy equilibrium will be fully revealing.

Recall Milgrom's (1981) criticism of fully-revealing rational expectations equilibria (REE) as giving no clue how an equilibrium price that aggregates agents' information comes to be. Given this context, it is interesting to observe that our revealing equilibria are Walrasian, hence rational expectations equilibria. The bonus is that the auction analysis gives a concrete game-theoretic story about where the price comes from. Indeed, working in a simple continuum setting allows us to draw very suggestive pictures, looking very much like demand and supply, but with a dynamic overlay portraying the bidding and learning process.

In Sections 4 and 5 we extend our model to a more re-

alistic 2-dimensional setting in which buyers preferences have both a common-value and a private-value component. Of particular interest in terms of rational expectations equilibria, in Section 5 we allow for the possibility of both informed and uninformed buyers, and consider when the latter can learn from the bidding of the former. Mindful of the importance of decentralized knowledge, in the 2-dimensional extension we try to give relatively weak common-knowledge assumptions that still yield both full information aggregation and efficiency—trying to move as far as we can away from no aggregate uncertainty and toward complete ignorance about others’ private values.

1.2 The debt to Izmalkov

Besides the two seminal papers by Pesendorfer and Swinkels, the third main influence on the current work is a very fine paper by Sergei Izmalkov (2001). Unlike a standard English auction in which individuals are not permitted to reenter the bidding once they quit, Izmalkov analyzes an auction without this restriction, what he calls an *English auction with reentry*. He argues convincingly that English auctions with reentry are more realistic: in real-world auctions, one can always reenter as one pleases. More important for our purpose, he also shows that English auctions with reentry are more robust for aggregating common-value information.

Unlike a private-value auction, in a common-value auction the course of bidding conveys valuable information about other players’ signals, which may be revealed after a player has exited, causing him to regret his exit decision. Thus a no-reentry constraint can be binding.¹ Izmalkov gives condi-

tions under which an English auction with reentry will have an efficient equilibrium, conditions that are significantly weaker than those required for a standard English auction to be efficient. (See Example 5 below.)

Following Izmalkov’s lead, in the current paper we focus primarily on an English auction with reentry. Beyond this similarity, the analysis in the two papers is very different. Unlike the current paper, Izmalkov’s paper analyzes learning in an English auction with just a few bidders and just one indivisible object for sale. In this small-numbers setting, buyers have monopoly power in terms of their private information. So each buyer must consider how much of his information about the commodity’s true quality he is “giving away” or “revealing” to less informed buyers by following any given bidding strategy.

To ensure the possibility of equilibria that are efficient—in spite of agents’ monopoly power,—Izmalkov restricts his analysis to the case of 1-dimensional signals, and he assumes a single-crossing condition that figures prominently in the literature on auctions with a small number of participants (e.g., Crémer and McLean (1988), Maskin (1992)).² The exclusive

price formation in a pure private-value setting. As is well known, when an English auction is played in such a setting, each bidder has a dominant strategy: he remains in the bidding until the price-clock reaches his private value, then he permanently quits. So the restriction to no reentry does not bind. The other well-known but still remarkable fact is that if everyone plays their dominant strategy in a pure private-value English auction, the outcome will be Walrasian, hence efficient, even when there is complete ignorance about others’ types.

²The condition requires that a bidder’s signal has a greater influence on his own value than on some other bidder’s value. One can see how this condition softens monopoly problems: If one assumes that a monopolist values the goods he produces more than his customers value them, he will end up efficiently producing for himself rather than inefficiently

¹By contrast, the standard English auction is adequate for analyzing

focus on 1-dimensional signals is very restrictive, but vital for efficiency. Dasgupta and Maskin (1997) provide some simple examples showing that in general there do not exist efficient mechanisms when there are a small number of agents and signals are multidimensional. Jehiel and Moldovanu (2001) prove a general impossibility result that implies, when there are multidimensional signals and a small number of bidders, efficient equilibria are nongeneric.

Interestingly, in our large-numbers setting, the above single-crossing condition does not appear at all! Because there are many competing agents, any one tiny bidder has no monopoly power due to his private information; he is “informationally small,” to use the felicitous phrase due to McLean and Postlewaite (2000). Indeed, since we are working at the limit (in the continuum) rather than doing limiting arguments (asymptotics), any one tiny bidder is the extreme of informationally small, he is “informationally redundant” in the sense of Makowski and Ostroy (2001): his signal is completely uninformative *given* the union of all others’ signals. Our results show that, as far as information aggregation goes, the presence or absence of competition (information monopolies) makes a big difference for the existence of efficient and fully-revealing equilibria. The connection with the results of McLean and Postlewaite is further discussed in the conclusion, along with the mechanism-design approach to information aggregation.

The plan of the paper is as follows. We begin with a pure common-value setting. Section 2 specifies the basic environment, including the common-value signals individuals receive before the auction begins; it motivates and introduces the idea of an English auction with reentry; and it formalizes the game.

undersupplying the market.

Section 3 proves our two main results for a pure common-value setting, the existence and uniqueness of fully-revealing pure-strategy equilibria. Section 4 extends our conclusions to a more realistic environment in which each individual’s preferences have both a common-value and a private-value component. In this more realistic environment, it matters who gets units of the commodity, that is, the question of allocative efficiency arises. For the 2-dimensional model, we prove the existence and uniqueness of fully-revealing *and allocatively efficient* equilibria. Section 5 examines a further extension, a 2-dimensional model in which only some buyers have received an informative signal; we give conditions under which existence and uniqueness still hold. Section 6 concludes.

2. A pure common-value model

There is a continuum of buyers with Lebesgue measure 1 and a continuum of units of a homogeneous, indivisible commodity with Lebesgue measure k , where $0 < k < 1$. Each buyer puts a valuation v on a single unit, and only wants one unit. The value v is the same for all buyers and equals the quality of the commodity. While the true quality is initially unknown, it is common knowledge that Nature has picked v from a finite set \mathcal{V} containing L possible qualities indexed by ℓ , where

$$0 \leq v_1 < \cdots < v_\ell < \cdots < v_L.$$

Each buyer receives a signal $s \in \{0, 1\}$ about the commodity’s quality, with the good signal ($s = 1$) being more likely the higher the true quality. Let η_ℓ denote the fraction of agents receiving the good signal when the commodity’s quality is v_ℓ . We assume

no aggregate uncertainty: the fraction of buyers receiving a good signal always equals η_ℓ when the commodity’s true quality is v_ℓ , where $\eta_\ell > 0$ for all ℓ and η_ℓ is strictly increasing with v_ℓ .

Regard this as an idealized expression of the law of large numbers, describing the limiting economy resulting when buyers’ signals are independent Bernoulli trials with η_ℓ the probability of a good signal. Notice, even for the lowest quality commodity v_1 , the probability of a good signal η_1 can be high; thus an individual who has received a good signal may still be very much in the dark about the commodity’s true quality.

Remark 1. P&S allow for an arbitrary number of possible signals, including the binary case we are focusing on. The binary case, with its coarse 0-1 signal, is the hardest for discovering the commodity’s true quality; in this sense, it is the most remarkable. Further, at least insofar as existence goes, there is no effective loss of generality in focusing on this case. To illustrate, suppose there are many possible signals $s \in \mathcal{S}$ with higher signals being more informative (perhaps even a continuum of possible different signals). One can always partition \mathcal{S} into two sets \mathcal{S}_0 and \mathcal{S}_1 consisting, respectively, of the “relatively bad” and “relatively good” signals. If each agent only acts on the basis of this coarsening of his full information s —whether $s \in \mathcal{S}_0$ or $s \in \mathcal{S}_1$,—we would return to the binary model. If full information aggregation is possible in the coarsened binary model—where individuals effectively are throwing away some of their information,—a fortiori it is possible when they act on the basis of their full information.

P&S analyze the possibility of information aggregation in

the symmetric mixed-strategy equilibria of a sealed-bid auction. Below we will be interested in information aggregation in the symmetric *pure-strategy* equilibria of an English—hence open rather than sealed-bid—auction.

2.1 A standard English auction

We first consider a standard English auction—that is, one without reentry. This will motivate the usefulness of permitting reentry. It also will allow us to formally present the basic structure of the auction game in a simple setting. This structure will—with some needful modifications—carry over to the more complex game with reentry.

Following the common practice, starting from a price of zero, we assume the price is raised continuously by an idealized automated auctioneer, a “price clock.” Let c stand for the choice to continue bidding and q for the choice to quit bidding. In a standard English auction, any potential buyer’s only choice at any point in the game is whether to continue or to quit, given the history of bidding preceding that point. Once he has quit, he cannot rejoin the bidding.

Let \bar{P} represent the maximum possible bid (exogenously given), where $\bar{P} > \max v_\ell$. We will say there is *news* at price $P \in [0, \bar{P}]$ (or, synonymously, P is a news event) if a positive mass of buyers quit bidding at P . In the English auctions we will consider, there will be no news at most prices P ; indeed, there will only be a finite number of prices involving news. But news events will be important because they will permit buyers who are still in the bidding to infer something (learn) about others’ signals. Formally, a *history* is a pair $h = (\mathcal{P}, Q)$ consisting of a finite set of prices $\mathcal{P} \subset [0, \bar{P}]$ and a function $Q : \mathcal{P} \rightarrow (0, 1]$. \mathcal{P} lists all the prices P from zero up to $\max \mathcal{P}$

that involve news, and $Q(P) > 0$ equals the mass of buyers who quit bidding at each price $P \in \mathcal{P}$. It is to be understood that for any $P \leq \max \mathcal{P}$, if $P \notin \mathcal{P}$ then no buyers quit at P ; in this sense a history gives a complete news report, at least up to the price $P = \max \mathcal{P}$. We permit an empty history, that is, any history $h = (\mathcal{P}, Q)$ with $\mathcal{P} = \emptyset$, interpreted as a history in which no one has yet quit.

Given any nonempty history $h = (\mathcal{P}, Q)$ and price $P \in [0, \max \mathcal{P}]$, the *demand* at P given h is

$$D(P, h) = 1 - \sum_{\substack{b \in \mathcal{P} \\ b \leq P}} Q(b),$$

that is, the mass of buyers still in the bidding at P . The standard English auction ends (the price-clock stops) once the mass of buyers still bidding equals k or less; hence for any nonempty history $h = (\mathcal{P}, Q)$:

$$D(P, h) > k \quad \text{for all } P < \max \mathcal{P};$$

and h is a *complete history* if

$$D(\max \mathcal{P}, h) \leq k.$$

Given any complete history of bidding, the equilibrium price will be $P^* \equiv \max \mathcal{P}$, which is the price that everyone who gets a unit must pay. (The probability of getting a unit will be specified shortly.)

A complete history unfolds as the price-clock proceeds. In particular, given any complete history $h = (\mathcal{P}, Q)$ and any price $P \in [0, \max \mathcal{P}]$, the *current history* when the price-clock reaches P , denoted $h \mid P^-$, is the partial history $h' = (\mathcal{P}', Q')$ such that $\mathcal{P}' = \mathcal{P} \cap [0, P)$ (the news events preceding P) and

$Q' = Q \mid \mathcal{P}'$ (the restriction of Q to the sub-domain \mathcal{P}'). The *information set* I any individual will find himself at when the clock reaches P is a triple of the form

$$I = (P, h \mid P^-, \text{buyer status}),$$

describing the current state of the price clock, the current history of bidding, and the current status of the buyer—either *active* or *inactive*. A bidder *must* play quit (q) whenever he finds himself in an information set where he has inactive status. When does this occur? A bidder becomes “inactive” after he quits at some price P ; and in the standard English auction he remains inactive for the rest of the game. Each bidder also has inactive status in any information set I with $P = \bar{P}$; so, if he has not quit before the price clock reaches \bar{P} , he must quit at this point.

Let \mathcal{I} denote the set of all possible information sets for each buyer in the standard English auction. A *pure strategy* for a buyer is a function

$$\sigma : \mathcal{I} \rightarrow \{c, q\}$$

specifying his choice at every possible information set, with the proviso that $\sigma(I) = q$ whenever he has inactive status in I

A complete history $h = (\mathcal{P}, Q)$ leads to the equilibrium price $P^* \equiv \max \mathcal{P}$, the price that all buyers who get a unit must pay. Let $x(\sigma, h)$ denote the probability of getting a unit if one plays σ and the complete history of bidding is h . In the standard English auction, anyone who has not quit at P^* gets a unit of the commodity with probability 1. On the other hand, if he quit before P^* , he gets a unit with probability 0. Finally, if the individual only quits once the price reaches P^* ,

then

$$x(\sigma, h) = \frac{k - D(P^*, h)}{Q(P^*)},$$

that is, he has an equal chance of getting one of the remaining units. (The numerator above equals the number of units still available given the mass of bidders who have not yet quit at P^* , while the denominator equals the mass of individuals who quit at P^* .)

We will be interested in *symmetric* pure strategy equilibria (σ^0, σ^1) in which all buyers with signal $s \in \{0, 1\}$ play the same strategy σ^s . Assuming everyone plays according to (σ^0, σ^1) , let $h_\ell \equiv (\mathcal{P}_\ell, Q_\ell)$ denote the complete history of bidding that will result when $v = v_\ell$, hence when η_ℓ individuals play σ^1 and $1 - \eta_\ell$ play σ^0 ; and let $P_\ell \equiv \max \mathcal{P}_\ell$ denote the equilibrium price that will result when $v = v_\ell$.

To go one step further, continue to suppose individuals play according to (σ^0, σ^1) , and let I be any possible information set for a bidder (whether on the equilibrium path or off). Let $\text{Prob}_\ell(s, I)$ be his probability belief that $v = v_\ell$ given he received signal s and finds himself at I . In accord with sequential rationality, whenever possible we require $\text{Prob}_\ell(s, I)$ to be consistent with Bayes' rule and the fact that bidders are playing (σ^0, σ^1) . Similarly, let $h_\ell(s, I)$ be his conjecture about what the complete history of bidding will be if $v = v_\ell$, given he received signal s and finds himself at I . If $\text{Prob}_\ell(s, I) = 0$, this conjecture may be arbitrary. But if $\text{Prob}_\ell(s, I) > 0$, in accord with sequential rationality, we require $h_\ell(s, I)$ to be consistent with the hypothesis that others will continue to play (σ^0, σ^1) from I onward. In particular, if I is on the equilibrium path for $v = v_\ell$, then $h_\ell(s, I) = h_\ell$.

The *expected payoff* of a buyer with signal s if he plays σ while all others play (σ^0, σ^1) and he finds himself at I is:

$$\pi^s(\sigma | I) = \sum_{\ell} \text{Prob}_\ell(s, I)(v_\ell - P_\ell)x(\sigma, h_\ell(s, I)).$$

The strategies (σ^0, σ^1) form an *equilibrium* if, for each signal s , each information set I on the equilibrium path, and all possible deviations σ starting from I :

$$\pi^s(\sigma^s | I) \geq \pi^s(\sigma | I).$$

The equilibrium is *fully revealing* if

$$P_\ell = v_\ell \quad \text{for all } \ell.$$

Remark 2. The above definition of equilibrium only requires optimal behavior *on* the equilibrium path. By contrast, in finite-player games, a perfect Bayesian equilibrium also requires optimal behavior at information sets *off* the equilibrium path. The reason for this is that in finite-player games any one player, by deviating, can move the game to a position off the equilibrium path,—and we want to be sure that such a deviation will not be profitable for him. By contrast, in our game with a continuum of infinitesimal players, any one bidder's deviation does not affect the (aggregate) history of play, hence cannot affect others' behavior. That is, by deviating, he cannot move the game to a position off the equilibrium path. This will make life easier for us, since we will only have to worry about profitable deviations on the equilibrium path.

The following two examples will show that, even without reentry, there can exist pure-strategy equilibria that are fully revealing.

Example 1 (a fully revealing equilibrium without learning from others' bidding) Suppose a new model car is introduced with unknown quality. There are half as many cars as potential buyers, so the mean supply is $k = .5$. The car is either of low quality $v_1 = 40$ or high quality $v_2 = 80$. Each individual takes the car for a test drive before bidding, resulting in either a good or bad driving experience. The probability of a good experience is $\eta_1 = .4$ (respectively, $\eta_2 = .8$) if the car is of low (high) quality. So

$$v_\ell = 100\eta_\ell \quad \text{for all } \ell.$$

Consider the symmetric pure-strategy equilibrium (σ^0, σ^1) in which each individual with a bad experience continues bidding until $P^0 = 40$ and then quits permanently, while each individual with a good experience stays in the bidding until $P^1 = 80$ and then quits permanently. Formally, $\sigma^0(I) = c$ at all information sets I in which the price clock is at $P < P^0$ and the player's status is active, and $\sigma^0(I) = q$ at all other information sets; and similarly for σ^1 . Then, as illustrated in Figure 1, the equilibrium price of a car will always reflect its true quality: the price will be 40 if $v = v_1$ and 80 if $v = v_2$.

Example 2 (a fully-revealing equilibrium with learning from others' bidding) This example is identical to Example 1 except now there are ten times as many potential buyers as cars, so the mean supply is $k = .1$. The consequence is that on average more people will have good test drives than there are cars available—even if the car is of low quality. Now the pure strategies in Example 1 would lead to a price of 80 regardless of the car's quality. So these strategies are no longer equilibrium strategies: individuals would end up over-paying if $v = v_1$. What will happen in equilibrium now? Can there

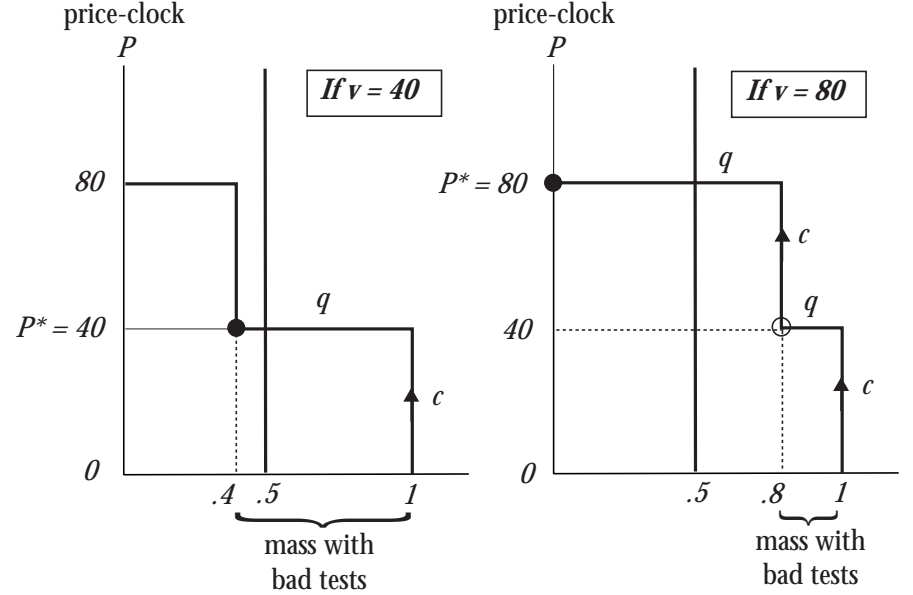


Figure 1

possibly be a fully-revealing equilibrium?

Suppose that individuals with bad signals quit at $P^0 < 40$. On the other hand, individuals with good signals quit at $P^1 = 40$ if at least 60% of all bidders quit before the price clock reaches 40; but they quit at $P^1 = 80$ if less than 60% of all bidders quit before the price reaches 40. Notice individuals with good signals now condition their behavior on the history of others' bidding, effectively learning from others: If bidding is not enthusiastic at prices below 40 — in the sense that many people quit before $P = 40$, — individuals with good signals assume that the commodity must be of low quality. As illustrated in Figure 2, these strategies again lead to a fully-revealing equilibrium; people with good signals learn at

the news event $P = P^0$ based on the size of $Q_\ell(P^0)$.

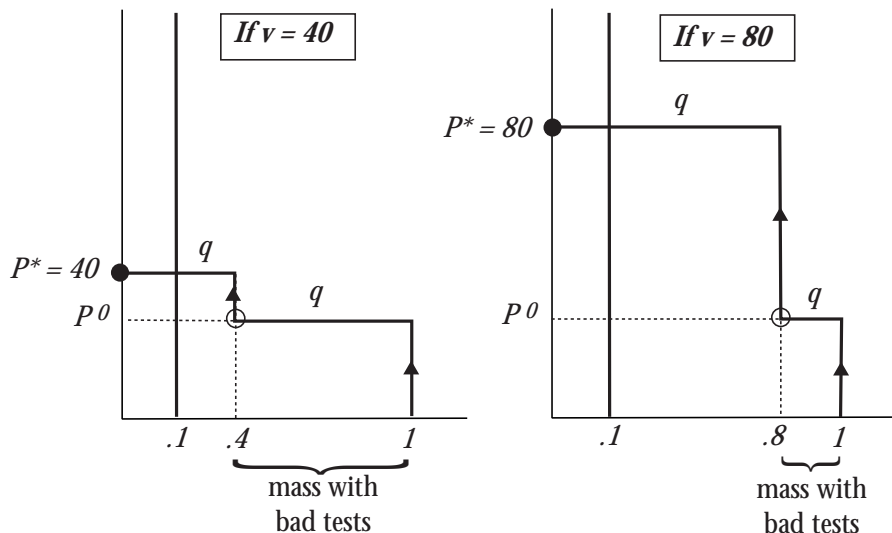


Figure 2

In the last example, it may seem strange that individuals with bad signals drop out of the bidding even before the price reaches 40 since the commodity is always worth at least 40. Suppose instead that individuals with bad signals remain in the bidding until $P^0 = 40$ and then they all quit, while all individuals with good signals continue bidding at $P = 40$. In this variation, if individuals with good signals observe 60% of all bidders quitting at $P = 40$, they would be sure that $v = 40$; hence, they would all want to leave “as soon as possible after $P = 40$.” But such a strategy is not well defined in the standard English auction (there is a closedness problem).

Following the lead in Izmalkov (2001), we will solve the problem by assuming the price-clock stops at least temporarily at any price P when someone quits. Anyone who is still in the bidding at P can quit while the clock is stopped. In terms of the above variation, when P reaches 40 the clock stops at least momentarily, and individuals with good signals can quit if they like after observing the mass of individuals that has just quit (while the clock was still running). Symmetrically, we also will permit individuals who have quit while the clock was running to re-enter the bidding when the clock stops. Hence, following Izmalkov, we will call the revised English auction an *English auction with reentry*. We now modify the rules of the game to permit reentry.³

2.2 An English auction with reentry

In an English auction with reentry, the clock stops at least temporarily whenever a positive mass of bidders quits. At this point, bidders who have not yet quit can decide to quit; and bidders who have previously quit can decide to reenter the bidding. Formally, let \mathcal{P} continue to denote a finite set of news events. To describe a history of play in an English auction with reentry we introduce two functions, $Q : \mathcal{P} \rightarrow (0, 1]$ and $R : \mathcal{P} \rightarrow [-1, 1]$, and we let a history be a triple $h = (\mathcal{P}, Q, R)$. $Q(P) > 0$ continues to equal the mass of bidders who quit at each P while the clock is moving, and $R(P)$ equals the mass of bidders who reenter the bidding after the clock has stopped at P . We assume once the clock stops at P , individuals can reenter even if they quit at prices strictly less than P , hence

³The reader is warned that, formally, Izmalkov’s English auction with reentry is a bit different than ours because his rules are formulated to deal with problems arising when there are only a small number of bidders.

$R(P)$ may exceed $Q(P)$. On the other hand, $R(P)$ may be negative since we permit additional people to quit (= negative reentry) once the clock stops at any news event.

The *net quits* at P is given by

$$Q^n(P) \equiv Q(P) - R(P).$$

In an English auction with reentry, $Q^n(P)$ plays much the same role as $Q(P)$ played in the standard English auction. An English auction with reentry ends (the price-clock stops permanently) once k or less buyers remain in the bidding—even after having a chance to reenter. Expressed formally, given any history $h = (\mathcal{P}, Q, R)$ and price $P \in [0, \max \mathcal{P}]$, define the *demand* at P given h as:

$$D(P, h) = 1 - \sum_{\substack{b \in \mathcal{P} \\ b \leq P}} Q^n(b),$$

that is, the mass of buyers still in the bidding at P given they have had a chance to reenter. Since bidding ends once $D(P, h) \leq k$, for any history h :

$$D(P, h) > k \quad \text{for all } P < \max \mathcal{P};$$

and h is a *complete history* if

$$D(\max \mathcal{P}, h) \leq k.$$

The set of all *possible histories*, denoted H , consists of all $h \equiv (\mathcal{P}, Q, R)$ such that either $\mathcal{P} = \emptyset$ (an empty history) or $\mathcal{P} \neq \emptyset$, $D(P, h) \in [0, 1]$ for all $P \in \mathcal{P}$, and $D(P, h) > k$ for all $P < \max \mathcal{P}$.

In a standard English auction, since the price-clock is always moving, the current state of the clock is completely described

by the price P at which it has arrived. But in an English auction with reentry, the current state of the clock must be described by a pair $(P, \textit{clock status})$, telling not just the price at which the clock has arrived, but also whether it is *moving* or *stopped*.

A complete history unfolds as the price-clock proceeds. In particular, given any complete history $h = (\mathcal{P}, Q, R)$ and price $P \in [0, \max \mathcal{P}]$, let $h \mid (P, \textit{moving})$ denote the *current history* of bidding when the clock reaches P and is still moving; it equals the partial history $h' \equiv (\mathcal{P}', Q', R')$ such that $\mathcal{P}' = \mathcal{P} \cap [0, P)$, $Q' = Q \mid \mathcal{P}'$ (that is, the restriction of Q to the specified sub-domain), and $R' = Q \mid \mathcal{P}'$. As in the standard English auction, we sometimes will denote the current history $h \mid (P, \textit{moving})$ more simply by $h \mid P^-$. Similarly, let $h \mid (P, \textit{stopped})$ denote the current history when the clock reaches P and has stopped; it equals the partial history h' such that $\mathcal{P}' = \mathcal{P} \cap [0, P]$, $Q' = Q \mid \mathcal{P}'$, and $R' = Q \mid \{P' \in \mathcal{P} : P' < P\}$. The idea is that, once the clock stops at P , everyone knows the mass of buyers that quit at P , but they do not yet know the mass that will reenter. We sometimes will denote $h \mid (P, \textit{stopped})$ more simply by $h \mid P$.

A pure strategy for a bidder specifies what action in $\{c, q\}$ he will choose in whatever information set he may find himself. Let \mathcal{I} denote the collection of all possible information sets for a bidder in the English auction with reentry. Any information set $I \in \mathcal{I}$ is a quadruple:

$$I = (P, \textit{current clock status}, \textit{current history of bidding}, \\ \textit{current bidder status}),$$

specifying the current position P of the price-clock; its current status, that is, whether it is *moving* or *stopped*; the cur-

rent history of bidding $h \mid (P, \text{clock status})$; and whether the bidder's current status is *active* or *inactive*. A bidder *must* play quit (q) whenever he finds himself in an information set where he has inactive status. When does this occur? A bidder becomes “inactive” after he quits at some price P ; and he remains inactive unless he re-enters the bidding at some price $P' \geq P$ when the clock has stopped; once he re-enters, he becomes active and remains so until he quits again. Since he can always reenter (play c) when the clock stops, a bidder's status is always active in information sets in which the clock has stopped; he only may be inactive when the clock is moving. There is only one exception to the above: in any information set with $P = \bar{P}$, the highest possible bid, the individual always has inactive status; so he must quit bidding once \bar{P} is reached. A buyer's *pure strategy* in the English auction with reentry is a function

$$\sigma : I \rightarrow \{c, q\}$$

specifying his choice at every possible information set, with the proviso that $\sigma(I) = q$ whenever his status is inactive in I .

To illustrate, let $h = (\mathcal{P}, Q, R)$ be a complete history with $P \in \mathcal{P}$. Consider two successive information sets

$$I = (P, \text{moving}, h \mid P^-, \text{active})$$

and

$$\bar{I} = (P, \text{stopped}, h \mid P, \text{active}).$$

If $\sigma(I) = \sigma(\bar{I})$, then if the clock stops at P , σ calls on the individual to just re-affirm his choice when the clock was moving. But if $\sigma(I) = c$ while $\sigma(\bar{I}) = q$ (resp. $\sigma(I) = q$ while $\sigma(\bar{I}) = c$), then σ calls on the individual to quit the bidding (resp. reenter the bidding) if the clock stops at P .

A complete history $h = (\mathcal{P}, Q, R)$ leads to the equilibrium price $P^* \equiv \max \mathcal{P}$, the price that all buyers who get a unit must pay. Let $x(\sigma, h)$ denote the probability of getting a unit if one plays σ and the complete history of bidding is h . In the English auction with reentry, the individual's decision when the clock has stopped can over-ride his decision while the clock was still moving. In particular, if he decides to continue bidding when the clock stops at P^* , he gets a unit of the commodity with probability 1, regardless of whether he previously quit. Formally expressed, if the individual finds himself in the information set $I^* = (P^*, \text{stopped}, h \mid P^*, \text{active})$, then

$$\sigma(I^*) = c \text{ implies } x(\sigma, h) = 1.$$

On the other hand, if he already quit before P^* and does not change his mind, he gets a unit with probability 0, just as in the standard English auction. Formally expressed, consider the information sets of the form

$$I(P) = (P, \text{moving}, h \mid P^-, \text{bidder status}).$$

Then $x(\sigma, h) = 0$ if $\sigma(I^*) = q$ and $\sigma(I(P)) = q$ for all $P \in [P', P^*]$ and some $P' < P^*$. Finally, if the individual only quits once the price reaches P^* , then the individual is rationed as in the standard English auction:

$$x(\sigma, h) = \frac{k - D(P^*, h)}{Q(P^*)}.$$
⁴

We will be interested in *symmetric equilibria* (σ^0, σ^1) in which each individual with signal s plays the same strategy σ^s .

⁴Formally, quitting only once the price reaches P^* means: (i) playing according to σ leads to an active status for all $P \in (P', P^*)$ and some $P' < P$, (ii) $\sigma(I(P)) = c$ for all $P \in (P', P^*)$, and (iii) $\sigma(I^*) = q$.

Assuming everyone plays according to (σ^0, σ^1) , as in the standard English auction let $h_\ell \equiv (\mathcal{P}_\ell, Q_\ell, R_\ell)$ denote the complete history of bidding that will result when $v = v_\ell$, hence when η_ℓ individuals play σ^1 and $(1 - \eta_\ell)$ play σ^0 ; and let $P_\ell \equiv \max \mathcal{P}_\ell$ denote the equilibrium price that will result when $v = v_\ell$.⁵ Again as in the standard English auction, let $\text{Prob}_\ell(s, I)$ be an individual's probability belief that $v = v_\ell$ given he received signal s and finds himself at information set I ; and let $h_\ell(s, I)$ be his conjecture about the complete history of bidding if $v = v_\ell$, given he received signal s and finds himself at I .

The *expected payoff* of a buyer with signal s if he plays σ while all others play (σ^0, σ^1) and he finds himself at I is:

$$\pi^s(\sigma | I) = \sum_{\ell} \text{Prob}_\ell(s, I)(v_\ell - P_\ell)x(\sigma, h_\ell(s, I)).$$

The strategies (σ^0, σ^1) form an *equilibrium* if, for each signal s , each information set I on the equilibrium path, and all possible deviations σ starting from I :

$$\pi^s(\sigma^s | I) \geq \pi^s(\sigma | I).$$

The equilibrium is *fully revealing* if

$$P_\ell = v_\ell \quad \text{for all } \ell.$$

⁵Given (σ^0, σ^1) , we can calculate the aggregate statistics $Q_\ell(P)$ and $R_\ell(P)$ for any news event P . For example, if P is the first news event when $v = v_\ell$ (so the history preceding P is empty), $Q_\ell(P)$ is the mass of buyers that, in equilibrium, plays q at P while the clock is moving. Similarly, $R_\ell(P)$ is the mass of buyers that plays q while the clock is moving at P , and then switches to c when it stops *minus* the mass that plays c while the clock is moving at P and then switches to q when it stops. Notice only individuals who switch at P when the clock stops affect the aggregate statistic $R_\ell(P)$.

As in the standard English auction, the above definition of equilibrium only requires that no one has a profitable deviation *on* the equilibrium path (recall Remark 2).

The following example shows how the possibility of reentry helps solve the closedness problem we discussed following Example 2.

Example 3 (a fully-revealing equilibrium with reentry) The example is identical to Example 2 except for buyers' strategies. Now individuals with bad signals do not quit until the price-clock reaches $P^0 = 40$, so $\sigma^s(I) = c$ for all s and at all information sets

$$I = I(P, h) \equiv (P, \text{moving}, h | P^-, \text{active})$$

with $P < 40$ and $h | P^-$ equal to an empty history. When the clock reaches $P = 40$, all individuals with bad signals quit while the clock is still moving while all individuals with good signals continue, so $\sigma^0(I(40, h_\ell)) = q$ and $\sigma^1(I(40, h_\ell)) = c$ for all ℓ . The price $P = 40$ is the first news event. If an individual with a good signal sees that 60% of all bidders quit at $P = 40$ while the clock was still moving, he infers the commodity must be of low quality, so he also quits once the clock stops at $P = 40$. So when $v = 40$ the equilibrium price will be $P^* = 40$. But if an individual with a good signal sees that less than 60% of all bidders quit at $P = 40$ while the clock was moving (enthusiastic bidding), he infers the commodity must be of high quality, so he continues bidding. In this case, since there are less units of the commodity available than number of eager bidders, the bidding continues until $P = 80$, at which point even individuals with good signals quit. So when $v = 80$ the equilibrium price will be $P^* = 80$. The fully-revealing equilibrium is illustrated in Figure 3. As in Example 2, people

with good signals learn at the news event $P = P^0$, based on the size of $Q_\ell(P^0)$. Unlike a standard English auction, in an English auction with reentry there is enough time for an individual with a good signal to leave “as soon as possible” if he sees the bidding is not enthusiastic at the first news event.

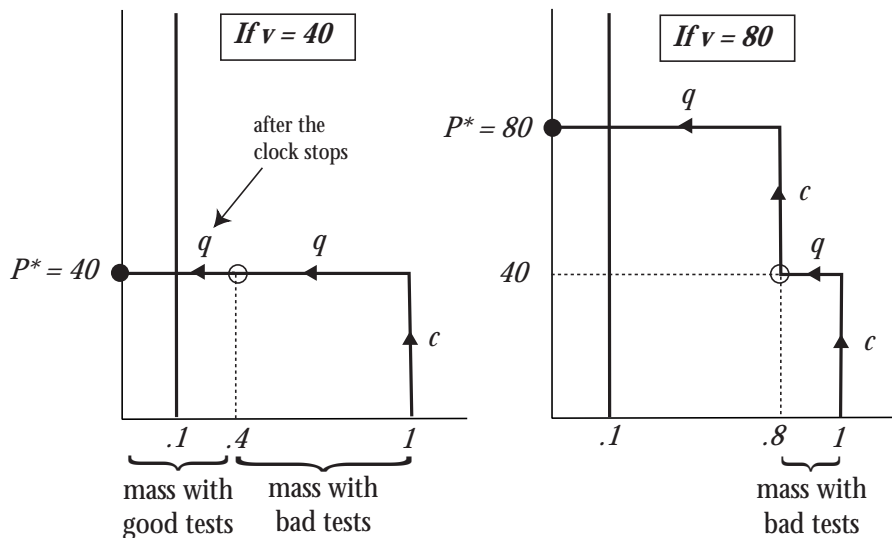


Figure 3

3. The existence and uniqueness of fully-revealing equilibria

We now generalize Example 3, to show that a fully-revealing equilibrium always exist in the English auction with reentry.

3.1 Existence

The proof of existence will be simple and constructive. Like in Example 3, the enthusiasm of bidding at the initial stages of the auction (in particular, at the first news event) will be completely informative about the commodity’s true quality. Thereafter, all individuals will bid just like in a private-value English auction since there is nothing more to learn.

Like in Pesendorfer and Swinkels (1997) and Makowski (2002), the equilibrium we construct will be symmetric; but unlike these papers, we will not require bidders to play mixed strategies. This highlights an important difference between sealed-bid and open-bid auctions: in the former, a bidder cannot learn about others’ signals by observing their bidding.

Theorem 1 (existence) *There always exists a fully-revealing, symmetric pure-strategy equilibrium in the English auction with reentry.*

Proof. In the equilibrium we will construct, the first news will occur when the price-clock reaches $P = v_1$, the lowest possible value; before that everyone only sees an empty history. In particular, if the true quality equals v_1 , the equilibrium history of play will be $h_1 \equiv (\mathcal{P}_1, Q_1, R_1)$ satisfying:

1. $\mathcal{P}_1 = \{v_1\}$ (there is only one news event);
2. $Q_1(v_1) = 1 - \eta_1$ because all individuals with bad signals quit at $P = v_1$ while the clock is running;
3. $R_1(v_1) = -\eta_1$ because all individuals with good signals also quit at v_1 once the clock stops, given the bad news that $1 - \eta_1$ people already quit.

On the other hand, if the true quality is $v_\ell > v_1$, the equilibrium history of play will be $h_\ell \equiv (\mathcal{P}_\ell, Q_\ell, R_\ell)$ satisfying:

1. $\mathcal{P}_\ell = \{v_1, v_\ell\}$ (so there will now be two news events);
2. $Q_\ell(v_1) = 1 - \eta_\ell$ because all individuals with bad signals still quit at $P = v_1$ while the clock is running;
3. $R_\ell(v_1) = 1 - \eta_\ell$ because all individuals with bad signals return to the bidding after seeing the good news that less than $1 - \eta_1$ quit;
4. $Q_\ell(v_\ell) = 1$ and $R_\ell(v_\ell) = 0$ because all individuals with good signals continue in the bidding until $P = v_\ell$, then they all quit as do all individuals with bad signals.

Notice the above histories lead to the equilibrium price $P_\ell = v_\ell$ for all ℓ , so the equilibrium we will construct is fully revealing (see Figure 4 below).

We now specify players' strategies leading to the above histories. Let $\sigma^0(I) = \sigma^1(I) = c$ at all information sets

$$I = I(P, h_\ell) \equiv (P, \text{moving}, h_\ell \mid P^-, \text{active})$$

with $P < v_1$. Then let $\sigma^0(I) = q$ while $\sigma^1(I) = c$ at the information set $I = I(v_1, h_\ell)$, leading to

$$Q_\ell(v_1) = 1 - \eta_\ell \quad \text{for all } \ell$$

as specified above.

Once the clock stops at v_1 , individuals will find themselves in some information set

$$I_\ell = (v_1, \text{stopped}, h_\ell \mid v_1, \text{active}),$$

which depends only on ℓ . Notice at this point on the equilibrium path, all buyers will know the commodity's true quality: $v = v_\ell$ if and only if the mass $1 - \eta_\ell$ quit at $P = v_1$ while the clock was moving. Let $\sigma^0(I_\ell) = \sigma^1(I_\ell) = q$ if $\ell = 1$, that is, if $1 - \eta_1$ buyers quit at v_1 while the clock was moving — leading to $R_1(v_1) = -\eta_1$ as specified above. Let $\sigma^0(I_\ell) = \sigma^1(I_\ell) = c$ if $\ell \neq 1$, that is, if less than $1 - \eta_1$ buyers quit at v_1 — leading to $R_\ell(v_1) = 1 - \eta_\ell$ as specified above.

If $v = v_\ell > v_1$, the clock will resume moving after stopping momentarily at v_1 . In this event, let all buyers continue bidding (play c) until $P = v_\ell$. Then let them all play quit while the clock is moving at $P = v_\ell$ (that is, let $\sigma^0(I) = \sigma^1(I) = q$ at $I = I(v_\ell, h_\ell)$), and let them continue to play quit once the clock stops at $P = v_\ell$. So the game ends.

If individuals play as above, $P_\ell = v_\ell$ for all ℓ . Hence everyone's payoff will be zero; and any deviation by one bidder also will yield him a zero payoff, so the above choices form an equilibrium. Players' choices at information sets off the equilibrium path can be chosen arbitrarily since they will not affect the equilibrium play and hence our conclusion. \square

The constructed equilibrium is illustrated in Figure 4. It is instructive to compare this equilibrium with an English auction in which everyone has *perfect information* about the commodity's true value v . Then, since no infinitesimal agent can influence the equilibrium price, each buyer would have a dominant strategy of just staying in the bidding until $P = v$, that is, it would be a dominant strategy for each buyer to play $\sigma(I) = c$ at all information sets I with $P < v$ and $\sigma(I) = q$ at all information sets I with $P \geq v$, as illustrated in Figure 5. In our constructed equilibrium, all individuals play this way for all prices $P > v_1$ since there is nothing left to learn once

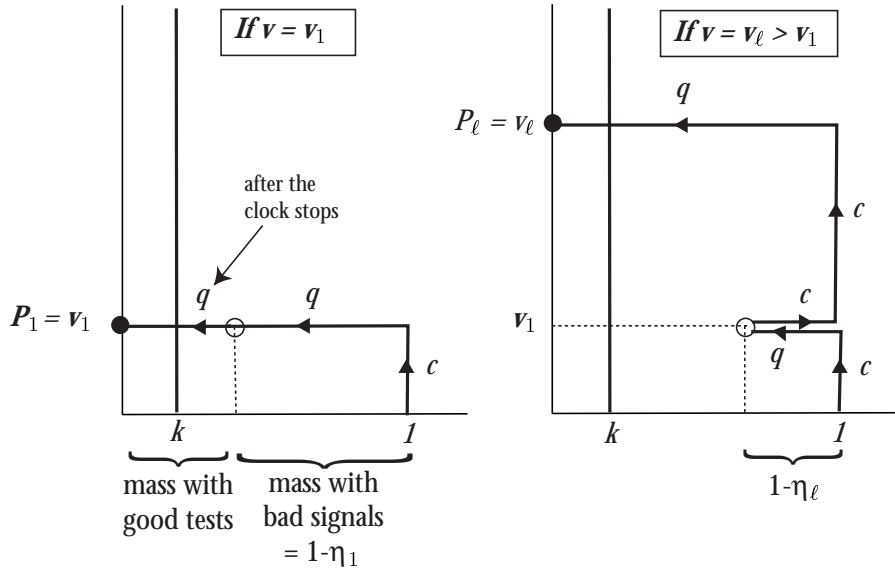


Figure 4

the price-clock starts again. To see this graphically, compare the right panel of Figure 4 with Figure 5.

3.2 Uniqueness

Suppose an individual is *sure* the price-clock will stop at P ; that is, $P \in \mathcal{P}_\ell$ for all ℓ . Then, in an English auction with reentry, it would be costless for the person to quit at P while the clock is running, and then change his mind when the clock stops and reenter the bidding. He might quit at P while the clock is moving and then reenter at P when the clock stops

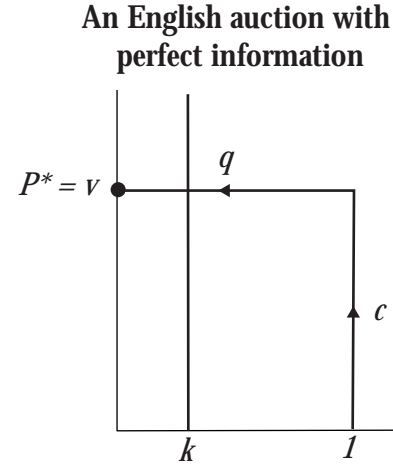


Figure 5

even if he is sure that he will reenter at P ; so quitting may be an entirely frivolous act. Notice there is no strategic reason to act in this way because any one individual's behavior does not affect the (aggregate) history of play, hence others' behavior. We will call a symmetric equilibrium "regular" if individuals do not engage in such frivolous quitting and reentering. In preparation, define the following two information sets:

$$I(P, h) = (P, \text{moving}, h \mid P^-, \text{active})$$

$$\bar{I}(P, h) = (P, \text{stopped}, h \mid P, \text{active}).$$

Given any symmetric equilibrium (σ^0, σ^1) , let H^* denote the set of all possible complete equilibrium histories, that is, $H^* = \{h_\ell : \ell = 1, \dots, L\}$.

DEFINITION 1. A symmetric equilibrium (σ^0, σ^1) is *regular* if

an individual's choice is

$$\sigma^s(I(P, h_\ell)) = q$$

at some information set $I(P, h_\ell)$ on the equilibrium path, implies it is not the case that

$$P \in \mathcal{P} \text{ and } \sigma^s(\bar{I}(P, h)) = c$$

for all histories $h = (\mathcal{P}, Q, R) \in H^*$ with the same partial history as h_ℓ , that is, with $h \upharpoonright P^- = h_\ell \upharpoonright P^-$.

So in a regular equilibrium no individual will quit at P if he is *sure* that the price-clock will stop at P and that he will then reenter the bidding. In other words, there are no frivolous quits in a regular equilibrium.

Theorem 2 (uniqueness) *Any regular, symmetric pure-strategy equilibrium is fully revealing.*

Proof. Let (σ^0, σ^1) be any regular, symmetric pure-strategy equilibrium. For the equilibrium play, let P^1 denote the lowest price at which some individuals will quit while the clock is moving. Hence all individuals will observe an empty history until the price-clock reaches P^1 , then $Q_\ell(P^1) > 0$ individuals will quit while the clock is still moving, where $Q_\ell(P^1)$ may depend on ℓ . In particular, if *only* individuals with signal $s = 0$ quit at P^1 while the clock is moving, then

$$Q_\ell(P^1) = 1 - \eta_\ell.$$

Similarly, if only individuals with signal $s = 1$ quit,

$$Q_\ell(P^1) = \eta_\ell.$$

In either of these cases, once individuals reach the information set $\bar{I}(P^1, h_\ell)$, everyone will know the true quality v just by observing the number of individuals who quit at P^1 while the clock was moving. On the other hand, if both types quit at P^1 while the clock is moving, $Q_\ell(P^1) = 1$ for all ℓ , which will be entirely uninformative about the commodity's true quality. So there is an all-or-nothing feature in any symmetric equilibrium: observing the mass quitting at P^1 is either completely informative or completely uninformative.

We first show that, if the equilibrium is regular, P^1 will *always* be informative.

Step 1: *If (σ^0, σ^1) is a regular equilibrium, observing $Q_\ell(P^1)$ will be completely informative.*

Proof. Suppose the contrary, that $Q_\ell(P^1) = 1$ for all ℓ . Then the current history when the clock stops, namely $h_\ell \upharpoonright P^1$, would not depend on ℓ . Hence for each individual with signal s , his choice when the clock stops at P^1 , namely $\sigma^s(\bar{I}(P^1, h_\ell))$, must be the same for all ℓ . In turn, regularity implies:

$$\sigma^s(\bar{I}(P^1, h_\ell)) = q \quad \text{for all } \ell \text{ and } s,$$

otherwise there would be an individual with some signal s who quits while the clock is still moving at P^1 even though he is *sure* he will continue bidding beyond P^1 .

So, for all ℓ the bidding must stop at P^1 . That is, if $Q_\ell(P^1)$ is uninformative, (σ^0, σ^1) must lead to a pure pooling equilibrium in which every quality sells at P^1 and everyone has an equal chance $x = k$ of getting a unit. In this equilibrium, the expected profit of an individual with signal s is given by

$$\Pi^s = \sum_{\ell} \text{Prob}(v_\ell \mid s)(v_\ell - P^1)x = [E(v \mid s) - P^1]x.$$

It must be that $\Pi^s \geq 0$ for each s , otherwise some individual could profitably deviate to quitting (without reentering) at $P = 0$. Since $E(v | s = 1) > E(v | s = 0)$, it follows that $\Pi^1 > 0$. But then, since $x < 1$, any individual with signal $s = 1$ has a profitable deviation: Continue bidding at the information set $\bar{I}(P^1, h_\ell)$, ensuring himself of a unit with probability 1, hence an expected profit of $E(v | s = 1) - P^1 > \Pi^1$. \square

We next show that since $Q_\ell(P^1)$ is informative, the commodity will never sell for less than its true value.

Step 2: *Since observing $Q_\ell(P^1)$ is completely informative, $P_\ell \geq v_\ell$ for all ℓ .*

Proof. Suppose $P_\ell < v_\ell$ for some ℓ . The price clock will stop at P_ℓ , where $P_\ell \geq P^1$, the first news event. If it turns out that the true quality is v_ℓ , everyone will know it once the clock stops at P^1 , hence they will know it when the clock stops at P_ℓ . Since $P_\ell < v_\ell$, in the equilibrium history h_ℓ some individuals must be deciding not to continue when the clock stops at P_ℓ , say it is individuals with signal s . Then, in the history h_ℓ , each individual with signal s will get a unit of the commodity with probability $x(\sigma^s, h_\ell) < 1$ (an individual gets a unit with probability 1 only by continuing in the bidding beyond $P = P_\ell$). Such an individual has a profitable deviation: Play c at the information set $\bar{I}(P_\ell, h_\ell)$ instead of q , ensuring himself of a unit with probability 1. \square

The previous two steps imply that $P_\ell \geq v_\ell$ for all ℓ , so everyone's equilibrium expected payoff will be nonpositive. Suppose $P_\ell > v_\ell$ for some ℓ . Then someone's expected payoff will be negative. This individual has a profitable deviation: Quit

at $P = 0$ without reentering, ensuring himself of a zero expected payoff. So in equilibrium

$$P_\ell = v_\ell \quad \text{for all } \ell,$$

as asserted. \square

4. An extension: Two-dimensional signals

In this section we extend our model to one in which each individual receives both a common-value and a private-value signal. We give conditions under which there will be (i) full information aggregation *and* (ii) an efficient allocation of resources. By contrast, in a pure common-value setting it does not matter which buyers get units of the commodity. Hence the question of allocative efficiency does not arise.

Formally, suppose now that each buyer's utility depends on both the common-value parameter $v \in \mathbb{R}_+$ and a private-value parameter $t \in \mathbb{R}_+$. In particular, if a buyer must pay P to get a unit, his final utility from a commodity with common-value v when he is of type t is given by

$$V(v, t) - P,$$

where the function V is strictly increasing in both its arguments. A simple example is the additive function $V(v, t) = v + t$. Below we will sometimes write $V_t(v)$ instead of $V(v, t)$.

Nature picks the distribution of private values μ from a set of possible probability distributions \mathcal{T} on $(0, \bar{t}]$; so $\bar{t} < \infty$ is the maximum possible private value. For simplicity, we will assume \mathcal{T} only contains distributions with finite supports.

Given any $\mu \in \mathcal{T}$, let $J(\mu)$ equal the number of types t in the support of μ , and let j index these types with

$$t_1 < \cdots < t_j < \cdots < t_{J(\mu)}.$$

A population μ can be represented by a vector:

$$\mu = (t_1, \dots, t_{J(\mu)}, m_1, \dots, m_{J(\mu)}),$$

where $m_j \in (0, 1]$ equals the mass of buyers of type t_j in the population. After picking a distribution μ from \mathcal{T} , Nature assigns a fraction m_j of all buyers the private value t_j , for $j = 1, \dots, J(\mu)$; so each buyer's private value "signal" is entirely informative about his true private value t .

When the auction begins, although each buyer knows his own private value, he only knows that others' private values have been picked from some distribution $\mu \in \mathcal{T}$. We will not assume that \mathcal{T} is a singleton, that is, we will not assume the population μ is common knowledge. We also will not need to make the strong Bayesian, common-priors assumption that Nature picks μ from a *commonly-known* probability distribution whose support equals \mathcal{T} . There would be *complete ignorance* about others' private values if \mathcal{T} includes all possible distributions. We will need to assume something more, although much less than that \mathcal{T} is a singleton. We will assume

some common knowledge about others' private values:

$m_1 = \underline{m}$, a constant, for all $\mu \in \mathcal{T}$, so the mass of individuals with the lowest private value is common knowledge.

We emphasize that knowing the mass m_1 of lowest-valuing individuals does not imply their private value t_1 is commonly known. Indeed, in the typical case we have in mind, when the

auction begins no one knows whether his type t equals $\min t_j$. Compared to P&S2's assumption that the entire distribution μ is common knowledge (hence both the masses m_j and values t_j), common knowledge of only m_1 is a relatively weak assumption. For example, it implies that before the auction begins no one will know what an efficient allocation of resources is for the population μ ! Nevertheless, we will see that an English auction with reentry can lead to an efficient allocation.

We continue to assume that there are L possible common values indexed by ℓ . In addition to receiving a private-value signal t , as before each buyer receives a signal $s \in \{0, 1\}$ about the common value, where η_ℓ is the probability of receiving a good signal if $v = v_\ell$, and where η_ℓ is strictly increasing with v_ℓ . We now also assume that each individual's common-value signal is independent of his private-value, more precisely

disintegrability: if μ is the distribution of private values and if the commodity's common value is $v_\ell \in \mathcal{V}$, a fraction η_ℓ of all buyers with private value t_j will receive a good signal, for all j .

Disintegrability implies that, to determine the commodity's true quality, it is sufficient to know the mass of good signals that individuals of any *one* type j receive. Below, for any population μ , the signals received by individuals with the lowest private value t_1 will play a key role.

Given any distribution of private values μ , let $j(\mu)$ equal the index of the type such that

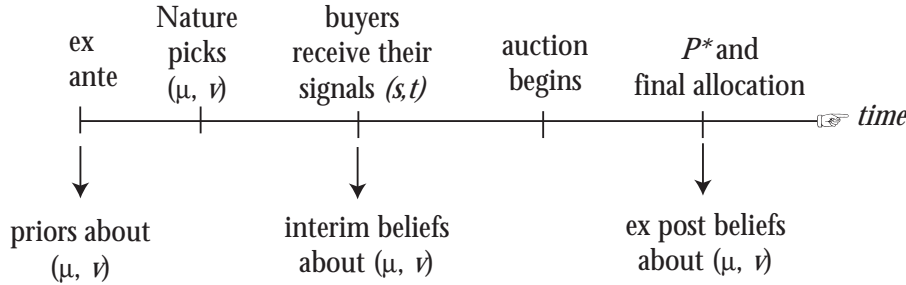
$$1 - \sum_{j=1}^{j=j(\mu)+1} m_j \leq k < 1 - \sum_{j=1}^{j=j(\mu)} m_j.$$

We will call individuals of type $j(\mu)$ the *marginal buyers* when the distribution of private values is μ . There is *allocative ef-*

efficiency for the population μ if all k units of the commodity are distributed, each individual with private value $t_j > t_{j(\mu)}$ gets a unit (with probability one), and each individual with private value $t_j < t_{j(\mu)}$ does not get a unit.

Extend the definition of a symmetric equilibrium for the English auction with reentry in an obvious way: Let σ_t^s be a pure strategy for a buyer of type t with signal s . A profile of such pure strategies $\{\sigma_t^s\} \equiv \{\sigma_t^s : s \in \{0, 1\} \text{ and } t \in (0, \bar{t}]\}$ is a symmetric equilibrium if no buyer with any signals (s, t) has a profitable deviation starting from any information set on the equilibrium path.

The timing of events in the 2-dimensional model is illustrated below. Unlike the 1-dimensional model, notice that a state of the world is now a pair $(\mu, v) \in \mathcal{T} \times \mathcal{V}$. We will say the



equilibrium $\{\sigma_t^s\}$ is *revealing* in the state (μ, v_ℓ) if, by the end of the auction, the true quality v_ℓ will be common knowledge in that state. The equilibrium is *always revealing* or, synonymously, *fully revealing* if it is revealing in all possible states $(\mu, v_\ell) \in \mathcal{T} \times \mathcal{V}$ that Nature might pick.

Let $W_\ell(\mu)$ denote the set of all possible Walrasian prices when the commodity's quality is v_ℓ and the population is μ , that is, when Nature has chosen the state (μ, v_ℓ) . As usual, $P \in W_\ell(\mu)$ means P is a market-clearing price when there is

perfect information that $v = v_\ell$ and buyers act as price-takers. We will call the equilibrium $\{\sigma_t^s\}$ *Walrasian* in (μ, v_ℓ) if both the equilibrium price and equilibrium allocation are Walrasian in the state (μ, v_ℓ) . The equilibrium is *always Walrasian* if it is Walrasian in all possible states in $\mathcal{T} \times \mathcal{V}$. So, if the equilibrium is always Walrasian and if $h_\ell = (P_\ell, Q_\ell, R_\ell)$ is the complete equilibrium history of bidding in the state (μ, v_ℓ) ,

$$P_\ell(\mu) \equiv \max \mathcal{P}_\ell \in W_\ell(\mu).$$

Of course, the First Theorem of Welfare Economics implies that if the equilibrium is always Walrasian, it will always be allocatively efficient.

Note: To avoid double subscripts, let us write the valuation of a marginal buyer for a unit of quality v_ℓ as $V_{t_\mu}(v_\ell) \equiv V_{t_{j(\mu)}}(v_\ell)$. If the Walrasian price is unique when the population is μ and $v = v_\ell$, then the Walrasian price will equal any marginal buyer's valuation: $P_\ell(\mu) \in W_\ell(\mu) \Rightarrow P_\ell(\mu) = V_{t_\mu}(v_\ell)$; on the other hand, if the Walrasian demand curve intersects the vertical supply over an interval, $W_\ell(\mu)$ will consist of a closed interval of prices, with $V_{t_\mu}(v_\ell)$ the minimum possible Walrasian price.

Analogous to the 1-dimensional model, we take for granted below that the maximum possible bid in the auction, namely \bar{P} , exceeds the maximum possible valuation: $\bar{P} > \max_\ell V(v_\ell, \bar{t})$.

Existence

Theorem 3 (existence) *In the model with 2-dimensional signals, there exists an equilibrium $\{\sigma_t^s\}$ that is always revealing and Walrasian.*

Proof. The proof will be constructive. To simplify notation, we will often write tj instead of t_j , and $t\mu$ instead of $t_{j(\mu)}$. Thus for example $V_{tj}(v_\ell) \equiv V_{t_j}(v_\ell)$. In the equilibrium we will construct, for any given μ , the value of the lowest possible quality to the lowest-valuing buyer, namely $V_{t_1}(v_1) \equiv \underline{V}(\mu)$, will play a crucial role, as will the mass of lowest-valuing buyers, namely $m_1 = \underline{m}$. When μ is given, we will write \underline{V} instead of $\underline{V}(\mu)$ to ease the notation. Please keep in mind that no buyer of any type t knows whether $t = t_1$ when the auction begins.

In the equilibrium, for any given μ , the first news will occur when the price-clock reaches $P = \underline{V}$, before that everyone only sees an empty history. In particular for any given μ , if the true quality is v_1 (the lowest possible quality), the equilibrium history of play $h_1 \equiv (\mathcal{P}_1, Q_1, R_1)$ will satisfy:

1. $\mathcal{P}_1 = \{V_{t_1}(v_1), V_{t_2}(v_1), \dots, V_{t_\mu}(v_1)\}$ (so there will be a news event at the valuation for v_1 of each type up to the marginal type $t\mu$).
2. $Q_1(V_{t_1}(v_1)) = (1 - \eta_1)\underline{m}$ because each individual with a bad signal of any type t plans to quit at $P = V_t(v_1)$ if the clock is running and no one has yet quit;
3. $R_1(V_{t_1}(v_1)) = -\eta_1\underline{m}$ because each individual with a good signal of any type t plans to quit when the clock stops at the first news event, say P^1 , *provided* $P^1 = V_t(v_1)$ and he gets the bad news that $(1 - \eta_1)\underline{m}$ people quit while the clock was running at P^1 .
4. $Q_1(V_{tk}(v_1)) = m_{tk}$ and $R_1(V_{tk}(v_1)) = 0$ for $k = 2, 3, \dots, j(\mu)$ because each individual of any type t plans to continue in the bidding until $P = V_t(v_1)$ provided the

first news event was at a price strictly less than $V_t(v_1)$ *and* he got the bad news that $(1 - \eta_1)\underline{m}$ people quit at the first news event.

On the other hand, for any given μ , if the true quality is $v_\ell > v_1$ the equilibrium history of play $h_\ell \equiv (\mathcal{P}_\ell, Q_\ell, R_\ell)$ will satisfy:

1. $\mathcal{P}_\ell = \{V_{t_1}(v_\ell), V_{t_1}(v_1), V_{t_2}(v_\ell), \dots, V_{t_\mu}(v_\ell)\}$ (so there will still be a news event at \underline{V} , and also a news event at the valuation for v_ℓ of each type up to the marginal type $t\mu$).
2. $Q_\ell(V_{t_1}(v_1)) = (1 - \eta_\ell)\underline{m}$ because each individual with a bad signal of any type t plans to quit at $P = V_t(v_1)$ if the clock is running and no one has yet quit;
3. $R_\ell(V_{t_1}(v_1)) = (1 - \eta_\ell)\underline{m}$ because each individual with a bad signal of any type t plans to reenter when the clock stops at the first news event *provided* he gets the good news that less than $(1 - \eta_1)\underline{m}$ people quit.
4. $Q_\ell(V_{tk}(v_\ell)) = m_{tk}$ and $R_\ell(V_{tk}(v_\ell)) = 0$ for $k = 1, 2, \dots, j(\mu)$ because each individual of any type t plans to continue in the bidding until $P = V_t(v_\ell)$ provided less than $(1 - \eta_1)\underline{m}$ people quit while the clock was still running at the first news event.

The above histories lead to an equilibrium play that is always revealing and Walrasian, illustrated in Figure 6. We now specify players' strategies leading to the above histories. Let σ_t^1 , the strategy of an individual of any type t who received a good signal, be such that the individual will remain in the bidding (play c) at least until the price clock stops at the first news event (that is, as long as he observes an empty

history). By contrast let σ_t^0 , the strategy of an individual of any type t who received a bad signal, be such that the individual will remain in the bidding (play c) at least until the price clock stops at the first news event *unless* the clock reaches $P = V_t(v_1)$ —his valuation for the lowest quality commodity—without stopping; in the latter event he quits at $P = V_t(v_1)$ while the clock is still running. Notice that, for any given μ , these strategies lead to the first news event occurring at $P = \underline{V}(\mu)$, the valuation of the lowest-valuing buyers in μ for the lowest-quality commodity; and they lead to

$$Q_\ell(\underline{V}) = (1 - \eta_\ell)\underline{m} \quad \text{for all } \ell,$$

that is, all individuals of type $t1$ with bad signals quit, as specified in the aggregate histories above.

Once the clock stops at \underline{V} , individuals will find themselves in some information set

$$I_\ell = (\underline{V}, \textit{stopped}, h_\ell \mid \underline{V}, \textit{active}),$$

which depends only on ℓ . Notice at this point on the equilibrium path, since \underline{m} is common knowledge, all buyers will know the commodity's true quality:

$v = v_\ell$ if and only if the mass $(1 - \eta_\ell)\underline{m}$ quit at the first news event while the clock was moving.

Starting from this point, each individual will have a dominant strategy: if he learns $v = v_\ell$, his dominant strategy is to continue in the bidding as long as $P < v_\ell$, and then to permanently quit if the clock reaches $P = v_\ell$. In our equilibrium, for any μ , all individuals play these dominant strategies starting from the point when the clock stops at \underline{V} . So, if it is learned that $v = v_1$, $\sigma_{t1}^0(I_\ell) = \sigma_{t1}^1(I_\ell) = q$ (all individuals

of the lowest-valuing type $t1$ quit) while all other types continue bidding — leading to $R_1(\underline{V}) = -\eta_1\underline{m}$ as specified in the aggregate history above. On the other hand, if it is learned that $v = v_\ell > v_1$, $\sigma_t^0(I_\ell) = \sigma_t^1(I_\ell) = c$ for *all* types t — leading to $R_\ell(v_1) = (1 - \eta_\ell)\underline{m}$ as specified above. Given all individuals are now playing their dominant strategies, the game will end — k or less bidders will remain in the bidding — once the price clock reaches $P = V_{t\mu}(v_\ell)$, the valuation for v_ℓ of the marginal buyers. So the equilibrium price will equal the smallest Walrasian price: $P_\ell = \min W_\ell(\mu)$ for all ℓ .

If individuals play as above, the outcome will be fully revealing and always Walrasian. Further, each individual will indeed want to play as above since his Walrasian demand is satisfied if he does not deviate and since he cannot influence the Walrasian price by deviating; so there is no profitable deviation for him. \square

The constructed equilibrium is illustrated in Figure 6 (drawn assuming the Walrasian price is unique). It is interesting to observe that if $v = v_1$, for any population μ the equilibrium history of play traces out the Walrasian demand curve for the population up to its minimum Walrasian price P_1 . Similarly if $v = v_\ell > v_1$, except for the learning episode when the clock momentarily stops at $V_{t1}(v_1) \equiv \underline{V}(\mu)$, the history of play again traces out the Walrasian demand curve for the population up to its minimum Walrasian price P_ℓ .

It is worth emphasizing that before the auction begins, since no one knows the true quality v_ℓ or the distribution of private value μ that Nature has picked, no one knows what the Walrasian price is or even what an efficient allocation of resources is. The Walrasian price is revealed by the market process, involving both competition and learning. Thus, just a little bit

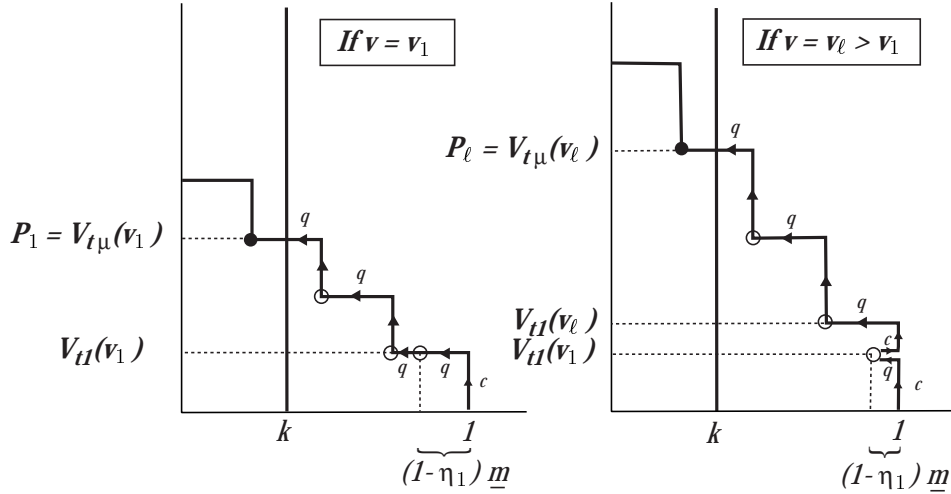


Figure 6

of common knowledge—just knowing m_1 —goes a long way! The proof of Theorem 3 shows the role that this common knowledge plays: If Nature has chosen $v_\ell \in \mathcal{V}$ and $\mu \in \mathcal{T}$, at the first news event the mass of quits while the clock is still running will equal

$$(1 - \eta_\ell)m_1.$$

Since everyone knows m_1 , everyone will infer $v = v_\ell$ if they observe this number of quits. But if there were complete ignorance about others' private values, each individual would be faced with an identification problem. If he does not know m_1 , how can he know whether the initial bidding is enthusiastic because η_ℓ is high or because m_1 is low? The possibility of identification problems will be illustrated in Example 4 below.

Uniqueness

Theorem 3 extends Theorem 1 to the case of 2-dimensional signals: Theorem 1 can be viewed as the special case of Theorem 3 in which $\mathcal{T} = \{\mu\}$, a singleton; and $\mu = (t, 1)$, so everyone has the same private value t in μ . We now show that under an appropriate regularity assumption, Theorem 2 (uniqueness) also can be extended to the case of 2-dimensional signals.

It seems natural to focus on equilibria in which individuals with the worst signals quit first. Formally expressed, consider a given equilibrium $\{\sigma_t^s\}$, and let

$$I_0(P) = (P, \text{moving}, h \mid P^-, \text{active})$$

be the information set in which the clock has reached P , is still moving, and the current history $h \mid P^-$ is an empty history (so the clock has not stopped at any price below P). Let

$$P_0(s, t) = \min\{P : \sigma_t^s(I_0(P)) = q\},$$

so $P_0(s, t)$ is the first price at which an individual with signals (s, t) plays quit if he sees no one quit before him.

DEFINITION 2. An equilibrium $\{\sigma_t^s\}$ is *regular* if $P_0(s, t)$ is strictly increasing in both s and t .

So in a regular equilibrium, for any distribution μ that Nature picks, individuals with a bad signal $s = 0$ and private value $t = t_1$ quit first.

Theorem 4 (uniqueness) *In the model with 2-dimensional signals, any regular equilibrium $\{\sigma_t^s\}$ is always revealing and Walrasian.*

Proof. Choose any population $\mu \in \mathcal{T}$. Let $h_\ell = (\mathcal{P}_\ell, Q_\ell, R_\ell)$ be the complete equilibrium history of bidding in the population μ if $v = v_\ell$, and let P^1 denote the price at which the clock first stops on the equilibrium path for μ . Regularity implies only individuals with signals $s = 0$ and $t = t_1$ will quit at P^1 while the clock is still moving, hence

$$Q_\ell(P^1) = (1 - \eta_\ell)\underline{m} \quad \text{for all } \ell.$$

It follows that, since \underline{m} is common knowledge, once individuals reach the information set $\bar{I}(P^1, h_\ell)$, everyone will know the true quality v_ℓ just by observing the mass of individuals that quit at P^1 while the clock was moving. That is, if the equilibrium is regular, the mass of quits at P^1 will be fully informative.

We now show that since $Q_\ell(P^1)$ is fully informative, the commodity will never sell for less than its minimum Walrasian price, that is,

$$P_\ell \geq \min W_\ell(\mu) \quad \text{for all } \ell.$$

Suppose the contrary, that $P_\ell < \min W_\ell(\mu)$ for some ℓ . If it turns out that the true quality is v_ℓ , everyone will know it once the clock stops at P^1 (the first news event), hence they will know it when the clock stops at P_ℓ . Let T_ℓ equal the set of types t in the support of μ such that $V(v_\ell, t) - P_\ell > 0$; and let J_ℓ equal the set of indexes j corresponding to these types. In the equilibrium history h_ℓ all types $t \in T_\ell$ will want to play c (continue) when the clock stops at P_ℓ , in order to ensure they will obtain a unit of the commodity with probability 1. But $P_\ell < \min W_\ell(\mu)$ implies $\sum_{j \in J_\ell} m_j > k$. Thus $D(P_\ell, h_\ell) > k$, contradicting h_ℓ is a complete history; that is, if $P_\ell < \min W_\ell(\mu)$, the price clock will not permanently stop moving at $P = P_\ell$.

Now suppose $P_\ell \notin W_\ell(\mu)$ for some ℓ . The above implies $P_\ell > \max W_\ell(\mu)$. Let \bar{T}_ℓ equal the set of types t in the support of μ such that $V(v_\ell, t) - P_\ell < 0$, and let \bar{J}_ℓ equal the set of indexes j corresponding to these types. There are two cases to consider: (i) $P_\ell > P^1$ or (ii) $P_\ell = P^1$. In the first case, $x(\sigma_t^s, h_\ell) = 0$ for all types $t \in \bar{T}_\ell$. (Otherwise any such type would have a profitable deviation: quit permanently once the clock stops at P^1 if it is revealed that $v = v_\ell$.) But $P_\ell > \max W_\ell(\mu)$ implies

$$1 - \sum_{j \in \bar{J}_\ell} m_j < k.$$

Hence, since $x(\sigma_t^s, h_\ell) = 0$ for all types $t \in \bar{T}_\ell$, either all these types quit permanently before the clock reaches P_ℓ (which implies the auction would end before P_ℓ is reached, a contradiction), or some of these types quit only once the clock reaches P_ℓ (which implies $x(\sigma_t^s, h_\ell) > 0$ for some $t \in \bar{T}_\ell$ since $D(P_\ell, h_\ell) < k$). So, either way, case (i) leads to a contradiction.

Only case (ii) remains possible. Accordingly suppose v_ℓ turns out to be the true quality, and $P_\ell = P^1 > \max W_\ell(\mu)$. We will again show a contradiction. Everyone will learn $v = v_\ell$ once the clock stops at P^1 . At this point buyers of all types $t \in \bar{T}_\ell$ will quit. But they will still have a positive probability of winning, hence of having to pay more for a unit than it is worth to them. Any such type t has a profitable deviation: Quit at $P = 0$ and stay out of the bidding until the clock first stops, hence until the true quality v_ℓ is revealed; if the clock first stops at P^1 , reenter the bidding at P^1 if and only if $P^1 < V(v_\ell, t)$; if $P^1 < V(v_\ell, t)$, continue in the bidding until the clock reaches $P = V(v_\ell, t)$, then quit permanently. This deviation will ensure the individual the maximum possible

payoff: for any $v_\ell \in \mathcal{V}$ and $\mu \in \mathcal{T}$, he will win with probability 1 whenever $P_\ell(\mu) < V(v_\ell, t)$, and he will win with probability 0 whenever $P_\ell(\mu) > V(v_\ell, t)$.

We conclude that any regular equilibrium is always revealing, and the equilibrium price is always Walrasian. Furthermore, the equilibrium allocation is also always Walrasian, otherwise the above deviation would be profitable. \square

That m_1 is common knowledge plays a crucial role in the proof of uniqueness. To highlight this, in the following example the mass of lowest-valuing buyers will be uncertain, leading to the possibility of an equilibrium that is not fully revealing.

Example 4 (the possibility of identification problems) Let $\mathcal{T} = \{\alpha, \beta\}$, so there are only two possible populations, where

$$\alpha = (t_1, t_2, .3, .7) \text{ and } \beta = (t_1, t_2, .7, .3),$$

with

$$t_1 = 0 \text{ and } t_2 = 1.$$

So the same types t inhabit both populations, but the mass of type 1 individuals is smaller in α than in β .

There are 2 possible qualities:

$$v_1 = 2 \text{ or } v_2 = 4, \text{ with } \eta_1 = .3 \text{ and } \eta_2 = .7.$$

Assume $V(v, t) = v + t$ and the mass of objects is $k = .9$. Since the mass of type 2 buyers is less than .9 in both α and β , the marginal buyers are of type $t = 0$ in both populations. If there were perfect information, the Walrasian price of a unit would be $P = 2$ if $v = 2$ or $P = 4$ if $v = 4$.

We now construct an equilibrium $\{\sigma_t^s\}$ that is not fully revealing. Choose $\{\sigma_t^s\}$ so that

$$P_0(s, t) = V(v_2, t) = 2 + t \quad \text{if } s = 0,$$

while $P_0(s, t) > 2 + t$ if $s = 1$. So individuals with signals $(s, t) = (0, 0)$ will always quit first (i.e., the equilibrium is regular), and the first news event will occur at $P^1 = 2$. Let

$$Q^1(\mu, v_\ell) \equiv (1 - \eta_\ell)m_1$$

denote the mass of buyers that quits at P^1 while the clock is still running, if the population is μ and the commodity's true quality is v_ℓ . In particular:

$$\begin{aligned} Q^1(\alpha, v_1) &= .09, & Q^1(\alpha, v_2) &= .21, \\ Q^1(\beta, v_1) &= .21, & Q^1(\beta, v_2) &= .49. \end{aligned}$$

The upshot is, once the clock stops at P^1 , everyone will know the true quality is high if they see that only 9% quit at P^1 (enthusiastic bidding). Similarly, everyone will know the true quality is low if they see 49% quit at P^1 (unenthusiastic bidding). But if they see 21% quit, no one will know if $v = v_1$ or $v = v_2$. Everyone faces an identification problem: Is the moderate number of quits due to both v and m_1 being high, or is it due to both v and m_1 being low?

If Q^1 is informative, let all buyers play their dominant strategies from P^1 onward, as in the proof of Theorem 3. Thus, when Q^1 is informative, the equilibrium outcome will be revealing and Walrasian. But if Q^1 is uninformative, let all buyers with signals $(s, t) = (0, 0)$ return to the bidding at P^1 and continue bidding until the price reaches

$$P^* = E(V \mid s = 0, t = 0).$$

At this point let all buyers with signals $(s, t) = (0, 0)$ quit permanently, while all other buyers continue in the bidding even when the clock stops at P^* . The upshot is that when Q^1 is uninformative, the equilibrium price will be P^* , hence neither revealing nor Walrasian.

To be more concrete, assume Nature picks μ and v independently, with $\text{Prob}(\alpha) = \text{Prob}(\beta) = .5$ and $\text{Prob}(v_1) = \text{Prob}(v_2) = .5$. Then $E(v_1 | s = 0) = .3$, hence $E(V | s = 0, t = 0) = .3(4) + .7(2) = 2.8$. So 50% of the time the equilibrium will not be revealing or Walrasian. In these states, the equilibrium price will be $P^* = 2.8$, that is, above the commodity's true value if $v = 2$, but below this value if $v = 4$. See Figure 7.

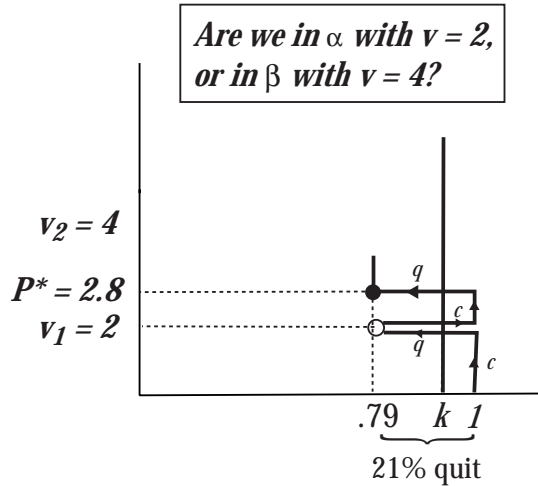


Figure 7

To conclude the example, we check that there are no profitable deviations when the outcome is not revealing. In these states, each buyer with signals $(s, t) = (0, 0)$ only receives

a unit with probability $x = 11/21$ (see Figure 7, recalling $k = .9$); but he is indifferent whether he gets a unit or not, so there is no profitable deviation for him. All other buyers get a unit with probability 1; since they strictly prefer this to being rationed, there is no profitable deviation for them either. It is worth noting that, even when the outcome is not revealing, it is still efficient in this example because each individual with $t = 1$ always gets a unit with probability 1 (cf. Example 6 below).

5. Informed and uninformed agents

Suppose that only some buyers get an informative signal. Will the uninformed be able to learn from the informed?

To answer this question, we now extend the model of Section 4 to include the possibility of uninformed agents. Assume there are three possible signals, in particular $s \in \{-1, 0, 1\}$, where $s = -1$ is interpreted as an uninformative signal about the commodity's quality (essentially receiving no signal). A *population* is now a probability distribution μ on $\mathbb{R}_+ \times \{\text{uninformed}, \text{informed}\}$, where buyers $(t, \text{uninformed})$ in the support of μ are buyers with private value t who will get the common-value signal $s = -1$, and buyers $(t, \text{informed}) \in \text{supp } \mu$ are buyers with private value t who will get an informative signal, either $s = 0$ or $s = 1$. Restricting ourselves to populations with finite supports, and letting $j = 1, \dots, J(\mu)$ index the private values in the support of μ , a population μ now can be described by a vector

$$\mu = (t_1, \dots, t_{J(\mu)}, m_1, \dots, m_{J(\mu)}, \gamma_1, \dots, \gamma_{J(\mu)}),$$

where $\gamma_j \in [0, 1]$ is the proportion of buyers with private value

t_j who will receive an informative signal; hence $\gamma_j m_j$ is the mass of such buyers who will be receive an informative signal. \mathcal{T} continues to denote the set of possible populations, where now a $\mu \in \mathcal{T}$ includes both informed and uninformed buyers. We continue to assume disintegrability in the sense that, for any state of the world $(\mu, v_\ell) \in \mathcal{T} \times \mathcal{V}$ that Nature picks, the mass of informed buyers with good signals will equal $\eta_\ell \gamma_j m_j$ for all j .

A symmetric *equilibrium* $\{\sigma_t^s\}$ now specifies a strategy σ_t^s for each buyer of every possible type $t \in (0, \bar{t})$ and with every possible signal $s \in \{-1, 0, 1\}$, including the uninformative signal. The following example illustrates the possibility of an equilibrium in which uninformed buyers learn from the bidding of the informed buyers. The example is based on an example in Izmalkov (2001), which in turn is based on an example in Perry and Reny (2001).

Example 5 In this example we assume the population μ is common knowledge, there are just two types with equal mass, and only buyers of type t_1 are informed. So, using our notation:

$$\mu = (t_1, t_2, .5, .5, 1, 0).$$

Assume the number of possible qualities L is very large, with $v_\ell = \ell/L$ for $\ell = 1, 2, \dots, L$; so the possible qualities range from $v_1 = 1/L \approx 0$ to $v_L = 1$. Also assume for any possible quality v :

$$V(v, t_1) = v \quad \text{and} \quad V(v, t_2) = v^2 + 2/9,$$

and the mass of objects $k = .5$. So for allocative efficiency all objects should go to type 1 individuals if the true quality is $v \in (1/3, 2/3)$, and all should go to type 2 individuals if $v < 1/3$ or $v > 2/3$.

The example is used by Izmalkov to motivate an English auction *with reentry*. The range of values v for which type 2 individuals should win is not connected. But in a standard English auction, it is unrealistically assumed that once an individual drops out of the bidding, he cannot reenter; so no standard English auction is efficient for all possible v in this example. However, once reentry is permitted, efficient equilibria do exist.

In particular, consider the following symmetric equilibrium.⁶ All type 2 buyers and all type 1 buyers with a good signal continue bidding until the first news event. All type 1 buyers with a bad signal continue bidding until the price-clock reaches $P = v_1 \equiv 1/L$, then they quit while the clock is still moving. So the number of quits at v_1 is entirely informative about the commodity's true quality: $.5\eta_\ell$ buyers will quit if and only if the commodity is of quality v_ℓ .

Suppose it is learned once the clock stops at v_1 that the true quality is v . Then, in equilibrium, all type 2 buyers continue bidding until the price clock reaches $V(v, t_2)$, at which point they all permanently quit. [We assume L is sufficiently large that the first news event $v_1 \equiv 1/L$ occurs before the price clock reaches $2/9 < \min_\ell V(v_\ell, t_2)$.] If it is learned that $v = v_1$, all type 1 buyers with good signals quit once the price-clock stops at $P = v_1$. But if it is learned that $v > v_1$, all type 1 buyers who quit while the clock was running, reenter the bidding once the clock stops; then, similar to the type 2 buyers, all type 1 buyers continue bidding until the price clock reaches $V(v, t_1)$,

⁶Although similar in spirit to Izmalkov, our equilibrium will differ in its details because Izmalkov assumes there is only a small number of buyers; in particular, in his example, there is only one buyer of type 1, one buyer of type 2, and one unit available, and each informed buyer receives a perfectly-informative signal rather than a very coarse signal.

then they all permanently quit.

As illustrated in Figure 8, these equilibrium strategies result in an outcome that is always revealing and Walrasian, hence efficient.

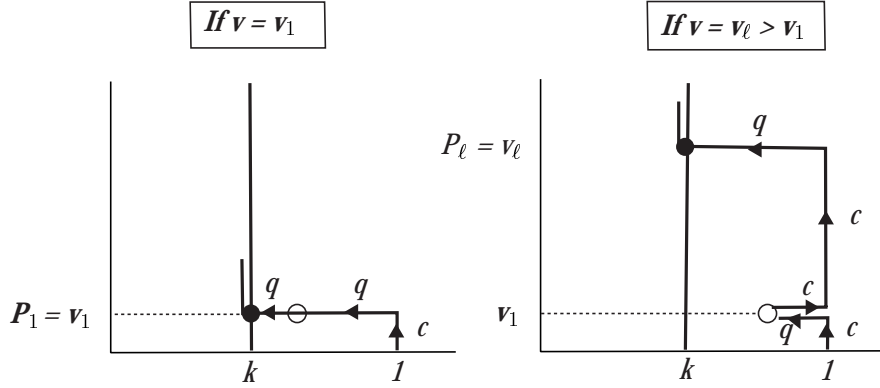


Figure 8

Recall our assumption in the previous section that $V(v, t)$ is strictly increasing in both v and t . Buyers' preferences in the above example illustrate that this assumption rules out some interesting possibilities. Thus, for the sake of generality, we shall now assume that V is strictly increasing in v , but not necessarily in t . This implies for example that, even for a fixed population, the identity of the marginal buyer may change with the commodity's quality v . We will continue to assume (just for convenience) that for any given population $\mu = (t_1, \dots, t_J, m_1, \dots, m_J, \gamma_1, \dots, \gamma_J) \in \mathcal{T}$, there is a *unique* type t that solves

$$\min_j V(v_1, t_j).$$

We will label this type t_1 ; it is the type in μ with the lowest

valuation for the lowest-quality good. More substantially, we also will assume that:

Some common knowledge: $(t_1, m_1, \gamma_1) = (\underline{t}, \underline{m}, \underline{\gamma})$, a constant, for all $\mu \in \mathcal{T}$, with $\underline{\gamma} > 0$.

That is, all aggregate details about the lowest-valuing buyers are commonly known, *and* there are some informed buyers among the lowest-valuing buyers. It is worth emphasizing that this common-knowledge assumption about others' private values is a lot weaker than assuming the entire distribution μ is common knowledge—the typical assumption in the literature on information aggregation.⁷ For example, if $1 - m_1 > k$, the above is consistent with no one knowing the Walrasian price for any quality v_ℓ before the auction begins; so, if the equilibrium is Walrasian, the market process will reveal this price, initially unknown to everyone. The above assumption also is consistent with no one knowing what an efficient allocation is before the auction begins. Nevertheless the above is stronger than our assumption in the previous section that only the mass m_1 is common knowledge. Some stronger assumption now is required to avoid identification problems, in particular, for buyers to be able to disentangle the bidding of uninformed and informed agents. The additional assumption that some lowest-valuing buyers are informed ($\gamma_1 > 0$) is trivially satisfied in the previous setting where everyone received

⁷In the literature with 1-dimensional signals and only a small number of bidders (e.g., Izmalkov), this assumption takes a little different form. Let $u_i(s_1, \dots, s_n)$ be the valuation of buyer i for the commodity when the profile of signals to all n buyers is (s_1, \dots, s_n) . In this literature it is assumed that the *functional form* of all individual's $u_i(\cdot)$'s is common knowledge — i.e., (u_1, \dots, u_n) is common knowledge, — although each individual only knows his own signal s_i .

an informative signal. Besides helping to overcome identification problems, this additional assumption is especially important for showing *uniqueness* rather than existence, helping to ensure that learning comes “soon enough” (see Example 6 below).

We can now generalize the lesson of Example 5.

Theorem 5 (existence) *In the model with 2-dimensional signals and only some informed buyers, there exists an equilibrium $\{\sigma_t^s\}$ that is always revealing and Walrasian.*

Proof. The proof involves only a small modification of the proof of Theorem 3.

Let $\sigma_{t_1}^0$, the strategy of a lowest-valuing buyer who receives a bad signal, satisfy: continue bidding until the price clock reaches $\underline{V} \equiv V(v_1, t_1)$, then quit while the clock is still moving. Let the strategies of all other types, regardless of their signal, satisfy: continue bidding at least until the clock stops at \underline{V} .

Notice that, for any given μ , these choices lead to the first news event occurring at $P = \underline{V}$, the valuation of the lowest-valuing buyers for the lowest-quality commodity; and they lead to

$$Q_\ell(\underline{V}) = (1 - \eta_\ell)\underline{\gamma}\underline{m} \quad \text{for all } \ell,$$

that is, all informed individuals of type t_1 with bad signals quit.

Once the clock stops at \underline{V} , individuals will find themselves in some information set

$$I_\ell = (\underline{V}, \text{stopped}, h_\ell \mid \underline{V}, \text{active}),$$

which depends only on ℓ . At this point on the equilibrium path, since $\underline{\gamma}$ and \underline{m} are common knowledge, all buyers will know the commodity’s true quality:

$v = v_\ell$ if and only if the mass $(1 - \eta_\ell)\underline{\gamma}\underline{m}$ quit at the first news event while the clock was moving.

From this point onward, the proof follows the same line as the proof of Theorem 3. Each individual will have a dominant strategy starting from the point when the price-clock stops at \underline{V} . As in the proof of Theorem 3, all individuals play these dominant strategies starting from \underline{V} . \square

The constructed equilibrium is again illustrated by Figure 6, except $(1 - \eta_\ell)\underline{m}$ must now be replaced with $(1 - \eta_\ell)\underline{\gamma}\underline{m}$.

We can also extend uniqueness. Analogous to the definition of regularity in the previous section, in the current generalized model we will call an equilibrium $\{\sigma_t^s\}$ *regular* if individuals with the lowest private value t_1 and worst common-value signal $s = 0$ quit first.

Theorem 6 (uniqueness) *In the model with 2-dimensional signals and only some informed buyers, any regular equilibrium $\{\sigma_t^s\}$ is always revealing and Walrasian.*

Proof. The proof involves only a small modification of the proof of Theorem 4.

Choose any population $\mu \in \mathcal{T}$. Let $h_\ell = (\mathcal{P}_\ell, Q_\ell, R_\ell)$ be the complete equilibrium history of bidding in the population μ if $v = v_\ell$, let $P_\ell \equiv \max \mathcal{P}_\ell$ be the equilibrium price in μ if $v = v_\ell$, and let P^1 denote the price at which the clock first stops on the equilibrium path. Regularity implies

$$Q_\ell(P^1) = (1 - \eta_\ell)\underline{\gamma}\underline{m} \quad \text{for all } \ell.$$

It follows that, since $\underline{\gamma}$ and \underline{m} are common knowledge, once individuals reach the information set $\bar{I}(P^1, h_\ell)$, everyone will

know the true quality v_ℓ just by observing the mass of individuals that quit at P^1 while the clock was moving. That is, if the equilibrium is regular, the mass of quits at P^1 will be fully informative.

From this point onward, the proof is identical to that of Theorem 4: One first shows that, since everyone knows the true quality once the clock stops at P^1 , $P_\ell \geq \min W_\ell(\mu)$ for all ℓ . Then one shows this weak inequality must be an equality. \square

When μ is uncertain, we have to worry about identification problems as illustrated in Example 4 in the previous section. When not everyone is informed, we also have to worry about learning coming “soon enough” — even when μ is common knowledge. Our assumption that some lowest-valuing buyers are informed ($\gamma_1 > 0$) ensures learning does come soon enough to permit full information aggregation. In the following example, this assumption will be violated — $\gamma_1 = 0$, — leading to the possibility of an equilibrium that is neither revealing nor efficient.

Example 6 (the possibility of inefficient equilibria without learning) Assume the population μ is common knowledge and is given by

$$\mu = (t_1, t_2, .8, .2, 0, 1).$$

So the mass of type 1 individuals is larger than the mass of type 2 individuals, and only the latter are informed. There are only 2 possible qualities, with

$$v_1 = 0 \quad \text{and} \quad v_2 = 4.$$

The private values satisfy for any possible v :

$$V(v, t_1) = v \quad \text{and} \quad V(v, t_2) = .5v + 1,$$

and the mass of objects is $k = .5$. So for efficiency, every type 2 individual should get a unit if $v = 0$, and all units should be rationed among the type 1 individuals if $v = 4$.

All individuals’ priors are that low quality is more likely, in particular:

$$\text{Prob}(v = 0) = .9.$$

Hence an uninformed type 1 individual’s willingness-to-pay for a unit is:

$$\tilde{P} \equiv E(V(v, t_1) \mid s = -1) = .4.$$

Consider the equilibrium strategies in which all uninformed individuals continue bidding until $P = \tilde{P}$, then they all permanently quit; but all informed individuals—including ones with bad signals—continue bidding even beyond $P = \tilde{P}$. These strategies lead to a pooling equilibrium in which either quality sells for \tilde{P} , and each type 2 individual always gets a unit. To check that these strategies indeed form an equilibrium, observe there is no incentive for a uninformed buyer to deviate since he learns nothing about informed buyers’ signals. There also is no incentive for any informed buyer to deviate since

$$\tilde{P} < 1 < E(V(v_\ell, t_2) \mid s = 0) < E(V(v_\ell, t_2) \mid s = 1).$$

In this equilibrium, the allocation is efficient when $v = 0$, but not when $v = 4$.

The above equilibrium is not unique. Indeed, for this economy there also exists an equilibrium that is always revealing and Walrasian, hence always efficient: Let all informed buyers with $s = 0$ quit (at least temporarily) at $P = 0$ while the clock is moving. The mass of quits at this price will be completely informative about the commodity’s true quality. Once the

clock stops at $P = 0$ and learning has occurred, let all buyers play their dominant strategies as in the proof of Theorem 5.

This example highlights the importance—for uniqueness—of our assumption that there are some *informed* buyers of type $t = t_1$.

6. Conclusion

Where do prices come from? How can the market process aggregate individuals' widely-dispersed common-value information? In this paper we have seen how an English auction can sometimes accomplish these tasks.

Recall the new car example in the Introduction. While writing “Pesendorfer and Swinkels in the Continuum,” I frequently discussed this example with some non-economist friends who wanted to know what I was doing; they eagerly pointed out to me that in “real life” new cars are not sold by sealed-bid auctions! Allowing for open bidding takes us a bit closer to reality. A potential buyer may well bid enthusiastically for a popular new-model car that is in short supply, offering the dealer more than the sticker price. Similarly, for unpopular slower-selling new cars, the dealer may well accept an offer substantially below the sticker price. Of course, an English auction only captures such bidding in a very stylized way, but it is a first step toward understanding the competitive market process more generally, which is our larger goal.

One important strand of the literature on information aggregation that we have barely mentioned is the mechanism-design approach. In our setting, since each buyer is “informationally redundant” (recall the Introduction), it is easy to construct a direct mechanism that is both efficient and

fully-revealing: No one buyer has an incentive to lie about his common-value signal since—given he is of measure zero—the information he reports to the mechanism designer (whether he received $s = 0$ or $s = 1$) will have no effect on the total mass of good and bad signals reported. Each individual will, of course, only have a weak incentive to truthfully reveal (that is, he will be indifferent whether he reports $s = 0$ or $s = 1$); nevertheless, it will be incentive compatible for him to tell the truth. Further, as long as the mechanism always chooses a Walrasian allocation, each individual also will have a (strong) incentive to truthfully reveal his private value, for the usual reasons. In the current paper, I will not prove the existence of an efficient and fully-revealing direct mechanism, for either the 1- or 2-dimensional models. Interested readers are directed to the nicely crafted and very educational paper by McLean and Postlewaite (2000). They consider the (harder) case of designing efficient mechanisms when individuals are “informationally small” but not necessarily “redundant”—analogous to doing asymptotics rather than working at the limit. Also see Jackson (1999).

It is worth pointing out that when buyers are informationally redundant, there exists a direct revelation mechanism that is both efficient and fully-revealing even when there is *complete ignorance* about others' private values (an observation also made by McLean and Postlewaite). Our need to assume at least some common knowledge about others' private values in Sections 4–5 arises because we are analyzing an indirect mechanism—hence analyzing how a *decentralized* process may lead to information aggregation, without the aid of a centralized mechanism designer who can costlessly process everyone's reports. A similar point arises in Izmalkov (2001) who, in addition to the usual single-crossing condition, must assume a

signal-intensity condition in order to ensure the existence of an efficient English auction. By contrast, only the single-crossing condition is required to ensure the existence of an efficient *direct* mechanism when there are a small number of bidders each with a 1-dimensional signal (Crémer and McLean (1985)).

References

- [1] Adam Brandenburger, *Knowledge and equilibrium in games*, Journal of Economic Perspectives **6** (1992), 83–101.
- [2] J. Crémer and Richard McLean, *Optimal selling strategies under uncertainty for a discriminating monopolist when demands are interdependent*, Econometrica **53** (1985), 345–361.
- [3] P. Dasgupta and Eric Maskin, *Efficient auctions*, Quarterly Journal of Economics **115** (1997), 341–389.
- [4] P. Dubey, A. Mas-Colell, and M. Shubik, *Efficiency properties of strategic market games: An axiomatic approach*, Journal of Economic Theory **22** (1980), 339–362.
- [5] Sergei Izmalkov, *English auctions with reentry*, Discussion Paper, Pennsylvania State University, 2001.
- [6] M. O. Jackson, *Efficiency and information aggregation in auctions with costly information*, Discussion Paper, Caltech, 1999.
- [7] Philippe Jehiel and Benny Moldovanu, *Efficient design with interdependent valuations*, Econometrica **69** (2001), 1237–1259.
- [8] Louis Makowski, *Pesendorfer and Swinkels in the continuum*, Discussion Paper, UC Davis, 2002.
- [9] Louis Makowski and Joseph M. Ostroy, *Perfect competition and the creativity of the market*, Journal of Economic Literature **39** (2001), 479–535.
- [10] Eric Maskin, *Auctions and privatization*, in H. Siebert (ed.), *Privatization*, Institut fuer Weltwirtschaft der Universitaet Kiel, Kiel, 115–136, 1992.
- [11] Richard McLean and Andrew Postlewaite, *Efficient auction mechanisms with multidimensional signals*, Discussion Paper, University of Pennsylvania, 2000.
- [12] Paul Milgrom, *A convergence theorem for competitive bidding with differential information*, Econometrica **47** (1979), 670–688.
- [13] ———, *Rational expectations, information aggregation, and competitive bidding*, Econometrica **49** (1981), 923–943.
- [14] M. Perry and P. Reny, *An efficient multi-unit ascending auction*, Discussion Paper, Hebrew University, 2001.
- [15] Wolfgang Pesendorfer and Jeroen M. Swinkels, *The loser’s curse and information aggregation in common value auctions*, Econometrica **65** (1997), 1247–1281.
- [16] ———, *Efficiency and information aggregation in auctions*, Discussion Paper, Northwestern University, 1998.
- [17] Xavier Vives, *Aggregation of information in large Cournot markets*, Econometrica **56** (1988).

- [18] Robert Wilson, *A bidding model of perfect competition*,
Review of Economic Studies **44** (1977), 511–518.