

## Midterm Exam - Answer Key

*Directions: Answer all questions; the questions are weighted equally. For full credit, you must provide **complete** explanations for your answers.*

- Greece is currently experiencing a severe public finance crisis. In particular, some analysts argue that the combination of high interest rates on Greek debt combined with a low growth rate of GDP for Greece imply that the path of Greek debt is not sustainable. Using the government budget constraint (and abstracting from money and inflation) provide an argument supporting this forecast.

ANSWER: To analyze the sustainability of Greece's debt we use the government's flow budget constraint:

$$P_t G_t + B_{t-1} R_{t-1} = P_t T_t + (B_t - B_{t-1}) + (M_t - M_{t-1}), \quad (1)$$

where  $P_t$  is the price level at time  $t$ ,  $G_t$  is government spending (in real terms) at time  $t$ ,  $B_t$  represent the dollar amount in outstanding government bonds at time  $t$ ,  $T_t$  denotes the tax revenue (in real terms) at time  $t$ , and  $R_{t-1}$  is the nominal interest rate on treasury bonds issued in period  $t - 1$ . We will ignore inflation and money and can therefore set  $P_t = 1$  and  $M_t = M_{t-1}$  for all time periods,  $t = 0, 1, 2, \dots$ . With these simplifying assumptions equation (1) reduces to

$$\begin{aligned} G_t + B_{t-1} r_{t-1} &= T_t + (B_t - B_{t-1}) \\ B_t &= (G_t - T_t) + (1 + r_{t-1}) B_{t-1} \\ B_t &= D_t + (1 + r_{t-1}) B_{t-1} \end{aligned} \quad (2)$$

where  $D_t$  denotes Greece's primary deficit and  $r_{t-1} \equiv R_{t-1} - \pi_{t-1} = R_{t-1}$  is the real interest rate on government debt issued in period  $t - 1$  (Remember, we are abstracting from inflation and hence  $\pi_{t-1} = 0$ ). In order to relate Greece's path of debt to its economic growth we divide both sides of equation (2) by Greece's GDP,  $Y_t$ .

$$\begin{aligned} \frac{B_t}{Y_t} &= \frac{D_t}{Y_t} + (1 + r_{t-1}) \frac{B_{t-1}}{Y_t} \\ \frac{B_t}{Y_t} &= \frac{D_t}{Y_t} + (1 + r_{t-1}) \frac{B_{t-1}}{Y_{t-1}} \frac{Y_{t-1}}{Y_t} \\ \frac{B_t}{Y_t} &= \frac{D_t}{Y_t} + \left( \frac{1 + r_{t-1}}{1 + n_{t-1}} \right) \frac{B_{t-1}}{Y_{t-1}}, \end{aligned} \quad (3)$$

where  $n_{t-1} \equiv \frac{Y_t}{Y_{t-1}} - 1$  denotes Greece's GDP growth rate. Assuming that the primary deficit as a fraction of GDP,  $d_t \equiv \frac{D_t}{Y_t}$ , the interest rate, as well as the rate of GDP growth stay constant, then the path of Greece's debt to GDP ratio,  $b_t \equiv \frac{B_t}{Y_t}$ , evolves according to the difference equation

$$b_t = d + \frac{1 + r}{1 + n} \cdot b_{t-1}. \quad (4)$$

To see how the interest rate,  $r$ , and GDP growth,  $n$ , determine Greece's path of debt let's assume that its current debt to GDP ratio was 100% of GDP, i.e.  $b_0 = 1$ . Further, let's assume that Greece manages to keep a zero primary deficit from now on, i.e.  $d = 0$ . From equation (4) it follows that the path of Greece's debt is

$$\begin{aligned} b_0 &= 1 \\ b_1 &= \frac{1 + r}{1 + n} \\ b_2 &= \left( \frac{1 + r}{1 + n} \right)^2 \\ &\vdots \\ b_t &= \left( \frac{1 + r}{1 + n} \right)^t. \end{aligned} \quad (5)$$

Hence, Greece's debt as a fraction of GDP in the very distant future is given by the limit

$$\lim_{t \rightarrow \infty} b_t = \begin{cases} 0 & \text{if } r < n, \text{ and} \\ \infty & \text{if } r > n. \end{cases} \quad (6)$$

In other words, if interest rates are higher than GDP growth rates, then even if Greece issues no further debt the interest payments on the current 100% of GDP debt burden keep accumulating faster than the Greek economy grows. This is why some analysts believe that Greece's current debt is not sustainable given the projections of slow GDP growth—not even if Greece were to maintain a zero primary deficit.

2. In his model of optimal taxes without a commitment mechanism, S. Fischer solved the economy backwards in time. That is, he first solved for optimal choices in the last period of the economy and then solved for optimal choices in the first period. From this perspective, answer the following:

- (a) In the second period of the economy, what determined the optimal tax rates on capital and labor. Explain.
- (b) The time consistent equilibrium was characterized by the intersection of two functions that summarized the backwards solution procedure. Explain what these functions describe and why their intersection implies a time-consistent equilibrium. (It is not necessary to write down the functions - simply give a verbal description of the relationships represented by the two functions.)

ANSWER: (a) The optimal tax rate in the second period is no tax on labor. The reason is that capital is inelastically supplied in the second period so there is no deadweight welfare loss to taxing capital. (b) The two functions are: (1) the optimal tax rate on capital (chosen by the government in period 2) and (2) the desired amount of savings (i.e. capital) chosen by households in period 1. The function in (1) is determined by the government taking as exogenous the capital stock chosen by households in the first period of the economy. That is  $\tau_k = f(k)$ . The function in (2) is determined by households taking as given the tax rate they expect in period 2 (in particular, that the tax rate on labor will be zero) and choosing capital. That is,  $k = g(\tau_k^e)$ . A time consistent solution requires the tax rate to be consistent with these two functions. That is,  $\tau_k = f(g(\tau_k))$ .

3. Assume that a household lives for two periods and has preferences given by

$$U(c_0, c_1) = -c_0^{-1} - \beta c_1^{-1}$$

where  $c_0$  denotes first period consumption,  $c_1$  is second period consumption, and  $0 < \beta < 1$  represents the time discount factor. The household receives no income in the first period and an income of  $Y_1$  in the second period. Consumption in the first period must be financed by borrowing - the interest rate is  $(1+r)$ . Assume that  $\beta(1+r) = 1$

- (a) Set up the household's maximization problem as a Lagrangian and show that the intertemporal marginal rate of substitution is equal to  $(1+r)$ . Interpret this result.
- (b) Derive the household's demand function for  $c_1$ . How responsive is  $c_1$  to a change in  $r$ ? Interpret your result.
- (c) Solve for the household's indirect utility function.

ANSWER: Since the household doesn't receive any income in period 0 the intertemporal budget constraint is given by

$$c_0 + \frac{c_1}{1+r} = \frac{Y_1}{1+r} \quad (7)$$

and we can represent the maximization problem as the following Lagrangian

$$\max_{c_0, c_1, \lambda} \mathcal{L}(c_0, c_1, \lambda) = -c_0^{-1} - \beta c_1^{-1} + \lambda \left[ \frac{Y_1}{1+r} - c_0 - \frac{c_1}{1+r} \right]. \quad (8)$$

The first order conditions are then

$$\frac{\partial \mathcal{L}(c_0, c_1, \lambda)}{\partial c_0} = c_0^{-2} - \lambda = 0 \quad \Leftrightarrow \quad c_0^{-2} = \lambda \quad (9)$$

$$\frac{\partial \mathcal{L}(c_0, c_1, \lambda)}{\partial c_1} = \beta c_1^{-2} - \frac{\lambda}{1+r} = 0 \quad (10)$$

$$\frac{\partial \mathcal{L}(c_0, c_1, \lambda)}{\partial \lambda} = \frac{Y_1}{1+r} - c_0 - \frac{c_1}{1+r} = 0 \quad (11)$$

Combining equations (9) and (10) yields

$$\begin{aligned}\beta c_1^{-2} - \frac{c_0^{-2}}{1+r} &= 0 \\ \frac{c_0^{-2}}{\beta c_1^{-2}} &= 1+r.\end{aligned}\tag{12}$$

The left hand side of equation (12) is the intertemporal marginal rate of substitution (IMRS) since

$$\text{IMRS} \equiv \frac{\frac{\partial U(c_0, c_1)}{\partial c_0}}{\frac{\partial U(c_0, c_1)}{\partial c_1}} = \frac{c_0^{-2}}{\beta c_1^{-2}}.\tag{13}$$

Therefore, equation (12) implies that, at an optimum, the IMRS is equal to  $1+r$ . This means that, at an optimum, the rate at which the consumers are willing to shift consumption from tomorrow to today has to exactly equal the rate at which they can borrow that additional unit of consumption. Graphically, this means that, at the optimal consumption profile  $(c_0^*, c_1^*)$ , the consumer's indifference curve is exactly tangent to the budget line.

(b) From equation (12) and the assumption that  $\beta(1+r) = 1$  it follows that the optimal consumption profile is flat and households choose to perfectly smooth their consumption:

$$\begin{aligned}\frac{c_0^{-2}}{c_1^{-2}} &= \beta(1+r) = 1 \\ c_0^{-2} &= c_1^{-2} \\ c_0 &= c_1 = c^*.\end{aligned}\tag{14}$$

Consequently, the budget constraint (7) or, equivalently, the FOC with respect to the Lagrange multiplier (11) implies that

$$\begin{aligned}c^* + \frac{c^*}{1+r} &= \frac{Y_1}{1+r} \\ \left[1 + \frac{1}{1+r}\right] c^* &= \frac{Y_1}{1+r} \\ \left[\frac{2+r}{1+r}\right] c^* &= \frac{Y_1}{1+r} \\ c^* &= \frac{Y_1}{2+r}\end{aligned}\tag{15}$$

This is the annuity value of the consumer's lifetime income. It is the constant payment whose present value is equal to the present value of the household's lifetime income. To better see this we can also rewrite equation (15) using the same notation as in class and in chapter 14 of Doepke, Lehnert and Sellgren's text

$$c^* = \frac{0 + \frac{Y_1}{1+r}}{1 + \frac{1}{1+r}} = \frac{\bar{Y}}{1 + \frac{1}{1+r}} = \bar{W}.\tag{16}$$

Also, since the household receives all her income in the second period of life she can only achieve her goal of perfect consumption smoothing by borrowing at the interest rate  $r$ . If this interest rate rises, and hence borrowing becomes more expensive, the optimal amount of consumption in both periods will fall. Formally, we can easily show that

$$\frac{dc^*}{dr} = -\frac{Y_1}{(2+r)^2} < 0\tag{17}$$

(c) The household's indirect utility function is the utility function evaluated at the optimal consumption choices:

$$V(Y_1, r) = -\frac{1}{c_0^*} - \beta \frac{1}{c_1^*} = -\frac{2+r}{Y_1} - \beta \frac{2+r}{Y_1} = -\frac{(1+\beta)(2+r)}{Y_1}\tag{18}$$

A quick inspection of the indirect utility function tells us that higher interest rates imply lower utility and higher income implies higher utility. Formally, we have that

$$\begin{aligned}\frac{\partial V(Y_1, r)}{\partial r} &= -\frac{(1+\beta)}{Y_1} < 0, \text{ and} \\ \frac{\partial V(Y_1, r)}{\partial Y_1} &= \frac{(1+\beta)(2+r)}{Y_1^2} > 0.\end{aligned}$$

Again, this is because the household can only smooth consumption by borrowing at the interest rate  $r$ . The higher this rate, the harder it is for the household to achieve her consumption smoothing objective. Obviously, more lifetime income will allow her to consume more in either period and hence enjoy more lifetime utility.

4. The government of Finlandia is faced with the Ramsey problem of choosing a sequence of excise taxes so that the utility of its citizens is maximized. It uses this revenue to pay for a war in the first period of the economy. The government knows that households' optimal consumption is given by:

$$c_t^* = \frac{\sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t}}{1 + \tau_t} \quad (19)$$

$Y_t$  denotes income and it is assumed that  $Y_t = 10$  in every period. The government has expenditures of  $G_0 = 2$  in the very first period (this is the war expenditures) and nothing thereafter; that is,  $G_t = 0$  for  $t \geq 1$ . Given this environment, answer the following

- (a) Find the optimal excise tax,  $\tau^*$ . (Hint: Use the conditions that the tax smoothing hypothesis implies taxes are constant,  $\tau_t = \tau^*$ , and the government's intertemporal budget constraint implies that the present discounted value of tax revenues (given by  $c_t \tau_t$  in every period) equals the present discounted value of government expenditures. You do not have to solve for the indirect utility function.)
- (b) Find the implied path of government debt. Does this satisfy the transversality condition?

ANSWER: There is a typo in this question. Consumption should have been given as:

$$c_t^* = \frac{\left(\frac{r}{1+r}\right) \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t}}{1 + \tau_t}$$

where the numerator is the annuity value of household's consumption. Since they receive 10 units every period, and we know that consumption and taxes will be constant, we have

$$c^* (1 + \tau^*) = 10$$

We also know that the present discounted value of tax revenues must equal the present discounted value of government expenditures:

$$\sum_{t=0}^{\infty} \frac{c^* \tau^*}{(1+r)^t} = \left(\frac{\tau^*}{1+\tau^*}\right) 10 \left(\frac{1+r}{r}\right) = 2 \quad (20)$$

Solving for  $\tau^*$  yields:

$$\tau^* = \frac{r}{5 + 4r}$$

Denote tax revenues (also constant) as  $T^* = c^* \tau^* = \left(\frac{\tau^*}{1+\tau^*}\right) 10$ . But note from eq.(20) that  $\left(\frac{\tau^*}{1+\tau^*}\right) = \left(\frac{1}{5}\right) \left(\frac{r}{1+r}\right)$ . So we have

$$T^* = 2 \left(\frac{r}{1+r}\right)$$

To find the path of debt, use the government budget constraint:

$$B_t = G_t + B_{t-1} (1+r) - T^*$$

In period 0 (assuming  $B_{-1} = 0$ ) we have:

$$B_0 = 2 - 2 \left(\frac{r}{1+r}\right) = \frac{2}{1+r}$$

In period 1 we have:

$$B_1 = 0 + \frac{2}{1+r}(1+r) - 2\left(\frac{r}{1+r}\right) = \frac{2}{1+r}$$

This implies that the debt stays constant at  $2/(1+r)$ . This clearly satisfies the transversality condition:

$$\lim_{t \rightarrow \infty} \frac{B_t}{(1+r)^t} = \lim_{t \rightarrow \infty} \frac{2}{(1+r)^{t+1}} = 0$$