A User's Guide to Solving Real Business Cycle Models

The typical real business cycle model is based upon an economy populated by identical infinitely-lived households and firms, so that economic choices are reflected in the decisions made by a single representative agent. It is assumed that both output and factor markets are characterized by perfect competition. Households sell capital, k_t, to firms at the rental rate of capital and sell labor, h_t, at the real wage rate. Each period firms choose capital and labor subject to a production function to maximize profits. Output is produced according to a constant-returns-to-scale production function that is subject to random technology shocks. Specifically $y_t = z_t f(k_t, h_t)$, where y_t is output and z_t is the technology shock. (The price of output is normalized to one.) Households' decisions are more complicated; given their initial capital stock, agents determine how much labor to supply and how much consumption and investment to purchase. These choices are made in order to maximize the expected value of lifetime utility. Households must forecast the future path of wages and the rental rate of capital. It is assumed that these forecasts are made rationallyA rational expectations equilibrium consists of sequences for consumption, capital, labor, output, wages, and the rental rate of capital such that factor and output markets clear.

While it is fairly straightforward to show that a competitive equilibrium exists, it is difficult to solve for the equilibrium sequences directly. Instead an indirect approach is taken in which the Pareto optimum for this economy is determined (this will be unique given the assumption of representative agents). As shown by Debreu (1954), the Pareto optimum as characterized by the optimal sequences for consumption, labor, and capital in this environment will be identical to that in a competitive equilibrium. Furthermore, factor prices are determined by the marginal products of capital and labor evaluated at the equilibrium quantities. (For a detailed exposition of the connection between the competitive equilibrium and Pareto optimum in a real business cycle model, see Prescott, 1986 **[4]**.We now provide an example of solving such a model.

I. DERIVING THE EQUILIBRIUM CONDITIONS

The first step in solving for the competitive equilibrium is to determine the Pareto optimum. To do this, the real business cycle model is recast as the following social planner's problem:

$$\begin{aligned}
&\max \quad E_1\left[\sum_{t=1}^{\infty} b^{\,t-1} U\left(c_t, 1-h_t\right)\right] \\
&\textit{subject to}: \\
(1) \qquad &c_t + i_t = z_t f\left(k_t, h_t\right) \equiv y_t . \\
&k_{t+1} = k_t\left(1-d\right) + i_t . \\
&z_{t+1} = z_t^{r} e_{t+1} . \\
&k_1 \textit{ is given.}
\end{aligned}$$

where $E_1[\cdot]$ denotes expectations conditional on information at $t = 1$, $0 < b < 1$ is agents' discount factor, c_t denotes consumption, $\left(1-h_t\right)$ is leisure (agents endowment of time is normalized to one), i_t is investment, and $0 < d < 1$ is the depreciation rate of capital. The exogenous technology shock is assumed to follow the autoregressive process given in the last equation; the autocorrelation parameter is $0 \leq r \leq 1$ and the innovation to technology is assumed to have a mean of one and standard deviation s_e. The first two constraints in (1) is the economy-wide resource constraint and the second is the law of motion for the capital stock.

Dynamic Programming Problem

This infinite horizon problem can be solved by exploiting its recursive structure. That is, the nature of the social planner's problem is the same every period: given the beginning-of-period capital stock and the current technology shock, choose consumption, labor and investment. Note that utility is assumed to be time-separable; that is the choices of consumption and labor at time *t* do not affect the marginal utilities of consumption and leisure in any other time period. Because of this recursive structure, it is useful to cast the maximization problem as the following dynamic programming problem (for a discussion of dynamic programming, see Sargent (1987)):

state variables at time *t*: $\left(k_t, z_t\right)$.

control variables at time *t*: $\left(c_t, h_t, k_{t+1}\right)$.

$$
\begin{aligned}
& v\left(k_t, z_t\right) = \max_{\left(c_t, k_{t+1}, h_t\right)} \left\{U\left(c_t, 1-h_t\right) + b\, E_t\left[v\left(k_{t+1}, z_{t+1}\right)\right]\right\} \\
(2) \qquad & \textit{subject to} \;\; c_t + k_{t+1} = z_t f\left(k_t, z_t\right) + k_t\left(1-d\right) \\
& \textit{and} \qquad\qquad z_{t+1} = z_t^r e_{t+1}
\end{aligned}
$$

(Note that investment has been eliminated by using the law of motion for the capital stock.) A solution to this problem must satisfy the following necessary conditions and resource constraint:

$$
\begin{aligned}
(N1) \qquad & U_{2,t} = U_{1,t} z_t f_{2,t}. \\
(N2) \qquad & U_{1,t} = b\, E_t\left\{U_{1,t+1}\left[z_{t+1} f_{1,t+1} + \left(1-d\right)\right]\right\}. \\
(RC) \qquad & k_{t+1} = z_t f\left(k_t, h_t\right) + k_t\left(1-d\right) - c_t.
\end{aligned}
$$

Where the notation $U_{i,t}\,; i = 1,2$ denotes the derivative of the utility function with respect to the *i*th argument evaluated at the quantities $\left(c_t, 1-h_t\right)$; $f_{i,t}\,; i = 1,2$ has an analogous interpretation. *N1* represents the intra-temporal efficiency condition (the labor-leisure tradeoff).Iit implies that the marginal rate of substitution between labor and consumption must equal the marginal product of labor. The second condition, *N2*, represents the intertemporal efficiency condition. The left-hand side represents the marginal cost in terms of utility of investing in more capital while the right-hand side represents the expected marginal utility gain; at an optimum these costs and benefits must be equal.

To simplify the analysis (again, see Prescott (1986 **[4]**) for a justification), assume the following functional forms:

$$U(c_t, 1-h_t) = \ln c_t + A(1-h_t); \quad f(k_t, z_t) = z_t k_t^a h_t^{1-a} .$$

(The assumption that utility is linear in leisure is based on Hansen's (1985 **[8]**) model . Then the three equilibrium conditions become

$$\begin{aligned} c_t &= \left[(1-a)\, z_t k_t^a h_t^{-a} / A\right]. \\ c_t^{-1} &= b\, E_t\left\{c_{t+1}^{-1}\left[a\, z_{t+1} k_{t+1}^{a-1} h_{t+1}^{1-a} + (1-d)\right]\right\}. \\ k_{t+1} &= z_t k_t^a h_t^{1-a} + k_t(1-d) - c_t . \end{aligned} \tag{3}$$

A steady-state equilibrium for this economy is one in which the technology shock is assumed to be constant so that there is no uncertainty, that is $z_t = 1$ for all *t*, and the values of capital, labor, and consumption are constant, $k_t = \bar{k}, h_t = \bar{h}, c_t = \bar{c}$ for all *t*. Imposing these steady-state conditions in (3), the steady-state values are found by solving the following steady-state equilibrium conditions:

$$(SS1) \quad \bar{c} = ((1-a)/A)\bar{k}^a \bar{h}^{-a} .$$

$$(SS2) \quad b^{-1} - 1 + d = a\, \bar{k}^{a-1} \bar{h}^{1-a} = a(\bar{y}/\bar{k}) .$$

$$(SS3) \quad d\, \bar{k} = \bar{k}^a \bar{h}^{1-a} - \bar{c} = \bar{y} - \bar{c} .$$

In the above expressions, $\bar{y}$ denotes the steady-state level of output.

<u>Calibration</u>

The next step in solving the model is to choose parameter values for the model. This is done through *calibration*: the set of parameters (d, b, A, a) are chosen so that the steady-state behavior of the model match the long-run characteristics of the data. The features of the data which do not exhibit cyclical characteristics are:

(1) $(1-a)$ = labor's average share of output.

(2) $b^{-1}-1$ = average risk-free real interest rate.

(3) Given (a, b) choose d so that the output-capital ratio (from (SS2)) is consistent with observation.

(4) The parameter *A* determines the time spent in work activity. To see this, multiply both sides of (SS1) by $\bar{h}$ and rearrange the expression to yield: $\bar{h} = \left[(1-a)/A\right](\bar{y}/\bar{c})$. But the steady-state resource constraint, (SS3), implies that $\dfrac{\bar{y}}{\bar{c}} = \dfrac{1}{1-d\left(\dfrac{\bar{k}}{\bar{y}}\right)}$ so that the output-consumption ratio is implied by the parameter values chosen in the previous three steps. Hence, the choice of *A* directly determines $\bar{h}$.

Typical parameter values based on postwar U.S. data (see Hansen and Wright (1992 **[4]**) are: $a = 0.36$ implying labor's share is 64%, $b = 0.99$ implying an annual riskless interest rate of 0.04%, $d = 0.025$ implying the capital-output ratio (where output is measured on a quarterly basis) of roughly 10, and *A* = 3 which implies that roughly 30% of time is spent in work activity. (These values will be used later in Section IV below.)

II. LINEARIZATION

The solution to the social planner's problem is characterized by a set of policy functions for capital, consumption, and labor; moreover, the solution exists and is unique; (see Prescott (1986 **[4]**). There is, however, no analytical solution. To make the model operational, therefore, an approximate numerical solution is found. One of the simplest methods is to take a linear approximation (i.e. a first-order Taylor series expansion) of the three equilibrium conditions and the law of motion of the technology shock around the steady-state values $\left(\bar{c},\bar{k},\bar{h},\bar{z}\right)$. Provided the stochastic behavior of the model does not push the economy too far from the steady-state behavior, the linear approximation will be a good one. (The discussion below follows closely that of Farmer (1994).) This technique is demonstrated below:[1]

Intratemporal efficiency condition:

The optimal labor-leisure choice is represented by condition *N1*:

$$c_t = \left[\left(1-a\right)/A\right] z_t k_t^a h_t^{-a} .$$

Linearizing around the steady-state values $\left(\bar{c},\bar{k},\bar{h},\bar{z}\right)$:

$$\begin{aligned}
\left(c_t-\bar{c}\right) &= a\left[\left(1-a\right)/A\right]\bar{k}^{a-1}\bar{h}^{-a}\left(k_t-\bar{k}\right) - a\left[\left(1-a\right)/A\right]\bar{k}^{a}\bar{h}^{-a-1}\left(h_t-\bar{h}\right) \\
&\quad + \left[\left(1-a\right)/A\right]\bar{k}^{a}\bar{h}^{-a}\left(z_t-\bar{z}\right) \\
&= a\left[\left(1-a\right)/A\right]\bar{k}^{a}\bar{h}^{-a}\frac{\left(k_t-\bar{k}\right)}{\bar{k}} - a\left[\left(1-a\right)/A\right]\bar{k}^{a}\bar{h}^{-a}\frac{\left(h_t-\bar{h}\right)}{\bar{h}} \\
&\quad + \left[\left(1-a\right)/A\right]\bar{k}^{a}\bar{h}^{-a}\frac{\left(z_t-\bar{z}\right)}{\bar{z}}
\end{aligned} \tag{4}$$

Note that in the last expression, all variables have been expressed as percentage deviations from the steady-state (the first two terms modify the respective derivatives while the last term uses the fact that $\bar{z}=1$ in steady-state). Consumption can be expressed as a percentage deviation from steady-state by using the steady-state condition $\bar{c}=\left[\left(1-a\right)/A\right]\bar{k}^{a}\bar{h}^{-a}$; dividing both sides of the

[1] Recall that the general form for the Taylor series expansion of a function around a point x* is:

$$f\left(x\right)-f\left(x^*\right)=f'\left(x^*\right)\left(x-x^*\right)+f''\left(x^*\right)\frac{\left(x-x^*\right)^2}{2!}+f'''\left(x^*\right)\frac{\left(x-x^*\right)^3}{3!}+\ldots$$

where *N*! denotes factorial.

equation by this expression and denoting percentage deviations from steady-state as $\tilde{x}$, eq. (4) can be written as:

$$\tilde{c}_t = a\ \tilde{k}_t - a\ \tilde{h}_t + \tilde{z}_t . \tag{5}$$

Intertemporal Efficiency Condition:

This efficiency condition is given by *N2*

$$c_t^{-1} = b\, E_t \left\{ c_{t+1}^{-1} \left[a\, z_{t+1} k_{t+1}^{a-1} h_{t+1}^{1-a} + (1-d) \right] \right\}$$

Again, linearizing around the steady-state and expressing all variables as percentage deviations from steady-state yields:

$$\begin{aligned} -\bar{c}^{-1}\tilde{c}_t = {} & -b\,\bar{c}^{-1}\left[a\,\bar{k}^{a-1}\bar{h}^{1-a} + (1-d)\right] E_t(\tilde{c}_{t+1}) + b\,\bar{c}^{-1} a\,(a-1)\bar{k}^{a-1}\bar{h}^{1-a} E_t(\tilde{k}_{t+1}) \\ & + b\,\bar{c}^{-1} a\,(1-a)\bar{k}^{a-1}\bar{h}^{1-a}\ E_t(\tilde{h}_{t+1}) + b\,\bar{c}^{-1} a\ \bar{k}^{a-1}\bar{h}^{1-a}\ E_t(\tilde{z}_{t+1}) \end{aligned}$$

Multiplying each side of the equation by $\bar{c}$ and using the steady-state condition (SS2) that

$$1 = b\ \left[a\,\bar{k}^{a-1}\bar{h}^{1-a} + (1-d)\right]$$

yields

$$\begin{aligned} -\tilde{c}_t = {} & -E_t(\tilde{c}_{t+1}) + b\,(a-1)a\,\bar{k}^{a-1}\bar{h}^{1-a} E_t(\tilde{k}_{t+1}) \\ & + b\,(1-a)a\,\bar{k}^{a-1}\bar{h}^{1-a} E_t(\tilde{h}_{t+1}) \\ & + b\,a\,\bar{k}^{a-1}\bar{h}^{1-a} E_t(\tilde{z}_{t+1}) \end{aligned} \tag{6}$$

Resource Constraint

Following the same procedure as before, linearizing the resource constraint around the steady-state yields

$$\tilde{k}_{t+1} = \left[a\,\bar{k}^{a-1}\bar{h}^{1-a} + (1-d)\right]\tilde{k}_t + (1-a)\bar{k}^{a-1}\bar{h}^{1-a}\tilde{h}_t + \bar{k}^{a-1}\bar{h}^{1-a}\tilde{z}_t - \left(\bar{c}/\bar{k}\right)\tilde{c}_t\,. \tag{7}$$

Technology Shock Process

The critical difference between the steady-state model and the real business cycle model is the assumption that technology shocks are random - the shocks follow the autoregressive process described in eq. (1). Linearizing the auto-regressive process for the technology shock results in:

$$\tilde{z}_{t+1} = r\,\tilde{z}_t + \tilde{e}_{t+1}\,. \tag{8}$$

Taking expectations of both sides:

$$E_t\left(\tilde{z}_{t+1}\right) = r\ \tilde{z}_t\,. \tag{9}$$

III. SOLUTION METHOD

The equations that define a rational expectations equilibrium (eqs. 5, 6, 7, 9) can be written as a vector expectational difference equation. Let $\mathbf{u}_t = \begin{pmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{h}_t \\ \tilde{z}_t \end{pmatrix}$ where bold print denotes a vector, then the linear system of equations can be written as:

$$\mathbf{A}\mathbf{u}_t = \mathbf{B}E_t\left(\mathbf{u}_{t+1}\right). \tag{10}$$

The matrices **A** and **B** are:

$$\mathbf{A} = \begin{pmatrix} 1 & -a & a & -1 \\ -1 & 0 & 0 & 0 \\ -\bar{c}/\bar{k} & a\,\bar{k}^{a-1}\bar{h}^{1-a} + 1 - d & (1-a)\bar{k}^{a-1}\bar{h}^{1-a} & \bar{k}^{a-1}\bar{h}^{1-a} \\ 0 & 0 & 0 & r \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & b(a-1)a\,\bar{k}^{a-1}\bar{h}^{1-a} & b(1-a)a\,\bar{k}^{a-1}\bar{h}^{1-a} & b\;a\,\bar{k}^{a-1}\bar{h}^{1-a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Premultiplying both sides of eq. (10) by $\mathbf{A}^{-1}$ yields:

(11) $$\mathbf{u}_t = \mathbf{A}^{-1}\mathbf{B}\,E_t(\mathbf{u}_{t+1}) .$$

The matrix $\mathbf{A}^{-1}\mathbf{B}$ can be decomposed as (see Hamilton (1994) for details):

(12) $$\mathbf{A}^{-1}\mathbf{B} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1} .$$

where $\mathbf{Q}$ is a matrix whose columns are the eigenvectors of $\mathbf{A}^{-1}\mathbf{B}$ and $\boldsymbol{\Lambda}$ is a diagonal matrix whose diagonal elements are the eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$. Using this decomposition and premultiplying both sides of the resulting expression in eq. (11) by $\mathbf{Q}^{-1}$ yields:

(13) $$\mathbf{Q}^{-1}\mathbf{u}_t \equiv \mathbf{d}_t = \boldsymbol{\Lambda}\,E_t(\mathbf{d}_{t+1}) = \boldsymbol{\Lambda}\,E_t\left(\mathbf{Q}^{-1}\mathbf{u}_{t+1}\right).$$

Note that the elements of the defined (*4* x *1*) column vector $\mathbf{d}_t$ are constructed from a linear combination of the elements in the rows of the (*4* x *4*) matrix $\mathbf{Q}^{-1}$ and the elements of the (*4* x *1*) column vector $\mathbf{u}_t$. Since $\boldsymbol{\Lambda}$ is a diagonal matrix, eq. (13) implies four independent equations:

$$d_{i,t} = l_i E_t\left(d_{i,t+1}\right);\ i = 1,2,3,4\ . \tag{14}$$

Since the equations in (14) must hold every period, it is possible to recursively substitute the expressions forward for *T* periods to yield:

$$d_{i,t} = l_i^T E_t\left(d_{i,t+T}\right);\ i = 1,2,3,4\ . \tag{15}$$

The l_i are four distinct eigenvalues associated with the four equilibrium conditions (eqs. 5 - 8). Since one of these conditions is the law of motion for the exogenous technology shock (eq. (8)), one of the eigenvalues will be r^{-1}. Also, the first rows of the matrices **A** and **B** are determined by the intratemporal efficiency condition; since this is not a dynamic relationship, one of the eigenvalues will be zero. The remaining two eigenvalues will bracket the value of unity as is typical for a saddle path equilibrium implied by the underlying stochastic growth framework. As implied by eq. (15), the stable, rational expectations solution to the expectational difference equation is associated with the eigenvalue with a value less than one. That is, if $l_i > 1$ then iterating forward implies $d_{i,t} \to \infty$ which is not a permissible equilibrium. Furthermore, for eq. (15) to hold for all *T* (again taking the limit of the right-hand side), in the stable case when $l < 1$, it must be the true that $d_{i,t} = 0$; this restriction provides the desired solution. That is, $d_{i,t} = 0$ imposes the linear restriction on $\left(\tilde{c}_t, \tilde{k}_t, \tilde{h}_t, \tilde{z}_t\right)$ which is consistent with a rational expectations solution. (Recall that $d_{i,t}$ represents a linear combination between the elements of a particular row of $\mathbf{Q}^{-1}$ and the elements of the vector $\mathbf{u}_t$.)

IV. A PARAMETRIC EXAMPLE

In this section, a parameterized version of the RBC model described above is solved. The following parameter values are used: $\left(b = 0.99, a = 0.36, d = 0.025, A = 3\right)$. These imply the following steady-state values: $\left(\bar{c} = 0.79, \bar{k} = 10.90, \bar{h} = 0.29, \bar{y} = 1.06\right)$. Note that these values imply that agents spend roughly 30% of their time in work activities and the capital-output ratio is approximately 10 (output is measured on quarterly basis); both of these values are broadly consistent with US experience (see McGrattan, 1994).

The remaining parameter values determine the behavior of the technology shock. These are estimated by constructing the Solow residual[2] and then detrending that series linearly. Specifically, the Solow residual is defined as $Z_t = \ln y_t - a \ln k_t - \left(1 - a\right) \ln h_t$. The Z_t series can then be regressed on a linear time trend (which is consistent with the assumption of constant technological progress) and the residual is identified as the technology shock z_t. Using this procedure on quarterly data over the period 60.1 - 94.4 resulted in an estimate of the serial correlation of z_t (the parameter r) to be 0.95. The variance of the shock to technology (i.e. the variance of $\tilde{e}_t$ in eq. (8)) was estimated to be 0.007. Note that the variance of the technology shock is not relevant in solving the linearized version of the model -- however, when the solution of the model is used to generate artificial time series in the simulation of the economy, this parameter value must be stipulated.

These values generated the following entries into the **A** and **B** matrices:

[2] The use of the Solow residual as a measure of technology shocks is discussed in Hoover and Salyer (1996).

$$\begin{pmatrix} 1 & -0.36 & 0.36 & -1 \\ -1 & 0 & 0 & 0 \\ -0.072 & 1.010 & 0.062 & 0.098 \\ 0 & 0 & 0 & 0.95 \end{pmatrix}\begin{pmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{h}_t \\ \tilde{z}_t \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & -0.022 & 0.022 & 0.035 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} E_t \begin{pmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \\ \tilde{h}_{t+1} \\ \tilde{z}_{t+1} \end{pmatrix}$$

Following the steps described in the previous section (pre-multiplying by $\mathbf{A}^{-1}$) yields the following:

$$\begin{pmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{h}_t \\ \tilde{z}_t \end{pmatrix} = \begin{pmatrix} 1 & 0.022 & -0.022 & -0.035 \\ 0.23 & 0.94 & -0.0051 & -0.27 \\ -2.55 & 0.87 & 0.057 & 2.75 \\ 0 & 0 & 0 & 1.05 \end{pmatrix} E_t \begin{pmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \\ \tilde{h}_{t+1} \\ \tilde{z}_{t+1} \end{pmatrix}$$

Next, decomposing $\mathbf{A}^{-1}\mathbf{B}$ into $\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1}$ and then pre-multiplying by $\mathbf{Q}^{-1}$ yields

$$\mathbf{Q}^{-1}\mathbf{u}_t = \begin{pmatrix} -2.18 & -0.048 & 0.048 & 24.26 \\ 0 & 0 & 0 & 23.01 \\ -2.50 & 1.36 & 0.056 & 1.10 \\ -2.62 & 0.94 & -0.94 & 2.62 \end{pmatrix}\begin{pmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{h}_t \\ \tilde{z}_t \end{pmatrix} = \Lambda E_t\left(\mathbf{Q}^{-1}\mathbf{u}_{t+1}\right) =$$

$$\begin{pmatrix} 1.062 & 0 & 0 & 0 \\ 0 & 1.05 & 0 & 0 \\ 0 & 0 & 0.93 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} E_t \left[\begin{pmatrix} -2.18 & -0.048 & 0.048 & 24.26 \\ 0 & 0 & 0 & 23.01 \\ -2.50 & 1.36 & 0.056 & 1.10 \\ -2.62 & 0.94 & -0.94 & 2.62 \end{pmatrix}\begin{pmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \\ \tilde{h}_{t+1} \\ \tilde{z}_{t+1} \end{pmatrix}\right]$$

The entries in the matrix $\boldsymbol{\Lambda}$ (i.e. the eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$) determine the solution. Note that the second diagonal entry is (accounting for rounding error) r^{-1}. The fourth row of $\boldsymbol{\Lambda}$ is associated with the intratemporal efficiency condition. These values are proportional to those given in the first row of the **A** matrix; consequently dividing all entries by (-2.62) returns the original intra-temporal efficiency condition. The remaining two entries in the $\boldsymbol{\Lambda}$ matrix are those related to the saddle path properties of the steady-state solution. Since a stable rational expectations solution is associated with an eigenvalue less than unity, the third row of the $\mathbf{Q}^{-1}$ matrix provides the linear restriction we are seeking. That is, the rational expectations solution is:

$$-2.50\tilde{c}_t + 1.36\tilde{h}_t + 0.056\tilde{k}_t + 1.10\tilde{z}_t = 0.$$

Or,

(16) $$\tilde{c}_t = 0.54\tilde{h}_t + 0.02\tilde{k}_t + 0.44\tilde{z}_t.$$

The law of motion for the capital stock (the parameter values are given in the third row of the **A** matrix) and the intratemporal efficiency condition provides two more equilibrium conditions:

(17) $$\tilde{k}_{t+1} = -0.07\tilde{c}_t + 1.01\tilde{k}_t + 0.06\tilde{h}_t + 0.10\tilde{z}_t.$$

(18) $$\tilde{h}_t = -2.78\tilde{c}_t + \tilde{k}_t + 2.78\tilde{z}_t.$$

A random number generator can next be used to produce a sequence of technology shocks. The above equilibrium equations can then be used to produce time series for capital, consumption, labor, and output.

V. ANALYZING OUTPUT FROM THE ARTIFICIAL ECONOMY

The solution to the model is characterized by eqs. (16)- (18) - given initial values for capital, and next generating a path for the exogenous technology shock $(\tilde{z}_t)$, these equations will produce time-series for $(\tilde{c}_t, \tilde{k}_t, \tilde{h}_t)$. Two other series that most macroeconomists are interested in, namely output and investment, can be generated by linearizing the production function and the resource constraint, respectively.

Specifically, for output, linearizing the assumed Cobb-Douglas production function (i.e. $y_t = z_t k_t^a h_t^{1-a}$ and using the calibrated value that $a = 0.36$) yields the following equation:

(19) $$\tilde{y}_t = \tilde{z}_t + 0.36\,\tilde{k}_t + 0.64\,\tilde{h}_t.$$

Finally, a linear approximation of the condition that, in equilibrium, output must equal the sum of consumption and investment can be expressed in the form as a percentage deviation from the steady state as:

(20) $$\tilde{i}_t = \frac{\bar{y}}{\bar{i}}\tilde{y}_t - \frac{\bar{c}}{\bar{i}}\tilde{c}_t .$$

Using the steady-state values employed in the numerical solution, the investment equation becomes:

(21) $$\tilde{i}_t = \frac{1.06}{0.27}\tilde{y}_t - \frac{0.79}{0.27}\tilde{c}_t = 3.92\,\tilde{y}_t - 2.92\,\tilde{c}_t .$$

Hence, equilibrium in this economy is described by the following set of equations

(E) $$\begin{aligned}
\tilde{c}_t &= 0.54\tilde{h}_t + 0.02\tilde{k}_t + 0.44\tilde{z}_t \\
\tilde{k}_{t+1} &= -0.07\tilde{c}_t + 1.01\tilde{k}_t + 0.06\tilde{h}_t + 0.10\tilde{z}_t \\
\tilde{h}_t &= -2.78\tilde{c}_t + \tilde{k}_t + 2.78\tilde{z}_t \\
\tilde{y}_t &= \tilde{z}_t + 0.36\,\tilde{k}_t + 0.64\,\tilde{h}_t \\
\tilde{i}_t &= 3.92\,\tilde{y}_t - 2.92\,\tilde{c}_t \\
\tilde{z}_t &= 0.95\,\tilde{z}_{t-1} + \tilde{e}_t
\end{aligned}$$

To generate the time series implied by the model, it is necessary to first generate a series for the innovations to the technology shock, i.e $\tilde{e}_t$. These are assumed to have a mean of zero and a variance that is consistent with the observed variance for the innovations, which, as mentioned above, is roughly 0.007. Then, initializing $\tilde{z}_t = 0$ and using a random number generator in order to generate the innovations, a path for the technology shocks is created. Next, assuming that all remaining values are initially at their steady-state (which implies that all initial values are set to zero), the system of equations above can be solved to produce the time path for the endogenous variables.

We generate artificial time paths for consumption, output, and investment. (3000 observations were created and only the last 120 were examined) These are shown in Figure 1.

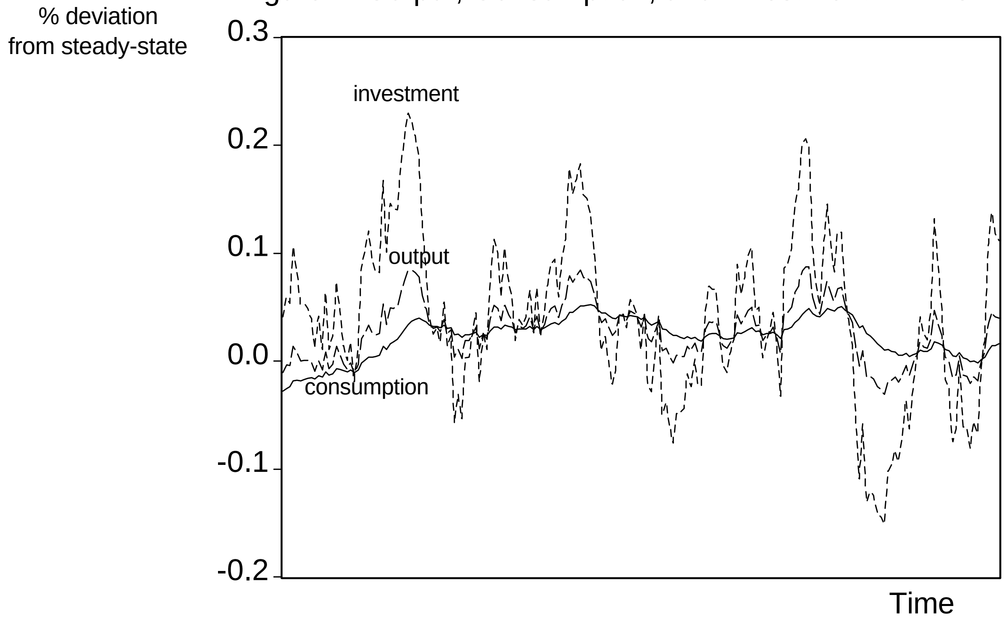
It is clear from Figure 1, as is also true in the actual data,that, the volatility of investment is greater than that of output, which is greater than that of consumption. To see this more precisely, the standard deviation of consumption, labor, and investment relative to output is reported in Table 1 along with the correlations of these series with output.

Table 1: Descriptive Statistics for U.S. and RBC Model[3]

		relative volatility	Corr(*x*, *y*)
consumption	model	0.52	0.82
	U.S. data	0.49	0.76
investment	model	2.86	0.95
	U.S. data	3.02	0.80
labor	model	0.65	0.89
	U.S. data	0.96	0.88

[3] Statistics for U.S. data are taken from Kydland and Prescott (1990 **[21]**), Tables I and II, p. 10-11.

Figure 1: Output, Consumption, and Investment in RBC Model



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