Professor Salyer, Economics 200E

1 Notes on the CCAPM

Since I had some difficulty writing the intuition for the sign of the risk premium of an asset, here are some notes on the risk premia in the CCAPM.

Suppose there are two assets in the economy: a riskless bond that costs 1 unit at time t and returns $R1_t$ units of consumption next period and a risky asset with (random) return of ρ_{t+1}. The associated necessary conditions are:

$$1 = R1_t \beta E_t \left[\frac{U'_{t+1}}{U'_t} \right] = R1_t \beta E_t \left(S_{t,t+1} \right) \tag{1}$$

$$1 = \beta E_t \left[S_{t,t+1} \rho_{t+1} \right] \tag{2}$$

Using the definition of covariance in eq.(2) yields:

$$1 = \beta \left[Cov_t \left(S_{t,t+1}, \rho_{t+1} \right) + E_t \left(S_{t,t+1} \right) E_t \left(\rho_{t+1} \right) \right] \tag{3}$$

Then, rearranging terms and using the definition of $R1_t$, one obtains:

$$E_t \left(\rho_{t+1} \right) - R1_t = -A_t Cov_t \left(S_{t,t+1}, \rho_{t+1} \right) \tag{4}$$

where $A_t = \beta R1_t > 0$.

The left hand side of eq.(4) is the definition of the (conditional) risk premium associated with the risky asset. The sign of the risk premium is the negative of the sign of the covariance between the marginal rate of substitution and the return of the risky asset. Note that since the only term that is random in the marginal rate of substitution is next period's marginal utility, the critical covariance is: $Cov_t \left(U'_{t+1}, \rho_{t+1} \right)$.

Now, the long awaited intuition: Risk averse agents do not want volatility in their marginal utility - a risky asset is, therefore, one that increases the volatility of U'. That is, if the return is high in those states in which consumption is already high (and, consequently, U' low), the asset's payout structure is exacerbating the volatility of the agent's marginal utility. Note that this payout structure implies that $Cov_t \left(U'_{t+1}, \rho_{t+1} \right) < 0$ so that the risk premium is positive. That is, the expected return on the asset must be greater than that on a riskless asset to compensate investors for the increased volatility in their marginal utility. (What kind of asset would carry a negative risk premium?) For those of you who have had some finance and studied the CAPM, note that this model extends the concept of risk in that model. There, it is the covariance between an asset's return and the market portfolio that determines the risk of an asset. This model (as its name implies) replaces the market portfolio with marginal utility.

2 Implications for the term premium

Suppose that, in addition to one period riskless bonds, two period bonds are traded. These bonds cost one unit of consumption in period t and return $R2_t^2$ upon maturity in period $t+2$. (Note that all returns are expressed as one-period yields.) The associated necessary condition is:

$$U_t' = R2_t^2 \beta^2 E_t \left(U_{t+2}' \right) \tag{5}$$

Or,

$$R2_t^2 = \frac{1}{\beta^2 E_t \left[\frac{U_{t+2}'}{U_t'} \right]} \tag{6}$$

The term premium asks whether the return from purchasing a long term bond (here 2-period) is greater than that on a one-period bond. Defining this in terms of the holding premium, the term premium is given by the following expression

$$TP_t = R2_t^2 E_t \left(\frac{1}{R1_{t+1}} \right) - R1_t \tag{7}$$

The first expression is the expected return from selling the two-period bond after one period. Note that the holding premium is identical to a risk premium since it is defined as "expected return minus certain return". Using eq.(6) and eq.(1) in eq. (7) allows the (condtional) term premium to be written as:

$$TP_t = \frac{1}{\beta^2 E_t \left[\frac{U_{t+2}'}{U_t'} \right]} E_t \left[\beta E_{t+1} \left(\frac{U_{t+2}'}{U_{t+1}'} \right) \right] - \frac{1}{\beta E_t \left(\frac{U_{t+1}'}{U_t'} \right)} \tag{8}$$

Note that $\frac{U_{t+2}'}{U_t'} = \frac{U_{t+2}'}{U_{t+1}'} \frac{U_{t+1}'}{U_t'}$. Using this in the above expression and using the earlier notation for the marginal rate of substitution yields:

$$TP_t = \beta^{-1} \left\{ \frac{E_t \left(S_{t+1,t+2} \right)}{E_t \left(S_{t,t+1} \cdot S_{t+1,t+2} \right)} - \frac{1}{E_t \left(S_{t,t+1} \right)} \right\}$$

Or, upon rearranging:

$$TP_t = \frac{E_t \left(S_{t,t+1} \right) E_t \left(S_{t+1,t+2} \right) - E_t \left(S_{t,t+1} \cdot S_{t+1,t+2} \right)}{\beta E_t \left(S_{t,t+1} \cdot S_{t+1,t+2} \right) E_t \left(S_{t,t+1} \right)} \tag{9}$$

or

$$TP_t = - \frac{Cov_t \left(S_{t,t+1}, S_{t+1,t+2} \right)}{\beta E_t \left(S_{t,t+1} \cdot S_{t+1,t+2} \right) E_t \left(S_{t,t+1} \right)} \tag{10}$$

The denominator in eq.(10) is positive so that the sign of the term premium is the negative of the sign of the covariance term. But note that the covariance term determines the autocorrelation of the agent's marginal rate of substitution. The expression implies that, since $TP_t > 0$ in the data, agent's marginal rate of substitution must be negatively autocorrelated. Interpret this result in terms of a risk premium.